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CONVERGENCE WITH RESPECT TO SOME σ - IDEALS

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Abstract

The paper concerns the problem of topologizing the space of measurable real-valued functions by convergence with respect to a totally imperfect σ - ideal of subsets of a perfect Polish space.

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Suppose that we are given a nonempty set X. In all that follows below, we consider σ - fields and σ - ideals of subsets of X.

To begin with, let us recall two definitions.

Definition 1. *(cf. [5]). Let $\mathcal{I}$ be a σ - ideal. We say that a property holds $\mathcal{I}$ - almost everywhere on X (abrr. $\mathcal{I}$ - a.e.) if the set of all points which do not have this property belongs to $\mathcal{I}$.*

Definition 2. *(cf. [5]). We say that a sequence $\{f_n\}_{n\in N}$ of real functions defined on X converges with respect to a σ - ideal $\mathcal{I}$ to a function f (abbr. $f_n \xrightarrow[n\to\infty]{\mathcal{I}} f$) if each subsequence of $\{f_n\}n \in N$ contains a subsequence converging $\mathcal{I}$ - a.e. on X to f.*

For a σ - field $\mathcal{S}$ and σ - ideal $\mathcal{I}$, denote by $\mathcal{S}\Delta\mathcal{I}$ the smallest σ - field containing both $\mathcal{S}$ and $\mathcal{I}$. It is well known that $\mathcal{S}\Delta\mathcal{I}$ is the collection of all the sets of the form $(A-B)\cup C$ where $A\in\mathcal{S}, B, C\in\mathcal{I}$; moreover, $\mathcal{S}\Delta\mathcal{I}=\mathcal{S}$ if and only if $\mathcal{I}\subset\mathcal{S}$.

Without difficulty we can check that, for every σ-field $\mathcal{S}$ and any σ-ideal $\mathcal{I}$, the set $\mathcal{L}^*(\mathcal{S},\mathcal{I})$ of all $\mathcal{S}\Delta\mathcal{I}$-measurable real functions defined on X, equipped with convergence with respect to $\mathcal{I}$, is an $\mathcal{L}^*$-space (cf.[1; Problem 1. 7. 18. p 90]). Hence it is posible to define the closure operator on $\mathcal{L}^*(\mathcal{S},\mathcal{I})$ by letting $f\in\bar{A}$ if and only if A contains a sequence converging with respect to $\mathcal{I}$ to the function f. Then $\bar{\phi}=\phi$, $A\subset\bar{A}$, $\overline{A\cup B}=\bar{A}\cup\bar{B}$ for any sets $A, B\subset\mathcal{L}^*(\mathcal{S},\mathcal{I})$ however, $\bar{\bar{A}}=\bar{A}$ holds for each $A\subset\mathcal{L}^*(\mathcal{S},\mathcal{I})$ if and only if the following condition, usually labelled by (L4), is satisfied:

(L4) If $f_j \xrightarrow[j\to\infty]{\mathcal{I}} f$ and $f_{j,n} \xrightarrow[n\to\infty]{\mathcal{I}} f_j$ for each $j\in N$, then there exist sequences $\{j_p\}_{p\in N'}$ $\{n_p\}_{p\in N}$ of positive integers such that $f_{j_p,n_p} \xrightarrow[j\to\infty]{} f$.

If (L4) is fulfilled, then topology introduced in $\mathcal{L}^*(\mathcal{S},\mathcal{I})$ in the sense described above is called the Fréchet topology.

The problem of topologizing the space of measurable functions was studied by Wagner (see [5]). There are both some examples of σ-ideals convergence with respect to them yields the Fréchet topology or not. Our considerations lead to the examples of measurable spaces for which it is not possible to introduce the Fréchet topology.

Let $(X,\mathcal{S})$ be a measurable space and $\mathcal{I}$ be an arbitrary σ-ideal of subsets of X.

The following theorem was proved by Wagner (see th. 1 in [5]):

Theorem A. *Suppose that* $\mathcal{I}\subset\mathcal{S}$ *and every family of disjoint sets in* $\mathcal{S}$ - $\mathcal{I}$ *is at most countable. Then convergence with respect to* $\mathcal{I}$ *yields the Fréchet topology in the space* $\mathcal{L}^*(\mathcal{S},\mathcal{I})$ *if and only if the pair* $(\mathcal{S},\mathcal{I})$ *fulfils condition (E).*

Condition (E) is equivalent to the condition (E') (see ([6]). So let us recall this definition.

Definition 3. *We shall say that a pair $(\mathcal{S},\mathcal{I})$ fulfils condition (E') if and only if for each set $D \in \mathcal{S}-\mathcal{I}$ and for each double sequence $\{B_{j,n}\}_{j,n\in N}$ of $\mathcal{S}$-measurable sets such that*

$$1^o \quad B_{j,n} \subset B_{j,n+1} \quad \text{for} \quad \text{any } j,n \in N$$
$$2^o \quad \bigcup_{n=1}^{\infty} B_{j,n} = D \quad \text{for} \quad \text{each } j \in N,$$

there exists a sequence $\{n_j\}_{j\in N}$ of natural numbers such that $\bigcap_{j=1}^{\infty} B_{j,n_j} \notin \mathcal{I}$.

Further, we are going to demonstrate the σ-ideals and σ-field fulfilling condition (E), but convergence with respect to them does not yield the Fréchet topology.

At the beginning let us pay attention to the following property:

Proposition 1. *The pair $(\mathcal{S},\mathcal{I})$ fulfils condition (E') if and only if the pair $(\mathcal{S}\Delta\mathcal{I},\mathcal{I})$ fulfils the condition (E').*

Proof. Necessity. Let us consider any set $D \in (\mathcal{S}\Delta\mathcal{I})-\mathcal{I}$ and $B_{j,n} \in \mathcal{S}\Delta\mathcal{I}$ $(j,n \in N)$ which satisfy conditions (1^o) and (2^o) of Definition 3. There exist sets $A_{j,n} \in \mathcal{S}$, $C_{j,n} \in \mathcal{I}$ and $E_{j,n} \in \mathcal{I}$ such that

$$B_{j,n} = (A_{j,n} - C_{j,n}) \cup E_{j,n}, \text{ where } j,n \in N.$$

Denote

$$D^* = \bigcap_{j=1}^{\infty} \bigcup_{n=1}^{\infty} A_{j,n}, B^*_{j,n} = \bigcup_{k=1}^{n} A_{j,k} \cap D^*$$

and

$$C = \bigcup_{j=1,n=1}^{\infty} C_{j,n} \cup \bigcup_{j=1,n=1}^{\infty} E_{j,n} \text{ for } j,n \in N.$$

Of course, $D^* \in \mathcal{S}$, $C \in \mathcal{I}$ and $B^*_{j,n} \in \mathcal{S}$ for $j,n \in N$. Moreover, $B^*_{j,n} \subset B^*_{j,n+1}$ and $D^* = \bigcup_{n=1}^{\infty} B^*_{j,n}$, whenever $j \in N$. If the pair $(\mathcal{S},\mathcal{I})$ fulfils (E'), then there exists a sequence $\{n_j\}_{j\in N}$ of natural numbers such that $\bigcap_{j=1}^{\infty} B^*_{j,n_j} \notin \mathcal{I}$. Simultaneously,

$$B^*_{j,n} \cup C \subset \bigcup_{k=1}^{n} (A_{j,k} \cup C) \subset \bigcup_{k=1}^{n} B_{j,k} \cup C = B_{j,n} \cup C$$

for any $j, n \in N$. Hence,

$$\bigcap_{j=1}^{\infty} B^*_{j,n_j} - \bigcap_{j=i}^{\infty} B_{j,n_j} \in \mathcal{I}$$

so that

$$\bigcap_{j=1}^{\infty} B_{j,n_j} \notin \mathcal{I}.$$

Sufficiency is obvious, because $\mathcal{S} \subset \mathcal{S}\Delta\mathcal{I}$.

Now, we shall define the totally imperfect σ-ideals.

Definition 4. *A σ-ideal $\mathcal{I}$ of subsets of topological space X is totally imperfect if and only if it does not contain nonempty perfect sets in X.*

It is worth mentioning that, there are many natural examples of totally imperfect σ-ideals (see [3]).

Theorem 1. *Let $\mathcal{S}$ be a σ-field of subsets of a perfect Polish space X containing the family of perfect sets in X and $\mathcal{I}$ be an arbitrary totally imperfect σ-ideal of subsets of X. If for each set $A \in \mathcal{S} - \mathcal{I}$ there exists a nonempty perfect set $A^* \subset A$, then the pair $(\mathcal{S}\Delta\mathcal{I}, \mathcal{I})$ fulfils condition (E).*

Our proof will be based on the following lemma which is easy to prove.

Lemma 1. *Let $B \subset X$ be an uncountable Borel set. Then, for each positive number ε, there exist nonempty perfect sets C and D, such that $C \cup D \subset B$, $C \cap D = \emptyset$ and $d(C) < \varepsilon$, $d(D) < \varepsilon$, where $d(C), d(D)$ denote, respectively, the diameters of the sets C, D.*

Proof of theorem 1: According to proposition 1 we shall proof that the pair $(\mathcal{S}, \mathcal{I})$ fulfils condition (E'). Let an arbitrary set $D \in \mathcal{S} - \mathcal{I}$ and an arbitrary double sequence $\{B_{j,n}\}_{j,n\in N}$ of $\mathcal{S}$-measurable sets fulfil (1^o) and (2^o) of condition (E').

We define by induction a decreasing sequence $\{D_j\}_{j\in N}$ of perfect sets and a sequence $\{n_j\}_{j\in N}$ of natural numbers such that $\bigcap_{j=1}^{\infty} D_j$ is a nonempty perfect set and, for each $j \in N$, $D_j \subset B_{j,n_j}$.

Consequently, we shall obtain that $\bigcap_{j=1}^{\infty} B_{j,n_j} \notin \mathcal{I}$.
The sequence $\{D_j\}_{j\in N}$, will be defined in such a way that the following conditions hold:

(a) for each $j \in N, D_j = D_{j,1} \cup D_{j,2} \cup \ldots \cup D_{j,2^j}$, where $D_{j,k} \cap D_{j,s} = \emptyset$ for $k \neq s, 1 \leq k, s \leq 2^j$ and $D_{j,k}$ is perfect, $d(D_{j,k} < \frac{1}{2^j}$ for each $1 \leq k \leq 2^j$,

(b) for any $j \in N$ and $1 \leq t \leq 2^{j-1}$ there exist $1 \leq k, s \leq 2^j, k \neq s$ such that $D_{j,k} \cup D_{j,s} \subset D_{j-1,t}$

Since $D = \bigcup_{n=1}^{\infty} B_{1,n}$, there exists a positive integer n_1 such that the set $B_{1,n_1} \in \mathcal{S} - \mathcal{I}$. By Lemma 1 there exist two nonempty perfect sets $D_{1,1}$, $D_{1,2}$ such that $D_{1,1} \subset B_{1,n_1}$, $D_{1,2} \subset B_{1,n_1}$, $d(D_{1,1}) \leq \frac{1}{2}$, $d(D_{1,1}) < \frac{1}{2}$ and $D_{1,1} \cap D_{1,2} = \emptyset$.

Let $D_1 = D_{1,1} \cup D_{1,2}$. Suppose we have defined the sets $D_1, D_2, \ldots \ldots D_{j-1}$ fulfilling the above conditions (a) and (b). Now, we shall define a set D_j. Since

$$D \supset D_{j-1} = D_{j-1,1} \cup D_{j-2,2} \cup \ldots \cup D_{j-1,2^{j-1}},$$

thus,for each $1 \leq k \leq 2^{j-1}$, we have

$$D_{j-1}, k = \bigcup_{n=1}^{\infty} (B_{j,n} \cap D_{j-1,k}),$$

so there exists a natural number n_k such that

$$B_{j,n_k} \cap D_{j-1,k} \notin \mathcal{I}.$$

Let $n_j = \max(n_1, n_2, \ldots n_{2^{j-1}})$. Since $B_{j,n_j} \supset B_{j,n_k}$ for each $1 \leq k \leq 2^{j-1}$, therefore $B_{j,n_j} \cap D_{j-1,k}, \in \mathcal{S} - \mathcal{I}$ for each $1 \leq k \leq 2^{j-1}$. Putting $\varepsilon = 2^{-j}$, by Lemma 1, for each set $B_{j,n_j} \cap D_{j-1,k}$, where $1 \leq k \leq 2^{j-1}$, there exist two nonempty perfect sets $D_{j,2k-1}, D_{j,2k}$ such that $D_{j,2k-1} \cup D_{j,2k} \subset B_{j,n_j} \cap D_{j-1,k}$, $D_{j,2k-1} \cap D_{j,2k} = \emptyset$ and $d(D_{j,2k-1}) < 2^{-j}$, $d(D_{j,2k}) < 2^{-j}$. Let

$$D_j = D_{j,1} \cup D_{j,2} \cup \ldots \cup D_{j,2^j-1} \cup D_{j,2^j}.$$

By virtue of this construction and the properties of the sets D_{j-1}, it is easy to see that the set D_j fulfils conditions (a) and (b). Applying the Cantor theorem, we have that $\bigcap_{j=1}^{\infty} D_j \neq \phi$. Simultaneously, this closed set is perfect (see the fusion lemma in [2]).

Applying the above theorem we conclude.

Corollary 1. *In a perfect Polish space X the pair $(\mathcal{B}\Delta\mathcal{I},\mathcal{I})$ fulfils condition (E), where $\mathcal{B}$ denotes the family of Borel sets in X and $\mathcal{I}$ an arbitrary totally imperfect σ-ideal of subsets of X.*

We can also apply Theorem 1 to the totally imperfect σ-ideal $\mathcal{I}_M$ of sets having the property (s_o) introduced by Marczewski in a Polish space (see [4]). This σ-ideal is included in the σ-field $\mathcal{S}_M$ of sets having the property (s).

Corollary 2. *In a perfect Polish space X the pair $(\mathcal{S}_M,\mathcal{I}_M)$ fulfils condition (E).*

The following theorem proves that condition (E) is not sufficient to introduce the Fréchet topology.

Theorem 2. *Let $\mathcal{S}$ be a σ-field of subsets of a perfect Polish space X containing the family of perfect sets in X. Then convergence with respect to an arbitrary totally imperfect σ-ideal $\mathcal{I}$ of subsets of X does not yield the Fréchet topology in the space $\mathcal{L}^*(\mathcal{S},\mathcal{I})$.*

Proof. We shall prove that condition (L4) is not fulfilled. Let a sequence $\{r_n\}_{n\in N}$ be dense in X and let $\{\varepsilon_n\}_{n\in N}$ be a decreasing sequence of natural numbers converging to 0. Let $f_{j,n} = \chi_{A_{j,n}} + \frac{1}{j}$, where $A_{j,n} = \bigcup_{i=1}^{j} K(r_i,\varepsilon_n)$ for any $j,n \in N$. The functions $f_{j,n}$ are $\mathcal{S}\Delta\mathcal{I}$-measurable for any $j,n \in N$ and $\lim_{n\to\infty} f_{j,n}(x) = \frac{1}{j}$ if and only if

$$x \notin \limsup_{n\in N} A_{j,n} = \bigcap_{j=1}^{\infty} \bigcup_{n=k}^{\infty} \bigcup_{i=1}^{j} K(r_i,\varepsilon_n) = \{r_1,\dots r_j\}.$$

Hence, $f_{j,n} \underset{n\to\infty}{\longrightarrow} f_j = \frac{1}{j}$ and $f_j \underset{j\to\infty}{\longrightarrow} f \equiv 0$. Let $\{j_p\}_{p\in N}$ be an arbitrary increasing sequence of natural numbers and $\{n_p\}_{p\in N}$ an arbitrary sequence of natural numbers. Then

$$\{x :\sim (\lim_{p\to\infty} f_{j_p,n_p}(x) = 0\} \supset \{x : \limsup_{p\to\infty} f_{j_p,n_p}(x) \geq 1\} \supset$$

$$\supset \bigcap_{k=1}^{\infty} \bigcup_{p=k+1}^{\infty} \{x : f_{j_p,n_p}(x) \geq 1\} \supset \bigcap_{k=1}^{\infty} \bigcup_{p=k+1}^{\infty} A_{j_p,n_p}.$$

The set $\bigcup_{p=k+1}^{\infty} A_{j_p,n_p} = \bigcup_{p=k+1}^{\infty} \bigcup_{i=1}^{j} K(x_i, \varepsilon_{n_p})$ is open and dense, so the set $\bigcap_{k=1}^{\infty} \bigcup_{p=k+1}^{\infty} A_{j_p,n_p}$ is Borel and residual. Hence, it contains a nonempty perfect set which implies that $\{x :\sim (\lim_{p\to\infty} f_{j_p,n_p}(x) = 0\} \notin \mathcal{I}$.

Corollary 3. *Convergence with respect to $\mathcal{I}$ does not yield the Fréchet topology in the space $\mathcal{L}^*(\mathcal{B},\mathcal{I})$.*

Corollary 4. *Convergence with respect to $\mathcal{I}_M$ does not yield the Fréchet topology in the space $\mathcal{L}^*(\mathcal{S}_M,\mathcal{I}_M)$.*

By Theorem A we have

Corollary 5. *There exist, respectively, uncountable families of pairwise, disjoint sets in $\mathcal{B}$ - $\mathcal{I}$ and $\mathcal{S}_M - \mathcal{I}_M$.*

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REZIME

KONVERGENCIJA U ODNOSU NA NEKE σ - IDEALE

U ovom radu se razmatra problem topologiziranja prostora merljivih realnih funkcija pomoću konvergencije u odnosu na totalno imperfektan σ – ideal skupova perfektnog Polish prostora.

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