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# COMMON FIXED POINTS OF COMMUTING SET-VALUED MAPPINGS

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**Abstract**

Let $F$ and $G$ be continuous, commuting mappings of a complete metric space $(X,d)$ into $B(X)$ satisfying the inequality

$$\delta(F^p x, G^p y) \leq \max\{c\delta(F^r x, G^s y), \frac{1}{2}\delta(F^r x, F^{r'} x), \frac{1}{2}\delta(G^s y, G^{s'} y) : 0 \leq r,\ s \leq p;\ 0 \leq r',\ s' < p\}$$

for all $x, y$ in $X$, where $0 \leq c < 1$ and $p$ is a fixed positive integer. It is proved that if $F$ and $G$ also map $B(X)$ into itself, then $F$ and $G$ have a unique common fixed point $z$.

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In the following, as in [1], we let $(X,d)$ be a complete metric space and let $B(X)$ be the set of all nonempty, bounded subsets of $X$. The function $\delta(A,B)$ with $A$ and $B$ in $B(X)$ is defined by

$$\delta(A,B) = \sup\{d(a,b) :\ a \in A,\ b \in B\}.$$

If $A$ consists of a single point $a$ we write

$$\delta(A,B) = \delta(a,B).$$

If $B$ also consist of a single point $b$ we write

$$\delta(A,B) = \delta(a,b) = d(a,b).$$

It follows immediately that

$$\begin{aligned} \delta(A,B) &= \delta(B,A) \geq 0, \\ \delta(A,B) &\leq \delta(A,C) + \delta(C,B) \end{aligned}$$

for all $A, B$ and $C$ in $B(X)$.

If now $\{A_n : n = 1, 2, \ldots\}$ is a sequence of sets in $B(X)$, we say that it converges to the subset $A$ of $X$ if

(i) each point $a$ in $A$ is the limit of some convergent sequence $\{a_n \in A_n : n = 1, 2, \ldots\}$,

(ii) for arbitrary $\varepsilon > 0$, there exists an integer $N$ such that $A_n \subset A_\varepsilon$ for $n > N$, where $A_\varepsilon$ is the union of all open spheres with centers in $A$ and radius $\varepsilon$.

The set $A$ is then said to be the limit of the sequence $\{A_n\}$.

The following lemma was proved in [1].

**Lemma 1.** *If $\{A_n\}$ and $\{B_n\}$ are sequences of bounded subsets of a complete metric space $(X, d)$ which converge to the bounded subsets $A$ and $B$ respectively, then the sequence $\{\delta(A_n, B_n)\}$ converges to $\delta(A, B)$.*

Now let $F$ be a mapping of a complete metric space $(X, d)$ into $B(X)$. We say that the mapping $F$ is continuous at a point $x$ in $X$ if whenever $\{x_n\}$ is a sequence of points in $X$ converging to $x$, the sequence $\{Fx_n\}$ in $B(X)$ converges to $Fx$ in $B(X)$. We say that $F$ is a continuous mapping of $X$ into $B(X)$ if $F$ is continuous at each point $x$ in $X$. We say that a point $z$ in $X$ is a fixed point of $F$ if $z$ is in $Fz$. If $A$ is any nonempty subsets of $X$ we define the set $FA$ by

$$FA = \bigcup_{a \in A} Fa.$$

If $G$ is a second mappings of $X$ into $B(X)$ we say that $F$ and $G$ commute if $FGx = GFx$ for all $x$ in $X$. It then follows that $FGA = GFA$ for all nonempty subsets $A$ of $X$.

We now prove the following theorem.

**Theorem 1.** *Let $F$ and $G$ be continuous, commuting mappings of a complete metric space $(X,d)$ into $B(X)$ satisfying the inequality*

$$(1)\quad \delta(F^p x, G^p y) \leq \max\{c\delta(F^r x, G^s y), \tfrac{1}{2}\delta(F^r x, F^{r'} x), \tfrac{1}{2}\delta(G^s y, G^{s'} y) : 0 \leq r,\ s \leq p;\ 0 \leq r',\ s' < p\}$$

*for all $x, y$ in $X$ where $0 \leq c < 1$ and $p$ is a fixed positive integer. If $F$ and $G$ also map $B(X)$ into itself, then $F$ and $G$ have a unique common fixed point $z$. Further $Fz = Gz = \{z\}$.*

*Proof.* We will first of all assume, without loss of generality, that $c > \frac{1}{2}$. This will mean that $(1-c)/c < 1$.

Since we are supposing that $F$ and $G$ map $B(X)$ into itself we note that both sides of inequality (1) are finite. Further, if $A$ and $B$ are any sets in $B(X)$ then it follows that

$$(2)\quad \delta(F^p A, G^p B) \leq \max\{c\delta(F^r A, G^s B), \tfrac{1}{2}\delta(F^r A, F^{r'} A), \tfrac{1}{2}\delta(G^s B, G^{s'} B) : 0 \leq r, s \leq p;\ 0 \leq r',\ s' < p\},$$

both sides of the inequality again being finite.

Now let $x$ be an arbitrary point in $X$ and put $X_{mn} = F^m G^n x$ for $m, n = 0, 1, 2, \ldots$, where $X_{00} = x$. Let us suppose that the set of real numbers $\{K_n : n = 0, 1, 2, \ldots\}$ is unbounded, where

$$K_n = \max\{\delta(X_{n-i,i}, X_{pp}) : 0 \leq i \leq n\}.$$

Then there exists an integer $n \geq 2p$ such that

$$(3)\quad (1-c)K_n > c.\max\{K_r, \delta(X_{ps}, X_{ps'}), \delta(X_{sp}, X_{s'p}) : 0 \leq r, s, s' \leq p\}$$

and

$$(4)\quad K_n > \max\{K_r : 0 \leq r < n\}.$$

Inequality (3) implies that

$$(5)\quad K_n > \max\{\delta(X_{ps}, X_{ps'}), \delta(X_{sp}, X_{s'p}) : 0 \leq s, s' \leq p\}$$

since $(1-c)/c < 1$.

Inequalities (3) and (4) imply that

$$\begin{aligned} c\delta(X_{r-i,i}, X_{ps}) &\leq c\delta(X_{r-i,i}, X_{pp}) + c\delta(X_{pp}, X_{ps}) \\ &< cK_n + (1-c)K_n \\ &= K_n \end{aligned}$$

for $0 \leq i \leq r \leq n$ and $0 \leq s \leq p$. Similarly

$$c\delta(X_{r-i,i}, X_{sp}) < K_n$$

for $0 \leq i \leq r \leq n$ and $0 \leq s \leq p$ and so

(6) $K_n > c.\max\{\delta(X_{r-i,i}, X_{ps}), \delta(X_{r-i,i}, X_{sp}) : 0 \leq i \leq r \leq n; 0 \leq s \leq p\}$.

In the case when $0 \leq i \leq p$, inequality (4) implies that

$$\frac{1}{2}\delta(X_{r-i,i}, X_{r'-i,i}) \leq \frac{1}{2}\delta(X_{r-i,i}, X_{pp}) + \frac{1}{2}\delta(X_{pp}, X_{r'-i,i}) < K_n$$

for $n - p \leq r \leq n$ and $n - p \leq r' < n$ and so

(7)
$$K_n > \frac{1}{2}.\max\{\delta(X_{r-i,i}, X_{r'-i,i}) : 0 \leq i \leq p; \; n-p \leq r \leq n; \; n-p \leq r' \leq n\}.$$

Similarly, when $p < i \leq n$, inequality (4) implies that

(8)
$$K_n > \frac{1}{2}.\max\{\delta(X_{n-i,r}, X_{n-i,r'}) : p < i \leq n; \; i-p \leq r \leq i; \; i-p \leq r' < i\}.$$

On using inequality (2) it follows that

$$\begin{aligned} \delta(X_{n-i,i}, X_{pp}) \leq \; & \max\{c\delta(X_{r-i,i}, X_{ps}), \frac{1}{2}\delta(X_{r-i,i}, X_{r'-i,i}), \frac{1}{2}\delta(X_{ps}, X_{ps'}) : \\ & n-p \leq r \leq n; n-p \leq r' < n; 0 \leq s \leq p; 0 \leq s < p\} \end{aligned}$$

for $0 \leq i \leq p$. Inequalities (5), (6) and (7) now imply that

(9)
$$\max\{\delta(X_{n-i,i}, X_{pp}) : 0 \leq i \leq p\} < K_n.$$

Again on using inequality (2) it follows that

$$\delta(X_{n-i,i}, X_{pp}) = \delta(X_{pp}, X_{n-i,i}) \leq$$

$$\leq \max\{c\delta(X_{sp}, X_{n-i,r}),\ \frac{1}{2}\delta(X_{sp}, X_{s'p}),\ \frac{1}{2}\delta(X_{n-i,r}, X_{n-i,r'})\ :$$
$$i-p \leq r \leq i;\ i-p \leq r' < i;\ 0 \leq s \leq p;\ 0 \leq s' < p\}$$

for $p < i \leq n$. Inequalities (5), (6) and (8) now imply that

$$\max\{\delta(X_{n-i,i}, X_{pp})\ :\ i < p \leq n\} < K_n \tag{10}$$

and inequalities (9) and (10) together imply that $K_n < K_n$, a contradiction.

Thus

$$\sup\{K_n :\ n = 0,1,2,\ldots\} = \sup\{\delta(X_{n-i,i}, X_{pp}) : 0 \leq i \leq n; n = 0,1,2,...\} =$$
$$= \sup\{\delta(X_{mn}, X_{pp}) : m,n = 0,1,2,...\} < \infty$$

and so

$$\sup\{\delta(X_{mn}, X_{hk})\ :\ m,n,h,k = 0,1,2,\ldots\} \leq$$
$$\leq \sup\{\delta(X_{mn}, X_{pp}) + \delta(X_{pp}, X_{hk})\ :\ m,n,h,k = 0,1,2,\ldots\} =$$
$$= M < \infty.$$

We now note that since we are assuming that $c > \frac{1}{2}$ the following inequality holds

$$\delta(F^pA, G^pB)\ \leq\ c.\max\{\delta(F^rA, G^sB),\ \delta(F^rA, F^{r'}A),\ \delta(G^sB, G^{s'}B) :\ 0 \leq r,\ r',\ s,\ s' \leq p\} \tag{11}$$

for all $A, B$ in $B(X)$. For arbitrary $\varepsilon > 0$, choose an integer $N$ such that $c^NM < \varepsilon$. Then if $m,n,h,k \geq Np$ we have with repeated use of inequality (11)

$$\begin{aligned}\delta(X_{mn}, X_{kh})\ &\leq\ c.\max\{\delta(X_{rn}, X_{hj}),\ \delta(X_{rn}, X_{r'n}),\ \delta(X_{hj}, X_{hj'})\ :\\ &\qquad m-p \leq r,\ r' \leq m;\ k-p \leq j,\ j' \leq k\}\\ &\leq\ c.\max\{\delta(X_{rs}, X_{ij}),\ \delta(X_{rs}, X_{r's'}),\ \delta(X_{ij}, X_{i'j'})\ :\\ &\qquad m-p \leq r,\ r' \leq m;\ n-p \leq s,\ s' \leq n;\\ &\qquad h-p \leq i,\ i' \leq h;\ k-p \leq j,\ j' \leq k\}\end{aligned}$$

$$
\begin{aligned}
&\leq c^2.\max\{\delta(X_{rs},X_{ij}),\ \delta(X_{rs},X_{r's'}),\ \delta(X_{ij},X_{i'j'}) : \\
&\qquad m-2p \leq r,\ r' \leq m;\ n-2p \leq s,\ s' \leq n; \\
&\qquad h-2p \leq i,\ i' \leq h;\ k-2p \leq j,\ j' \leq k\} \\
&\leq c^N.\max\{\delta(X_{rs},X_{ij}),\ \delta(X_{rs},X_{r's'}),\ \delta(X_{ij},\ X_{i'j'}) : \\
&\qquad m-Np \leq r,\ r' \leq m;\ n-Np \leq s,\ s' \leq n; \\
&\qquad h-Np \leq i,\ i' \leq h;\ k-Np \leq j,\ j' \leq k\} \\
&\leq c^N M < \varepsilon.
\end{aligned}
$$

Choosing a point $x_n$ in $X_{nn}$ for $n = 1, 2, \ldots$ it follows that

$$d(x_m, x_n) \leq \delta(X_{mm}, X_{nn}) < \varepsilon$$

for $m, n > Np$. The sequence $\{x_n\}$ is therefore a Cauchy sequence in the complete metric space $X$ and so has a limit $z$ in $X$. Further

$$
\begin{aligned}
\delta(z, Fx_n) &\leq d(z, x_m) + \delta(x_m, Fx_n) \\
&\leq d(z, x_m) + \delta(X_{mm}, X_{n+1,n})
\end{aligned}
$$

since $x_m$ is in $X_{mm}$ and $Fx_n$ is in $X_{n+1,n}$. Thus

$$\delta(z, Fx_n) < d(z, x_m) + \varepsilon$$

for $m, n+1 > Np$. Letting $m$ tend to infinity it follows that

$$\delta(z, Fx_n) \leq \varepsilon$$

for $n + 1 > Np$. Using the continuity of $F$ and the lemma, it follows on letting $n$ tend to infinity that

$$\delta(z, Fz) \leq \varepsilon.$$

Since $\varepsilon$ is arbitrary, $\delta(z, Fz) = 0$ and so we must have $Fz = \{z\}$.

We can prove similarly that there exists a point $z'$ in $X$ such that $Gz' = \{z'\}$. Then

$$
\begin{aligned}
d(z, z') &= \delta(F^p z, G^p z') \\
&\leq c.\max\{\delta(F^r z, G^s z'),\ \delta(F^r z, F^{r'} z),\ \delta(G^s z', G^{s'} z') : \\
&\qquad 0 \leq r, r', s, s' \leq p\} \\
&= cd(z, z')
\end{aligned}
$$

and so $z = z'$. Then point $z$ is therefore a common fixed point of $F$ and $G$.

Now suppose that $F$ and $G$ have a second common fixed point $w$ so that $F^rG^sw$ is contained in $F^pG^pw$ for $r, s = 0, 1, 2, \ldots, p$. Then on using inequlity (11) we have

$$\delta(F^pG^pw, F^pG^pw) = \delta(F^pG^pw, G^pF^pw) \leq$$
$$\leq c.\max\{\delta(F^rG^pw, G^sF^pw),\ \delta(F^rG^pw, F^{r'}G^pw),\ \delta(G^sF^pw, G^{s'}F^pw):$$
$$0 \leq r, r', s, s' \leq p\} =$$
$$= c\delta(F^pG^pw, F^pG^pw)$$

and so $\delta(F^pG^pw, F^pG^pw) = 0$. It follows that the set $F^pG^pw$ consists of a single point which must be $w$. This means that $Fw = Gw = \{w\}$. Thus

$$d(z, w) = \delta(F^pz, G^pw) \leq$$
$$\leq c.\max\{\delta(F^r, G^sw),\ \delta(F^rz, F^{r'}z),\ \delta(G^sw, G^{s'}w):\ 0 \leq r, r', s, s' \leq p\} =$$
$$= cd(z, w)$$

and it follows that the common fixed point $z$ of $F$ and $G$ is unique. This completes the proof of the theorem.

**Corollary 1.** *Let $S$ and $T$ be continuous, commuting mappings of a complete metric space $(X, d)$ into itself satisfying the inequality*

$$(12) \qquad d(S^px, T^py) \leq \max\{cd(S^rx, T^sy),\ \frac{1}{2}d(S^rx, S^{r'}x),\ \frac{1}{2}d(T^sy, T^{s'}y):\ 0 \leq r, r', s, s' \leq p\}$$

*for all $x, y$ in $X$, where $0 \leq c \leq 1$ and $p$ is a fixed positive integer. Then $S$ and $T$ have a unique common fixed point $z$. Further $z$ is the unique fixed point of $S$ and $T$.*

*Proof.* Define mappings $F$ and $G$ of $X$ into $B(X)$ by putting

$$Fx = \{Sx\}, \qquad Gx = \{Tx\}$$

for all $x$ in $X$. The conditions of the theorem are satisfied for $F$ and $G$ since when $r = r' = s = s' = p$

$$d(F^rx, F^{r'}x) = d(G^sy, G^{s'}y) = 0.$$

$F$ and $G$ therefore have a unique common fixed point $z$ and $z$ is then of course the unique common fixed point of $S$ and $T$.

Now suppose that $S$ has a second fixed point $w$. Then on using inequality (12)

$$d(w,z) = d(S^p w, T^p z) \le \max\{cd(w,z),\ \frac{1}{2}d(w,z)\}$$

and the uniqueness of $z$ follows. Similarly $z$ is the unique fixed point of $T$.

**Theorem 2.** *Let $F$ and $G$ be commuting mappings of a complete metric space $(X,d)$ into $B(X)$ satisfying the inequality*

$$(13) \quad \delta(F^p x, Gy) \le \max\{c\delta(F^r x, G^s y),\ \frac{1}{2}\delta(F^r x, F^{r'} x),\ \frac{1}{2}\delta(y, Gy) : 0 \le r \le p;\ 0 \le r' < p;\ s = 0,\ 1\}$$

*for all $x, y$ in $X$, where $0 \le c < 1$ and $p$ is a fixed positive integer. If $F$ is continuous and if $F$ and $G$ also map $B(X)$ into itself, then $F$ and $G$ have a unique common fixed point $z$. Further $Fz = Gz = \{z\}$.*

*Proof.* Since $F$ is continuous and $F$ and $G$ obviously satisfy inequality (2) it follows as in the proof of theorem 1 that $F$ has a fixed point $z$ and $Fz = \{z\}$. Further on using inequality (13)

$$\begin{aligned} \delta(z, Gz) &= \delta(F^p z, Gz) \\ &\le \max\{c\delta(z, Gz),\ \frac{1}{2}\delta(z, Gz)\} \end{aligned}$$

and it follows that $Gz = \{z\}$. The uniqueness of $z$ follows easily. This completes the proof of the theorem.

The corollary follows immediately.

**Corollary 2.** *Let $S$ and $T$ be commuting mappings of a complete metric space $(X,d)$ into itself satisfying the inequality*

$$d(S^p x, Ty) \le \max\{cd(S^r x, T^s y),\ \frac{1}{2}d(S^r x, S^{r'} x),\ \frac{1}{2}d(y, Sy) : 0 \le r, r' \le p;\ s = 0, 1\}$$

*for all $x, y$ in $X$, where $0 \le c < 1$ and $p$ is a fixed positive integer. If $S$ is continuous then $S$ and $T$ have a unique common fixed point $z$. Further $z$ is the unique fixed point of $S$ and $T$.*

The next theorem and its corollary follow easily.

**Theorem 3.** *Let $F$ and $G$ be commuting mappings of a complete metric space $(X,d)$ into $B(X)$ satisfying the inequality*

$$\delta(Fx,Gy) \leq \max\{cd(x,y),\ c\delta(x,Gy),\ c\delta(y,Fx),\ \frac{1}{2}\delta(x,Fx),\ \frac{1}{2}\delta(y,Gy)\}$$

*for all $x,y$ in $X$, where $0 \leq c < 1$. If $F$ and $G$ also map $B(X)$ into itself, then $F$ and $G$ have a unique common fixed point $z$. Further $Fz = Gz = \{z\}$.*

**Corollary 3.** *Let $S$ and $T$ be commuting mappings of a complete metric space $(X,d)$ into itself satisfying the inequality*

$$d(Sx,Ty) \leq \max\{cd(x,y),\ cd(x,Ty),\ cd(y,Sx),\ \frac{1}{2}d(x,Sx),\ \frac{1}{2}d(y,Ty)\}$$

*for all $x,y$ in $X$, where $0 \leq c < 1$. Then $S$ and $T$ have a unique common fixed point $z$. Further $z$ is the unique fixed point of $S$ and $T$.*

The result of the above corollary was given in [2].

We finally show that although the mappings $F$ and $G$ in theorems 1, 2 and 3 necessarily have a unique common fixed point it is possible for either $F$ or $G$ to have a second fixed point. To see this let $X = \{x,y,z\}$ with the metric $d$ defined by

$$d(x,x) = d(y,y) = d(z,z) = 0,$$
$$d(x,y) = d(x,z) = 1,\quad d(y,z) = 2.$$

Define mappings $F$ and $G$ on $X$ by

$$Fx = Fy = \{x\},\quad Fz = \{y,z\},$$
$$Gx = Gy = Gz = \{x\}.$$

The conditions of the theorem are satisfied with $c = \frac{1}{2}$ but $F$ has two fixed points $x$ and $z$.

## References

[1] Fisher, B.: Common fixed points of mappings and set-valued mappings, Rostock. Math. Kolloq., 18 (1981), 69-77.

[2] Fisher, B.: A fixed point theorem for commuting mappings, Bull. Malaysian Math. Soc., (2) 5 (1982), 65-67.

**REZIME**

ZAJEDNIČKE NEPOKRETNE TAČKE KOMUTIRAJUĆIH SKUPOVNIH FUNKCIJA

Neka su $F$ i $G$ neprekidna, komutirajuća preslikavanja kompletnog metričkog prostora $(X, d)$ u $B(X)$ za koje važi nejednakost

$$\begin{aligned}\delta(F^p x, G^p y) \quad \leq \quad & \max\{c\delta(F^r x, G^s y), \frac{1}{2}\delta(F^r x, F^{r'} x), \frac{1}{2}\delta(G^s y, G^{s'} y) : \\ & 0 \leq r,\ s \leq p;\ 0 \leq r',\ s' < p\}\end{aligned}$$

za sve $x, y$ u $X$, gde je $0 \leq c < 1$ i $p$ fiksiran pozitivan ceo broj. Dokazano je da ako $F, G : B(X) \to B(X)$ tada $F$ i $G$ imaju jedinstvenu nepokretnu tačku $z$.

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