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OSCILLATIONS AND THE ASYMPTOTIC BEHAVIOUR OF CERTAIN SECOND ORDER NEUTRAL DIFFERENCE EQUATIONS

Mirko Budinčević
Institute of Mathematics, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

The paper deals with the oscillatory and asymptotic behaviour of solutions of second order neutral difference equations with variable coefficients and some generalizations of such equations.

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1. Introduction

This paper deals with the oscillatory and asymptotic behaviour of solutions of the second order neutral difference equation of the form

$$(1) \qquad \Delta(a(n)\Delta(x(n)+px(n-n_0)))+q(n)f(x(n-m_0))=0,$$

where $\{a(n)\}$ and $\{q(n)\}$ are positive sequences and Δ is the forward difference operator defined by $\Delta y(n)=y(n+1)-y(n)$. p is a constant and the function f is considered subject to the condition

(0) f is nondecreasing and $uf(u)>0$ for $u\neq 0$.

By a solution of (1) we mean a real sequence $\{y(n)\}$ satisfying (1). Throughout this paper, we usually refer to a solution $\{y(n)\}$ simply as a solution y and consider only nontrivial solutions.

A real sequence $\{r(n)\}$ *eventually* has some property if there exists $N \geq 0$ such that $r(n)$ has this property for $n = N, N+1, \ldots$.

A nontrivial solution y of (1) is said to be *oscillatory* if $y(n)$ changes its sing infinitely many times. Otherwise, y is said to be **nonoscillatory** an equation is called **oscillatory** if all its solutions are oscillatory. Otherwise, it is called **nonoscillatory**.

Similarly, as in the theory of differential equations, in a neutral difference equation the highest difference of the unknown function appears with the argument n (the present state of the system) and with one or more retarded arguments (the past state of the system). Investigations of such systems, beside their theoretical interest, have some importance for application (see [1] and [2]).

There is much current interest in the oscillation theory of differential equations of the neutral type see ([3],[4],[5] and [6]). As far as the author is aware, not to much has been done on the theory of difference equations. This, and the fact that the results for adequate difference equations could be quite different, are motivations for this paper.

2. Preliminaries

In what follows we shall use the following lemmas which give useful information about the bounds for nonoscillatory solutions of the following equation:

$$\Delta(a(n)\Delta z(n)) + q(n)f(z(n)) = 0, \; n = 0, 1, \ldots, \ldots \tag{2}$$

Lemma 1. *([7]) Consider (2) subject to conditions (0),*
$(C_1) q(n) \geq 0$ *for* $n = 0, 1, \ldots,$ *and* $q(n)$ *is not eventually zero,*
$a(n) > 0$ *and*
(C_2) $\sum_{n=0}^{\infty} \frac{1}{a(n)} < \infty$.

Then, every nonoscillatory solution y of (2) satisfies eventually the following estimate

$$A\rho(n) \leq | y(n) | \leq B$$

for some positive constants A and B (depending on y), where

$$\rho(n) = \sum_{i=n}^{\infty} \frac{1}{a(i)}.$$

Lemma 2. *([8]) Consider (2) subject to conditions*$(0), (C_1), a(n) > 0$ *and* $(C_3)\ \sum_{n=0}^{\infty} \frac{1}{a(n)} = \infty$.

Then, every nonoscillatory solution y of (2) satisfies eventually the following estimate

$$C \leq | y(n) | \leq DR(n)$$

for some positive constants C and D (depending on y), where

$$R(n) = \sum_{i=0}^{n} \frac{1}{a(i)}.$$

Near this *apriori* estimate we next need

Lemma 3. *Suppose that* $x(n) > 0$ *eventually and define*

$$z(n) = \sum_{i=0}^{k} p_i x(n - m_i),\ p_i > 0. \tag{3}$$

If $p_0 > \sum_{i=1}^{k} p_i \equiv P$, *then* $z(n) \to C \geq 0$ *if and only if* $x(n) \to \frac{C}{P+p_i}, n \to \infty$.

Proof. Suppose that $z(n) \to C$. Let $\overline{\lim}_{s\to\infty} x(n) = \lim_{\ell\to\infty} x(n_\ell) = \frac{C+q_1}{P+p_o}$ and

$$\lim_{n\to\infty} x(n) = \lim_{s\to\infty} x(n_s) = \frac{C - q_2}{P + p_0}.$$

We shall prove that $q_1 = q_2 = 0$.
a) suppose that $q_1 \geq q_2 \geq 0$ and $q_1 > 0$. Taking $n = n_\ell + m_0$, (3) implies that

$$C = \frac{C + q_1}{P + p_0} p_0 + \sum_{i=1}^{k} \frac{C + \delta_i}{P + p_0} p_i,$$

where $-q_2 - \epsilon < \delta_i < q_1 + \epsilon$ eventually for every $\epsilon > 0$. The above equation implies

$$p_0 q_1 = - \sum_{i=1}^{k} p_i \delta_i < \sum_{i=1}^{k} p_i (q_2 + \epsilon).$$

Choosing $\epsilon < \frac{p_0 - P}{P} q_1$ we get $q_1 < q_2$, a contradiction.

b) Suppose that $q_2 \geq q_1 \geq 0$ and $q_2 > 0$. Taking $n = n_s + m_0$, (3) implies that

$$C = \frac{C - q_2}{P + p_0} p_0 + \sum_{i=1}^{k} \frac{C + \delta_i}{P + p_0} p_i,$$

where $-q_2 - \epsilon < \delta_i < q_1 + \epsilon$ eventually for every $\epsilon > 0$. The above equation implies

$$p_0 q_2 = \sum_{i=1}^{k} p_i \delta_i < \sum_{i=1}^{k} p_i (q_1 + \epsilon).$$

Choosing $\epsilon < \frac{p_0 - P}{P} q_2$ we get $q_2 < q_1$, a contradiction.

As the convergence of $x(n)$ implies the convergence of $x(n - m_i)$ to the same limit, the proof of the second part of the theorem is obvious.

Remark 1. If $p_0 = P$, $x(n) = 2 + (-1)^n$ for appropriate m_i could be a counter-example.

3. Oscillations and asymptotic behaviour

Consider the second order neutral difference equation

$$\Delta(a(n)\Delta(x(n) + px(n - n_0))) + q(n)f(x(n - m_0)) = 0, \; n = 0, 1, ...$$

where $a(n), q(n) > 0$; $0 \leq p < 1$: $n_0, m_0 \in N$ and f satisfies (0).

Theorem 1. *If*

$$\sum_{n=k_0}^{\infty} q(n) = \infty \quad \textit{and} \quad \sum_{n=k_0}^{\infty} \frac{1}{a(n)} \sum_{k=k_0}^{n} q(k) = \infty$$

then every solution x of (1) is either oscillatory or else $x(n) \to 0$ as $n \to \infty$.

Proof. Let x be a nonoscillatiry solution of (1). Suppose, without loss of generality, that $x(n) > 0$ eventually. This implies that $x(n - n_0) > 0$ and $x(n - m_0) > 0$ eventually. Set

$$z(n) = x(n) + px(n - n_0), \tag{4}$$

then $z(n) > x(n) > 0$ eventually.
According to (1) we have $\Delta(a(n)\Delta z(n)) < 0$ eventually. Thus, either $\Delta z(n) > 0$ or $\Delta z(n) < 0$ eventually.

a) Assume that $\Delta z(n) > 0$ eventually. It follows that $\Delta z(n - m_0) > 0$ eventually and by (4)

$$x(n - m_0) = z(n - m_0) - px(n - n_0 - m_0) \geq$$

$$\geq z(n - m_0) - pz(n - n_0 - m_0) \geq z(n - m_0) - pz(n - m_0),$$

which implies

$$x(n - m_0) \geq (1 - p)z(n - m_0) = p_1 z(n - m_0).$$

Define a positive sequence w such that

$$w(n) = \frac{a(n)\Delta z(n)}{f(p_1 z(n - m_0))},$$

then

$$\Delta w(n) = \frac{\Delta(a(n)\Delta z(n))}{f(p_1 z(n - m_0))} - \frac{\Delta f(p_1 z(n - m_0))a(n + 1)\Delta z(n + 1)}{f(p_1 z(n - m_0))f(p_1 z(n - m_0 + 1))} \leq -q(n).$$

Summing the above inequality from N to n we get

$$w(n) \leq w(N) - \sum_{k=n}^{N} q(k) \to -\infty,\ n \to \infty,$$

which is a contradiction.

b) Assume that $\Delta z(n) < 0$ eventually. Then $\lim_{n\to\infty} z(n) = c$ and suppose that $c > 0$. According to Lemma 3 $\lim_{n\to\infty} x(n - m_0) = \frac{C}{1+p}$ which implies that $x(n - m_0) \geq \frac{C}{2(1+p)}$ eventually. Thus

$$\Delta(a(n)\Delta z(n)) \leq -f(\frac{c}{2(1 + p)})q(n)$$

eventually. Summing the above inequality from N to n we get

$$a(n)\Delta z(n) \leq a(N)\Delta z(N) - f(\frac{c}{2(1 + p)}) \sum_{k=N}^{n} q(k).$$

Using the assumptions of the theorem it yields to

$$a(n)\Delta z(n) \le -\frac{1}{2} f(\frac{c}{2(1+p)}) \sum_{k=N}^{n} q(k)$$

eventually. Dividing by $a(n)$ and summing from N to n we get

$$z(n) \le z(N) - \frac{1}{2} f(\frac{c}{2(1+p)}) \sum_{k=N}^{n} \frac{1}{a(s)} \sum_{k=N}^{s} q(k) \to \infty,$$

while $n \to \infty$, and this is a contradiction to the fact that $z(n) \to c > 0$. Thus $c = 0$ and the proof is complete.

The next theorem guarantees that all the solutions of (1) are oscillatory.

Theorem 2. *If* $\sum_{n=k_0}^{\infty} q(n) = \infty$ *and* $\sum_{n=k_0}^{\infty} \frac{1}{a(n)} = \infty$, *then equation (1) is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1) and let $x(n)$ be of a positive sign eventually. As in the proof of Theorem 1, we define $z(n)$. The second condition of the theorem, as it was shown in [8], implies that $\Delta z(n) > 0$ eventually . As the proof follows the same line as in case a) of Theorem 1, it will be omitted.

Remark 2. Accorrding to Lemma 3, we can generalize the assertions of Theorem 1 and the Theorem 2 to the case

$$(5)\Delta(a(n)\Delta(x(n) + \sum_{i=1}^{k} p_i x(n-m_i))) + q(n)f(x(n-m_0)) = 0,\ n = 0, 1,$$

while $\sum_{i=1}^{k} p_i < 1$.

A natural question is what happens when $\sum^{\infty} \frac{1}{a(n)}$ converges. The answer gives the following

Theorem 3. *Consider equation (1) where* $a(n), q(n) > 0;\ 0 \le p < \infty$, $p \ne 1$; f *satisfies (0);*

$$\sum_{n=k_0}^{\infty} \frac{1}{a(n)} < \infty \text{ and } \sum_{n=k_0}^{\infty} \frac{1}{a(n)} \sum_{k=k_0}^{n} q(k) = \infty.$$

Then, every solution x *of (1) is either oscillatory or else* $x(n) \to 0$ *since* $n \to \infty$.

Proof. As in the proof of Theorem 1 we introduce $z(n)$ and differentiate two cases:

a) Assume that $\Delta z(n) > 0$ eventually. According to Lemma 1 we have that $\lim_{n\to\infty} z(n) = c > 0$, which by Lemma 3 gives that $\lim_{n\to\infty} x(n) = \frac{C}{1+p}$ and the estimate $x(n-m_0) > \frac{z(n-m_0)}{2(1+p)}$ eventually. Conditions $\sum_{n=k_0}^{\infty} \frac{1}{a(n)} < \infty$ and $\sum_{n=k_0}^{\infty} \frac{1}{a(n)} \sum_{n=k}^{n} q(k) = \infty$ imply that $\sum_{n=k_0}^{\infty} q(n) = \infty$ and we can proceed as in the proof of case a) of Theorem 1.

b) Assume that $\Delta z(n) < 0$ eventually. According to the observations given in case a) the proof follows the same line as in the proof of case b) of Theorem 1.

Remark 3. In the light of Lemma 3 we are able to generalize the assertions of Theorem 3 on the difference equation (5) if $0 \le p_i$, $\sum_{i=1}^{k} p_i < 1$ or

$$p_j > 1 + \sum_{\substack{i=1 \\ i \ne j}}^{k} p_i$$

for some fixed j.

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REZIME

OSCILACIJE I ASIMPTOTSKO PONAŠANJE NEKIH NEUTRALNIH DIFERENCNIH JEDNAČINA DRUGOG REDA

U radu se posmatra neutralna diferencna jednačina drugog reda

$$\Delta(a(n)\Delta(x(n)+px(n-n_0)))+q(n)f(x(n-m_0))=0, \ n=0,1,\ldots$$

$a(n), q(n) > 0$; $p \geq 0$ a f je neopadajuća funkcija takva da je $uf(u) > 0$ za $u \neq 0$.
Daju se dovoljni uslovi da bi sva rešenja bila oscilatorna ili da bi još eventualno težila nuli.

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