Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 21, 1 (1991), 117-125

Review of Research
Faculty of Science
Mathematics Series

ON SEQUENTIAL MEASURES OF NONCOMPACTNESS IN TOPOLOGICAL VECTOR SPACES

Lothar Kaniok
Pādagogische Hochschule "Karl Friedrich Wilhelm Wander" Dresden
Sektion Mathematik
Wigardstraße 17, DDR, Dresden, 8060

Abstract

The notions of the sequential φ–measure of noncompactness and of the sequentially (φ, γ)–condensing mapping in topological vector spaces are introduced. Some examples of sequential φ–measures of noncompactness are given.

AMS Mathematics Subject Classification (1980): 47H10
Key words and phrases: measure of noncompactness, condensing mapping, topological vector space.

1. Introduction

If a measure of noncompactness is not defined on all, but only on the countable subsets of a locally convex space, then Sadovskij call it a sequential measure of noncompactness ([6]). Proceeding on [6; 1.4] and [3; Definition 4], we shall introduce the notion of the sequential φ–measure of noncompactness in topological vector spaces. Referring to this, we shall consider some examples of φ–measures of noncompactness. At this, the φ–measure auf noncompactness $J_{\mathcal{U}}$ introduced in [5] is of exceptional importance, just as the well-known Istrătescu's measure of noncompactness (cp. [1]).

Finally, we shall define the notion of the sequentially (φ,γ)−condensing mapping in topological vector spaces. At this, the condition

$$\varphi(\gamma(T(M))) \geq \gamma(M) \Rightarrow \overline{T(M)} \text{ is compact } \ (M \subseteq D(T)),$$

which is fundamental for a (φ,γ)−condensing mapping T (cp. [3], [5]), is relaxed by assuming that it holds only for at most countable sets $M \subseteq D(T)$. We shall show, that every sequentially $(\varphi, J_{\mathcal{U}})$−condensing mapping is $(\varphi, J_{\mathcal{U}})$−condensing, too.

2. Notions and definitions

In this paper every topological vector space will be assumed to be separated and real. Let E be a topological vector space and $K \subseteq E$. By $\mathcal{U}(E)$ we shall denote a fundamental system of circled, closed neighbourhoods of zero in E and by $\mathcal{F}_{\mathcal{U}}$ the set of all nonnegative functions on $\mathcal{U}(E)$ with the natural order. Moreover, by $\overline{K}$, co$K$, $\overline{\text{co}}K$ and ∂K we shall denote the closed hull, the convex hull, the closed convex hull and the boundary of K, respectively. We define $2^K := \{M \subseteq K : M \neq \emptyset\}$, $b(K) := \{M \in 2^K : M \text{ is bounded}\}$, $\text{cc}(K) := \{M \in 2^K : M \text{ is closed in } K,\ M \text{ is convex}\}$ and $\text{fucc}(E) := \{K \subseteq E : K = \bigcup_{i \in I} K_i,\ I \text{ is finite, } K_i \in \text{cc}(E) \text{ for all } i \in I\}$.

We say that $K \in 2^E$ is of *Zima's type* ([3]), iff for every $U \in \mathcal{U}(E)$ there exists $V \in \mathcal{U}(E)$ so that $\text{co}(V \cap (K - K)) \subseteq U$. Special examples of sets of Zima's type are given in [2] and [3]. K is said to be ***starshaped, relative to some*** $u \in K$, iff $tx + (1-t)u \in K$ for all $x \in K$ and all $t \in [0,1]$.

Let $K \in 2^E$, $M \in b(\overline{\text{co}}K)$ and $U \in \mathcal{U}(E)$. As in [5] we define

$$\begin{aligned} J(M,U) \ &:= \ \sup\{a \geq 0 : M \text{ contains a countable set } \{x_n : n \in \mathbb{N}\} \\ &\qquad \text{such that } x_i - x_k \notin aU \text{ for } i \neq k\} \end{aligned}$$

($\sup \emptyset = 0$, by definition). By $[J_{\mathcal{U}}(M)](U) := J(M,U)$ a mapping $J_{\mathcal{U}} : b(\overline{\text{co}}K) \to \mathcal{F}_{\mathcal{U}}$ is defined.

Now, let (E,d) be a metric space, $K \in 2^E$ and $M \in b(\overline{\text{co}}K)$. We define ([1]):

$$\begin{aligned} \alpha(M) \ &:= \ \inf\{a > 0 : M \text{ has a finite } a-\text{net in } E\}, \\ \beta(M) \ &:= \ \inf\{a > 0 : M \text{ has a finite } a-\text{net in } M\}, \end{aligned}$$

$$J(M) := \sup\{a > 0 : M \text{ contains an infinite } a - \text{discrete set}\} \quad (\sup \emptyset = 0)$$

and

$$\chi(M) := \inf\{a > 0 : M \text{ has a finite } a - \text{cover}\}.$$

The mappings α, β, J, χ are the well-known Hausdorff's, inner Hausdorff's, Istrătescu's and Kuratowski's function, respectively. For fundamental properties of these functions see [1], [2], [3], [6], [7], respectively.

In [1] Daneš asked the question ([1; Problem 1]): Is J algebraically subadditive? The following result is the answer to this problem.

Proposition 1. *Let (E, d) be a metric vector space and $K \in 2^E$. Then, the inequality*

$$J(M + N) \leq J(M) + J(N) \qquad (M \in b(\overline{co}K),\ N \in b(\overline{co}K))$$

holds.

Proof. Without loss of generality we may assume that $J(M + N) > \max\{J(M),\ J(N)\}$. Let $\varepsilon > 0$ such that $J(M) + \varepsilon =: a < b := J(M + N) - \varepsilon$. Then there is an infinite set $\{x_n + y_n : n \in \mathrm{N}\} \subseteq M + N$ such that $d(x_i + y_i,\ x_k + y_k) > b$ for $i \neq k$. Since $J(M) > a$, there is an infinite set

$$\{x_{n_j} + y_{n_j} : j \in \mathrm{N}\} \subseteq \{x_n + y_n : n \in \mathrm{N}\}$$

with

$$b < d(x_{n_i} + y_{n_i}, x_{n_k} + y_{n_k}) \leq d(x_{n_i}, x_{n_k}) + d(y_{n_i}, y_{n_k}) \leq a + d(y_{n_i}, y_{n_k})$$

for $i \neq k$.

Then there must be $d(y_{n_i}, y_{n_k}) > b - a$ for $i \neq k$. From this the assertion follows.

3. Sequential φ–measures of noncompactness in topological vector spaces

In the following, let E be a topological vector space, $K \in 2^E$, A a partially ordered set with the partial ordering $\leq$, $\varphi : A \to A$ and $\mathcal{M}$ a system of subsets of $\overline{\text{co}}K$ such that:

$$M \in \mathcal{M} \Rightarrow (\overline{M} \in \mathcal{M}, \text{co}M \in \mathcal{M}, M \cup \{u\} \in \mathcal{M}(u \in K), N \in \mathcal{M}(N \subseteq M)),$$

if we don't state additional claims.

Moreover we denote by $\mathcal{M}_s$ the system of all at most countable sets of $\mathcal{M}$.

Definition 1. ([3]) *The mapping* $\gamma : \mathcal{M} \to A$ *is said to be a* $\varphi-$measure of noncompactness on K, *iff the following conditions are satisfied:*

$$(1) \quad \gamma(\overline{M}) = \gamma(M \cup \{u\}) = \gamma(M) \geq \gamma(N) \qquad (M \in \mathcal{M},\ N \subseteq M,\ u \in K),$$

$$(2) \qquad \gamma(\text{co}M) \leq \varphi(\gamma(M)) \qquad (M \in \mathcal{M}).$$

If $\varphi(t) = t \quad (t \in A)$, *then* γ *is called measure of noncompactness on* K.

Examples.

(a) Let K be a set of Zima's type with $K \in b(E)$, which is starshaped, relative to some $u \in K$. For every $U \in \mathcal{U}(E)$ we can choose a neighbourhood $W_u \in \mathcal{U}(E)$ (fixed) such that $J(\text{co } M, U) \leq J(M, W_u)$ for every $M \subseteq K$ ([5; Corollary to Lemma 2]). By $v(U) := W_u(U \in \mathcal{U}(E))$ and $\varphi^*(f) := f \circ v(f \in \mathcal{F}_{\mathcal{U}})$ we define a mapping φ^* of $\mathcal{F}_{\mathcal{U}}$ into $\mathcal{F}_{\mathcal{U}}$ and $J_{\mathcal{U}}$ is a φ^*- measure of noncompactness on K ([5; Proposition 2.]).

(b) Let (E,p) be a paranormed space (s. [2], [3]). E is a metrizable topological vector space. The fundamental system of neighbourhoods of zero in E is given by the family $\mathcal{V} = \{V_r : \ r > 0\}$, where $V_r := \{x \in E : p(x) < r\}$.

Let $K \in 2^E$ and $\varphi : (0,\infty) \to (0,\infty)$. The set K is said to be of $Z_\varphi-$*type* ([3]), iff, for every $r > 0$, $\text{co}(V_r \cap (K - K) \subseteq V_{\varphi(r)}$. In [3] Hadžić gave an example of a set of $Z_\varphi-$type. Every set, which is of $Z_\varphi-$type, is of Zima's type also, if $\inf\{\varphi(r) : r > 0\} = 0 \quad$ ([3]).

Let K be a convex set with $K \in b(E)$. Hadžić proved in [3] that the inner Hausdorff's and the Kuratowski's function satisfy the condition (2) from Definition 1, if K is of $Z_\varphi-$type and $\varphi : [0,\infty) \to [0,\infty)$ is a right continuous and a continuous mapping, respectively. The mapping χ satisfies (1) from Definition 1 also, but β is not monotone, in general (s. [1]).

On K the following inequality holds ([1; Proposition 1]):

$$\alpha \leq \beta \leq J \leq \chi \leq 2\alpha.$$

Hence, the Istrătescu's function J and the Kuratowski's function χ are $2\varphi-$measures of noncompactness on K, if K is of $Z_\varphi-$type and $\varphi : [0,\infty) \to [0,\infty)$ is a right continuous, non decreasing mapping. If, in addition, φ is positive homogeneous then the Hausdorff's function α is a $2\varphi-$measure of noncompactness on such a set K, too.

Definition 2. *We say, that the mapping $\gamma : \mathcal{M}_s \to A$ is a sequential $\varphi-$measure of noncompactness on K, iff, for all $M, N \in \mathcal{M}_s$ the following conditions hold:*

$$(1) \qquad \gamma(N) \leq \gamma(M) = \gamma(M \cup \{u\}) \qquad (N \subseteq \overline{M},\ u \in K),$$

$$(2) \qquad \gamma(N) \leq \varphi(\gamma(M)) \qquad (N \subseteq \mathrm{co}M)$$

If $\varphi(t) = t (t \in A)$ then γ will be called a sequential measure of noncompactness on K.

Remark. Such properties as the subadditivity, the algebraic subadditivity and the positive homogenity have the same importance, which is well-known from the theory of the measures of noncompactness, for sequential $\varphi-$measures of noncompactness, too.

It is obvious that every $\varphi-$measure of noncompactness is a sequential $\varphi-$measure of noncompactness also. Sadovskij proved ([6]) that in a metric vector space every sequential measure of noncompactness generates a measure of noncompactness in specified way by the preservation of the basic properties. Analogously to Theorem 1.1. from [6] we can prove the

Theorem 1. *Let E be a metric vector space, $K \in 2^E, \varphi : A \to A$ a non decreasing mapping and γ a sequential $\varphi-$measure of noncompactness on K. Then, by*

$$\text{(i)} \qquad \tilde{\gamma}(M) := \sup\{\gamma(Z) : Z \in \mathcal{M}_s,\ Z \subseteq M\}\ (M \in \mathcal{M})$$

a $\varphi-$measure of noncompactness on K is defined.

If the implication ($M \in \mathcal{M}$, $N \in \mathcal{M} \Rightarrow M \cup N \in \mathcal{M}$, $M+N \in \mathcal{M}$, $cM \in \mathcal{M}$ $(c \in \mathbf{R})$) holds and the mapping γ is positively homogeneous, subadditive or algebraically subadditive, then $\tilde{\gamma}$ has the corresponding property.

Proof. (cp. [6; Theorem 1.1.]).

(1) The relations

$$\tilde{\gamma}(N) \leq \tilde{\gamma}(M \cup \{u\}) = \tilde{\gamma}(M) \leq \tilde{\gamma}(\overline{M}) \qquad (M \in \mathcal{M},\ N \subseteq M,\ u \in K)$$

and the positive homogenity, the subadditivity and the algebraic subadditivity follow from the definition of $\tilde{\gamma}$ and from the equivalent properties of γ.

We shall prove the inequality $\tilde{\gamma}(\overline{M}) \leq \tilde{\gamma}(M)$ $(M \in \mathcal{M})$. The assertion is clear for finite sets. Let M be an infinite set of $\mathcal{M}$ and $Z := \{z_n\}$an at most countable subset of $\overline{M}$. Then there are $x_{nm} \in M$ such that $\lim_{m\to\infty} x_{nm} =$ $(z_n \in Z)$. We denote the set of all x_{nm} by Z_0. Z_0 is a countable subset of M with $Z \subseteq \overline{Z}_0$. Therefore $\gamma(Z) \leq \gamma(Z_0)$ and $\tilde{\gamma}(\overline{M}) \leq \tilde{\gamma}(M)$.

(2) Let $M \in \mathcal{M}$ and $Z = \{z_n\}$ an at most countable subset of co M. For every $z_n \in Z$ there exist $x_{n1}, \ldots, x_{nm(n)} \in M$ and $c_{n1}, \ldots, c_{nm(n)} \geq 0$ with $\sum_{k=1}^{m(n)} c_{nk} = 1$ such that $z_n = \sum_{k=1}^{m(n)} c_{nk} x_{nk}$. Now we denote the set of all $x_{nk} (k \in \{1, \ldots, m(n)\})$ by Z_0. Z_0 is an at most countable subset of M with $Z \subseteq \text{co}\ Z_0$. Hence $\gamma(Z) \leq \varphi(\gamma(Z_0))$. Thus $\tilde{\gamma}(\text{co}\ M) \leq \varphi(\tilde{\gamma}(M))$.

The question of the unicity of the extension of a sequential φ−measure of noncompactness from $\mathcal{M}_s$ onto $\mathcal{M}$ arises. In particular, Sadovskij asked the question ([6]):
Is $\alpha = \tilde{\alpha}$, if the Hausdorff's measure of noncompactness α is extended to $\tilde{\alpha}$ according to (i)? The equality holds in separable spaces ([6; p. 96]). In arbitrary metric vector spaces the question is unanswered.

Now we deal with this problem and give a partial answer.

Lemma 1. *Let $M \in b(\overline{\text{co}}K)$ and $U \in \mathcal{U}(E)$. It is*

$$J(M,U) = \max\{J(Z,U) : Z \subseteq M,\ Z \text{ is at most countable}\}.$$

Proof. Since $J_{\mathcal{U}}$ is monotone ([5]), the inequality $\sup\{J(Z,U) : Z \subseteq M,\ Z$ is at most countable $\} \leq J(M,U)$ holds. It is easy to see that the equality holds, if $J(M,U) = 0$. Let $\varepsilon > 0$ and $J(M,U) > 0$. There exists a countable subset $Z = \{z_n : n \in \mathbb{N}\}$ of M such that $z_m - z_n \notin (1+\varepsilon)^{-1} \cdot J(M,U)U$ for $m \neq n$. Hence $J(Z,U) \geq (1+\varepsilon)^{-1} \cdot J(M,U)$. Thus $\sup\{J(Z,U) : Z \subseteq M,\ Z$ is at most countable$\} \geq J(M,U)$, too. Finally, we can write max instead of sup, since always from $\lim_{n\to\infty} J(Z_n, U) = J(M, U)$ follows $J(\bigcup_{n\in\mathbf{N}} Z_n, U) = J(M, U)$.

Corollary 1. *The equality $J_{\mathcal{U}} = \tilde{J}_{\mathcal{U}}$ (s. (i)) holds on every set $K \in 2^E$ of an arbitrary topological vector space E.*

Remark. In the same way as in the proof of the above Lemma it can be proved that for the Istrătescu's measure of noncompactness J the equality $J = \tilde{J}$ (s. (i) with max instead of sup) is true on every set $K \in 2^E$ of a metric space E. Since the functions α and χ are monotone and $\alpha \leq \beta \leq J \leq \chi \leq 2\alpha$ ([1]) the following inequalities hold in any metric space:

$$\frac{1}{2}\gamma \leq \tilde{\gamma} \leq \gamma \quad (\gamma \in \{\alpha, \chi\}) \qquad \text{and} \qquad \frac{1}{2}\beta \leq \tilde{\beta} \leq 2\beta.$$

The same relations are true for the well-known Hausdorff's and Kuratovski's measure of noncompactness in locally convex spaces. This is a partial answer to the above-mentioned problem from [6; p. 96].

4. Sequentially (φ, γ)–condensing maps

The same agreements from the section 2 let be true for $E, K, A, \varphi, \mathcal{M}$ and $\mathcal{M}_s$. Moreover, let $M \in 2^K$ and $F : M \to \mathrm{cc}(K)$ be an upper semicontinuous mapping.

Definition 3 *(cp. [5]). Let γ be a φ–measure of noncompactness on K. The mapping F is said to be a (φ, γ)– condensing mapping, iff for every $N \subseteq M$ the following implication holds:*

(ii) $$\gamma(N) \leq \varphi(\gamma(F(N))) \Rightarrow \overline{F(N)} \text{ is compact.}$$

We call F sequentially (φ, γ)–condensing, iff (ii) *holds for every at most countable subset N of M.*

In [5] we gave an example of a $(\varphi^*, J_{\mathcal{U}})$–condensing mapping in an arbitrary topological vector space, where φ^* is the mapping defined in the example (a) of section 3.

It is obvious that a (φ,γ)–condensing mapping is sequentially (φ,γ)–-condensing as well.

Theorem 2. *In any topological vector space a sequentially $(\varphi, J_{\mathcal{U}})$–condensing mapping $F : M \to \mathrm{cc}(K)$ is $(\varphi, J_{\mathcal{U}})$–condensing.*

Proof. Let $N \subseteq M$ and let $F(N)$ be not relatively compact. From the above Lemma it follows that there is a countable, not relatively compact subset Z of $F(N)$ such that $J_{\mathcal{U}}(Z) = J_{\mathcal{U}}(F(N))$. We choose an at most countable subset Z_0 of N with $F(Z_0) = Z$. Since $\overline{F(Z_0)}$ is not compact and F is sequentially $(\varphi, J_{\mathcal{U}})$–condensing,we obtain

$$J_{\mathcal{U}}(N) \geq J_{\mathcal{U}}(Z_0) > \varphi(J_{\mathcal{U}}(F(Z_0))) = \varphi(J_{\mathcal{U}}(Z)) = \varphi(J_{\mathcal{U}}(F(N))).$$

Hence, the inequality $J_{\mathcal{U}}(N) \leq \varphi(J_{\mathcal{U}}(F(N)))$ $(N \subseteq M)$ implies the compactness of $\overline{F(N)}$. This means that F is $(\varphi, J_{\mathcal{U}})$–condensing.

Remark. Just as in Theorem 2 it can be proved that in any metric vector space a sequentially (φ, J)–condensing mapping is (φ, J)–condensing, since $J = \tilde{J}$.

Now, from [5; Corollary to Theorem 1 and Corollary to Theorem 2] we obtain the following fixed point theorem.

Theorem 3. *Let K be a set of Zima's type with $K \in b(E)$, which is star-shaped, relative to some $u \in K$, and φ^* the mapping defined in the example* (a) *of the section 2. Moreover, one of the following conditions holds:*

(1) Let $K \in \mathrm{fucc}(E)$, $U \subseteq K$ an in K closed neighbourhood of u and $F : U \to \mathrm{cc}(K)$ a sequentially $(\varphi^*, J_{\mathcal{U}})$–condensing mapping with $x \notin tF(x) + (1-t)u \quad (x \in \partial_K U,\ t \in (0,1))$.

(2) Let $F : K \to \mathrm{cc}(K)$ be a sequentially $(\varphi^*, J_{\mathcal{U}})$–condensing mapping.

Then F has a fixed point.

For continuous single-valued maps the notion of the sequentially (φ,γ) - condensing mapping can be defined as in Definition 3 by proceeding on the assumption that γ is a sequential φ–measure of noncompactness.

Using Theorem 1, where in (i) "sup" can be replaced by "max" also (cp. with the proof of the above Lemma), with the methods of the proof of Theorem 2 it can be proved the

Theorem 4. *Let E be a metric vector space, $K \in 2^E$, $M \in 2^K$, $\varphi : A \to A$ non decreasing and γ a sequentially $\varphi-$measure of noncompactness on K. Then a sequentially $(\varphi,\gamma)-$condensing mapping $G : M \to K$ is $(\varphi,\tilde{\gamma})-$condensing as well, where $\tilde{\gamma}$ is the $\varphi-$measure of noncompactness defined by* (i).

References

[1] Daneš, J.: On the Istrătescu's measure of noncompactness, Bull. Math. Soc. R. S. Roumanie, 16(64), No. 4, 1972.

[2] Hadžić, O.: Fixed point theory in topological vector spaces, Institute of Mathematics, Novi Sad, 1984, 327 pp.

[3] Hadžić, O.: Some properties of measures of noncompactness in paranormed spaces, Proc. Amer. Math. Soc., Vol. 102, No. 4, 1988, 843-849.

[4] Janos, L., Martelli, M.: Sequentially condensing maps, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 16,1(1986), 85-94.

[5] Kaniok, L.: On measures of noncompactness in general topological vector spaces, Comment. Math. Univ. Carolinae, 31,3(1990), 479-487.

[6] Sadovskij, B. N.: On measures of noncompactness and densifying operators, Probl. Mat. Anal. Slozn. Sistem, Vol. 2, Voronez, 1968 (in Russian).

[7] Sadovskij, B. N.: Asymptotically compact and densifying operators, Usp. Mat. Nauk 27, No. 1, 1972 (in Russian).

REZIME

O SEKVENCIJALNOJ MERI NEKOMPAKTNOSTI U VEKTORSKO TOPOLOŠKIM PROSTORIMA

Uvedeni su pojmovi sekvencijalne φ - mere nekompaktnosti i sekvencijalnog (φ,γ) - kondenzujućeg preslikavanja u vektorsko topološkim prostorima. Dati su neki primeri sekvencijalnih φ - mera nekompaktnosti.

Received by the editors February 26, 1990.