Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 21, 1 (1991), 173-191

Review of Research
Faculty of Science
Mathematics Series

ON PERIODIC SOLUTIONS OF DIFFERENTIAL INCLUSIONS WITH UNBOUNDED OPERATORS IN BANACH SPACES

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Abstract

In the paper an operator method is suggested using the fixed points index theory of condensing multivalued maps for the investigation of the existence of mild periodic solutions of the differential inclusion

$$x'(t) \in Ax(t) + F(t, x(t)),$$

where A is a linear operator and F is a multivalued map in a separable Banach space. Some applications to the existence of optimal periodic solutions of control systems in a Banach space are considered.

AMS Mathematics Subject Classification (1980): 34C25, 34A60, 34G20, 47H09, 49A27.
Key words and phrases: Differential inclusion, periodic solution, measure of noncompactness, condensing map, multivalued map, fixed points index, optimal periodic control.

1. Introduction

Let Y be a topological space; 2^Y denote a collection of all the subsets of Y. Then

$$P(Y) = \{D \in 2^Y : D \neq \emptyset\},$$

$$Pb(Y) = \{D \in P(Y) : \; D \text{ is bounded }\},$$

$$C(Y) = \{D \in P(Y) : \; D \text{ is closed }\},$$

$$K(Y) = \{D \in P(Y) : \; D \text{ is compact }\}.$$

If Y is a subspace of a topological vector space, then $Cv(Y)[Kv(Y)]$ denote a collection of all the nonempty closed convex [compact convex] subsets of Y.

Let $\mathcal{E}$ be an infinite-dimensional Banach space. A function $\varphi : 2^{\mathcal{E}} \to \mathbf{R}_+$ is said to be a measure of noncompactness in $\mathcal{E}$ if

$$\varphi(\overline{co}\Omega) = \varphi(\Omega)$$

for every $\Omega \in 2^{\mathcal{E}}$ (see, for example, [1],[2]). One of the well-known examples in the Hausdorff measure of noncompactness:

$$\chi(\Omega) = \inf\{\epsilon > 0 : \Omega \text{ has a finite } \epsilon\text{-net }\}.$$

We shall later use its following properties.

$\chi 1)$ If $\Omega_1 \subset \Omega_2$, then $\chi(\Omega_1) \leq \chi(\Omega_2)$.

$\chi 2)$ $\chi(\Omega_1 + \Omega_2) \leq \chi(\Omega_1) + \chi(\Omega_2)$ for all $\Omega_1, \Omega_2 \in 2^{\mathcal{E}}$.

$\chi 3)$ $\chi(Q) = \chi(S) = 1$, where $Q = \{x \in \mathcal{E} : \|x\| \leq 1\}$, $S = \partial Q$.

The symbol E in the sequel will denote a separable Banach space. Then the following properties should be noted. Let the interval $[0,T]$ be equipped with the Lebesgue measure.

$\chi 4)$ **The Kisielewicz lemma.** ([10]) Let $\{g_n\}_{n=1}^{\infty}$ be an integrally bounded sequence of mesurable functions from $[0,T]$ into E. Then $\omega : [0,T] \to \mathbf{R}_+$,

$$\omega(t) = \chi(\{g_n(t)\}_{n=1}^{\infty})$$

is a summable function and

$$\chi(\{\int_{\mathcal{T}} g_n(s)ds\}_{n=1}^{\infty}) \leq \int_{\mathcal{T}} \omega(s)ds$$

for every measurable set $\mathcal{T} \subset [0,T]$.

In the sequel, the space $C([0,T];E)$ will be denoted as $\mathcal{C}$.

$\chi5)$ If $\Omega \subset \mathcal{C}$ is bounded and equicontinuous, then

$$\psi(\Omega) := \sup_{t\in[0,T]} \chi(\Omega(t)) = \chi_C(\Omega),$$

where $\Omega(t) = \{y(t); y \in \Omega\}$ and χ_C is the Hausdorff measure of noncompactness in $\mathcal{E}$.

Let $L(E)$ be a space of all the bounded linear operators in E. The χ-norm of $B \in L(E)$ is defined as

$$\|B\|^{(\chi)} := \chi(BS),$$

where S is the unit sphere in E. We shall use the following properties of χ-norm.

Lemma 1.1. ([2]) *If $B \in L(E)$, then*

$$\chi(BD) \leq \|B\|^{(\chi)}\chi(D)$$

for every $D \in Pb(E)$.

Lemma 1.2. *Let $B \in L(E)$; $\|B\|^{(\chi)} < 1$ and $I - B$ is invertible. Then*

$$\|(I-B)^{-1}\|^{(\chi)} \leq (1 - \|B\|^{(\chi)})^{-1}.$$

Proof. Let $R = (I-B)^{-1}S$ and $\{r_n\}_{n=1}^{\infty}$ be a countable dense subset of R. Then the sequence $\{s_n\}_{n=1}^{\infty}$, $s_n = (I-B)r_n$ is dense in S, and using the properties of χ we have

$$1 = \chi(S) = \chi(\{s_n\}_{n=1}^{\infty}) = \chi(\{r_n - Br_n\}_{n=1}^{\infty}) \geq$$
$$\geq \chi(\{r_n\}_{n=1}^{\infty}) - \chi(\{Br_n\}_{n=1}^{\infty}) \geq \chi(R) - \|B\|^{(\chi)}\chi(R) =$$
$$= (1 - \|B\|^{(\chi)})\chi(R).$$

This implies the required inequality □.

A multivalued function (multifunction) $G : [0,T] \to K(E)$ is said to be measurable if it satisfies any of the following two equivalent conditions:

(i) the set $G^{-1}(V) = \{t \in [0,T] : G(t) \subset V\}$ is measurable for every open $V \subset E$;

(ii) there exists the sequence $\{g_n\}_{n=1}^{\infty}$ of measurable functions $g_n : [0,T] \to E$ such that $G(t) = \overline{\{g_n(t)\}_{n=1}^{\infty}}$ for all $t \in [0,T]$ (see, for example [6],[5]).

By the symbol S_G^1 we shall denote the set of all Bochner integrable selectors of the multifunction $G : [0,T] \to P(E)$, i.e.

$$S_G^1 = \{g \in L([0,T]; E) : g(t) \in G(t) \ \text{ a.e.}\}.$$

If $S_G^1 \neq \emptyset$, then the multifunction G is called integrable and

$$\int_{\mathcal{T}} G(s)ds := \{\int_{\mathcal{T}} g(s)ds : g \in s_G^1\}$$

for every measurable set $\mathcal{T} \subseteq [0,T]$. Clearly if G is measurable and integrably bounded (i.e. there exists $\alpha \in L_+^1([0,T])$ such that $\|G(t)\| := \max\{\|y\| : y \in G(t)\} \leq \alpha(t)$ *a.e.*), then G is integrable.

Let us now prove the following generalization of the Kisielewicz lemma (see $\chi 4$),

Lemma 1.3. *Let the multifunction $G : [0,T] \to Pb(E)$ be integrable, inegrably bounded and*

$$\chi(G(t)) \leq \beta(t) \ a.e. \ on \ [0,T],$$

where $\beta \in L_+^1([0,T])$. Then

$$\chi(\int_{\mathcal{T}} G(s)ds) \leq \int_{\mathcal{T}} \beta(s)ds$$

for every measurable set $\mathcal{T} \subseteq [0,T]$. In particular, if $\chi(G(\cdot)) \in L_+^1([0,T])$, then

$$\chi(\int_{\mathcal{T}} G(s)ds) \leq \int_{\mathcal{T}} \chi(G(s))ds.$$

Proof. Since the space E is separable, so is the space $L^1([0,T]; E)$. Hence the set S_G^1 contains the dense countable subset $\{g_n\}_{n=1}^{\infty}$. Then the set $\{\int_{\mathcal{T}} g_n(s)ds\}_{n=1}^{\infty}$

is dense in $\int_T G(s)ds$. From the properties of the measure of noncompactness χ we have

$$\chi(\int_T G(s)ds) = \chi(\{\int_T g_n(s)ds\}_{n=1}^{\infty})$$

and

$$\chi(\{g_n(t)\}_{n=1}^{\infty}) \leq \chi(G(t)) \leq \beta(t) \ a.e..$$

But, then from $\chi 4$, it follows that

$$\chi(\{\int_T g_n(s)ds\}_{n=1}^{\infty}) \leq \int_T \chi(g_n(s)\}_{n=1}^{\infty} ds \leq$$

$$\leq \int_T \beta(s)ds \ \ \square.$$

Let X, Y be topological spaces; a multivalued map (multimap) $H : X \to C(Y)$ is said to be; (i) closed if its graph

$$Gr_H = \{(x, y) \in X \times Y : y \in H(x)\}$$

is a closed subset in $X \times Y$; (ii) upper semicontinuous if $H^{-1}(V) = \{x \in X : H(x) \subset V\}$ is an open subset of X for every open $V \subset Y$. If multimap $H : X \to K(Y)$ is closed and compace (i.e. $\overline{H(X)}$ is compact) then H is upper semicontinuous (see, for example [5]).

Let X be a closed subset of a Banach space $\mathcal{E}, \varphi$ a measure of noncompactness in $\mathcal{E}$ and $k \geq 0$. A multimap $H : X \to K(\mathcal{E})$ (or a family of multimaps $G : X \times [0,1] \to K(\mathcal{E})$) is called (k, φ)-contraction (or (k, φ)-contractive) if, respectively,

$$\varphi(H(\mathcal{D})) \leq k\varphi(\mathcal{D}) \ \ or$$

$$\varphi(G(\mathcal{D} \times [0,1])) \leq k\varphi(\mathcal{D})$$

for every $\mathcal{D} \subseteq X$.

Now, let K be a convex closed subset of $\mathcal{E}, \Omega \subset K$ is open in relative topology, $\bar{\Omega}_K$ and $\partial\Omega_K$ denote the closure and the boundary of Ω relative to K. Let $\Gamma : \bar{\Omega}_K \to Kv(K)$ be a closed $(k, \chi_{\mathcal{E}})$ -contraction, where $0 \leq k < 1$ and $\chi_{\mathcal{E}}$ is the Hausdorff measure of noncompactness in $\mathcal{E}$. Assume that the fixed point set of $\Gamma |_{\Omega}$

$$Fix(\Gamma |_{\Omega}) = \{x \in \Omega : x \in \Gamma(x)\}$$

is compact. (It is sufficient to suppose that $Fix\Gamma \cap \partial\Omega_K = \emptyset$). Then the topological characteristic: the relative fixed points index $ind_K(\Gamma, \Omega)$ is defined (see [13]). It has the following fundamental properties.

1^0 If $\Gamma(x) \equiv y_0 \in K$ then

$$ind_K(\Gamma,\Omega) = \begin{cases} 1 & \text{if } y_0 \in \Omega \\ 0 & \text{if } y_0 \neq \Omega. \end{cases}$$

2^0 Let Ω_i $(i = 1,2,...)$ be open in K and mutually disjoint subsets of Ω and $Fix\Gamma \cap (\Omega \setminus \cup_{i=1}^{\infty} \Omega_i) = \emptyset$. Then, the indices $ind_K(\Gamma,\Omega_i)$ are defined, only a finite number of them does not vanish and

$$ind_K(\Gamma,\Omega) = \sum_{i=1}^{\infty} ind_K(\Gamma,\Omega_i).$$

3^0 If closed (k,χ_E)-contractions $\Gamma_0,\Gamma_1 : \bar{\Omega}_K \to Kv(K)$ are homotopic, i.e. there exists the closed (k,χ_E)-contractive family $G : \bar{\Omega}_K \times [0,1] \to Kv(K)$, such that $\cup_{\lambda\in[0,1]} Fix(G(.,\lambda)\mid_\Omega)$ is compact and $G(.,0) \equiv \Gamma_0, G(.,1) \equiv \Gamma_1$, then $ind_K(\Gamma_0,\Omega) = ind_K(\Gamma_1,\Omega)$.

4^0 If $ind_K(\Gamma,\Omega) \neq 0$, then Γ has a fixed point in Ω.

In this paper we study the existence of mild periodic solutions of the differential inclusion

(1) $$x''(t) \in Ax(t) + F(t,x(t))$$

in a separable Banach space E. Here A is a closed linear not necessarily bounded operator in E and F is a multimap from $\mathbf{R} \times E$ into E. Let F be T-periodic in the first variable $(T > 0)$, i.e. $F(t+T,.) \equiv F(T),.)$ for all $t \in \mathbf{R}$. In the sequal we shall consider the restriction $F\mid_{[0,T]\times E}$ denoting it by the same symbol F.

We shall assume that A generates an analytic semigroup e^{At} satisfying the estimation

(2) $$\|e^{At}\|^{(\chi)} \leq e^{-\delta t}$$

for all $t \in [0,T]$, where $\delta > 0$. It should be noted that condition (2) is satisfied if the resolvent of A is completely continuous at a certain point. It will be supposed also that 1 does not belong to the spectrum σe^{AT} of the operator e^{AT}; hence the operator $[I - e^{AT}]^{-1}$ is well defined.

Now we shall assume that the multimap $F : [0,T] \times E \to Kv(E)$ satisfies the following conditions.

F1) for every $e \in E$ the multifunction $F(.,e) : [0,T] \to Kv(E)$ admits a measurable selector;

F2) for almost all $t \in [0,T]$ the multimap $F(t,.) : E \to Kv(E)$ is upper semicontinuous;

F3) for every bounded $D \subset E$ there exists a function $a \in L^p_+([0,T])$, $p > 1$ such that

$$\|F(t,e)\| := \sup\{\|y\| : y \in F(t,e)\} \leq a(t)$$

for all $e \in D$ and almost all $t \in [0,T]$;

F4) for every bounded $D \subset E$ we have

$$\chi(F(\{t\} \times D)) \leq k\delta\chi(D)$$

for almost all $t \in [0,T]$, where $0 \leq k < 1$ and δ is the constant from estimation (2).

It is clear that condition F1 is satisfied if the multifunction $F(.,e)$ is measurable for all $e \in E$. From conditions F1- F3 it follows that $S^1_{F(.,x(.))} \neq \emptyset$ for every $x \in C$ (see, for example [17]).

In the sequel we shall consider as solutions of the inclusion (1) the mild solutions, i.e. the functions $x \in C$ of the form

$$x(t) = e^{AT}x(0) + \int_0^t e^{A(t-s)}f(s)ds,$$

where $f \in S^1_{F(.,x(.))}$. it should be noted that the existence of mild solutions for differential inclusions of form (1) is considered for example in the works of N.S. Papageorgiou [14], [16] and V.V.Obukhovskiĭ [12].

It is clear that the question on T-periodic solutions of inclusion (1) is reduced to the existence of solutions $x \in C$ satisfying the boundary condition of periodicity

$$x(0) = x(T). \tag{3}$$

In the sequal the solution $x \in C$ satisfying condition (3) will be called the T-periodic solution of (1).

Boundary value problems for differential inclusions with linear operators in a Banach space were considered earlier in the works of P.Zecca-P.Zezza [18] and N.Papageorgiou [15]. In this paper we study the periodic problem under other assumptions, using a method which is new (even in the case of differential equations) and is based on the fixed points index theory for condensing multimaps.

2. Integral multioperator and its properties

In order to investigate T-periodic solutions we shall consider the multivalued integral operator Γ in the space C defined in the following way

$$\Gamma(x) = \{y \in C : y(t) = e^{At}[I - e^{AT}]^{-1} \int_0^T e^{A(T-s)} f(s)ds +$$

$$+ \int_0^t e^{A(t-s)} f(s)ds,\ f \in S^1_{F(\cdot,x(\cdot))}\}.$$

Theorem 2.1. *A function $\chi \in C$ is a T-periodic solution of (1) iff it is a fixed point of* Γ.

Proof. Let $x \in C$ be a solution of (1) satisfying periodicity condition (3). Then

$$x(0) = e^{AT}x(0) + \int_0^T e^{A(T-s)} f(s)ds,$$

where $f \in S^1_{F(\cdot,x(\cdot))}$. Hence,

$$x(0) = [I - e^{AT}]^{-1} \int_0^T e^{A(T-s)} f(s)ds$$

and

$$x(t) = e^{At}[I - e^{AT}]^{-1} \int_0^T e^{A(T-s)} f(s)ds +$$

$$+ \int_0^t e^{A(t-s)} f(s)ds,$$

i.e. $x \in \Gamma(x)$.

The validity of condition (3) for the function $x \in Fix\Gamma$ can be verified directly. □

Let us now describe the main properties of the multioperator Γ. It follows immediately from the properties of the multimap F that the multioperator Γ has nonempty convex values.

Theorem 2.2. *The multioperator* Γ *is closed.*

Proof. Let

$$\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty} \subset C;\ y_n \in \Gamma(x_n)(n = 1, 2, ...),\ x_n \to x,\ y_n \to y.$$

We shall show that $y \in \Gamma(x)$.

By assumption we have for every $n \in \mathbf{N}$

$$y_n(t) = e^{At}[I - e^{AT}]^{-1} \int_0^T e^{A(T-s)} f_n(s)ds +$$

$$+ \int_0^t e^{A(t-s)} f_n(s)ds,$$

where $f_n \in S^1_{f(.,x_n(.))}$. From condition F4), it follows that

$$\chi(\{f_n(t)\}_{n=1}^{\infty}) \leq k\delta\chi(\{x_n(t)\}_{n=1}^{\infty}) = 0$$

for almost every $t \in [0, T]$, i.e. $\overline{\{f_n(t)\}_{n=1}^{\infty}}$ is compact in E a.e. $t \in [0, T]$. From the Diestel criterion (see [7], [17]) it then follows that the sequence $\{f_n\}_{n=1}^{\infty}$ is relatively weak compact in the space $L_1([0, T]; E)$. Therefore, we can assume that

$$f_n \underset{w}{\to} f \in L_1([0, T]; E).$$

We claim that $f \in S^1_{f(.,x(.))}$.

Indeed, according to the Mazur lemma (see, for example, [8]) the weak convergence $f_n \underset{w}{\to} f$ implies the existence of the double sequence of nonnegative numbers $\{\lambda_{ik}\}_{k=1}^{\infty}{}_{i=1}^{\infty}$, such that :

1) $\sum_{k=i}^{\infty} \lambda_{ik} = 1$ for all $i = 1, 2, ...$;

2) $\lambda_{ik} = 0$ for all $k \geq k_0(i)$;

3) the sequence $\{\tilde{f}_i\}_{i=1}^{\infty}$, $\tilde{f}_i(t) = \sum_{k=i}^{\infty} \lambda_{ik} f_k(t)$ converges to f with respect to the norm of $L^1([0, T]); E$. Passing if necessary to a subsequence, we can assume that $\{\tilde{f}_i\}_{i=1}^{\infty}$ converges to f almost everywhere on $[0, T]$.

From condition F2), it follows that for almost all $t \in [0, T]$ for a given $\epsilon > 0$ there exists the integer $i_0 = i_0(\epsilon, t)$, such that

$$F(t, x_i(t)) \subset V_\epsilon(F(t, x(t)))$$

for all $i \geq i_0$, where V_ϵ denotes the ϵ-neighbourhood of a set. But, then $f_i(t) \in V_\epsilon(F(t, x(t)))$ for $i \geq i_0$ and hence

$$\tilde{f}_i(t) \in V_\epsilon(F(t, x(t))),$$

by virtue of the convexity of $V_\epsilon(F(t,x(t)))$. Therefore,

$$f(t) \in F(t,x(t))$$

i.e. $f \in S^1_{F(.,x(.))}$.

For each $t \in [0,T]$ the map $g \mapsto \int_0^t e^{A(t-s)}g(s)ds$ is a continuous linear operator from $L^1([0,T];E)$ into E. It remains continuous if these spaces are endowed with weak topologies. Therefore, for each $t \in [0,T]$ the sequence $y_n(t)$ converges weakly to $e^{At}[I-e^{AT}]^{-1}\int_0^T e^{A(T-s)}f(s)ds+\int_0^t e^{A(t-s)}f(s)ds$. Since by assumption $y_n(t) \to y(t)$, we have $y \in \Gamma(x)$. □

Theorem 2.3. *For every bounded set $\Omega \subset C$ the set $\Gamma(\Omega)$ is bounded and equicontinuous.*

Proof. From condition F3), it follows that there exists a function $a \in L^p_+([0,T])$, $p > 1$, such that for every $x \in \Omega$ and $f \in S^1_{F(.,x(.))}$ we have

$$\|f(t)\| \leq a(t) \ a.e. \ t \in [0,T].$$

By virtue of the analyticity of the semigroup e^{At}, there exist constants C and γ such that

$$\|e^{At}\| \leq Ce^{\gamma t}. \tag{4}$$

Since by assumption $1 \notin \sigma e^{AT}$, we have

$$\|(I-e^{AT})^{-1}\| \leq K, \tag{5}$$

where $K > 0$.

Therefore, if $y \in \Gamma(\Omega)$, then

$$\|y(t)\| \leq KC^2e^{\gamma t}\int_0^T e^{\gamma(T-s)}a(s)ds+$$

$$+C\int_0^t e^{\gamma(T-s)}a(s)ds \leq \tilde{C}\|a\|_{L^p([0,T])},$$

where $\tilde{C}$ is a certain constant. Therefore, $\Gamma(\Omega)$ is bounded.
We now clain that the set $\Gamma(\Omega)$ is equicontinuous. To show this, we may remark that, according to [11], the negative fractional powers $A^{-\alpha}(0 < \alpha < 1)$ of A are defined and the following estimations hold

$$\|A^\alpha e^{At}\| \leq \frac{M(\alpha)}{t^\alpha} \tag{6}$$

$$\|A^{-\alpha}(e^{At}-I)\| \le \frac{M(1-\alpha)}{\alpha}t^{\alpha}, \tag{7}$$

where $M(\alpha)$ is a certain constant depending on α. Choose $\alpha \in (0,1)$ such that $\alpha q < 1$, where $1/p + 1/q = 1$. Then,

$$\int_0^t \frac{a(s)}{|t-s|^{\alpha}}ds \le \left(\int_0^t (a(s))^p ds\right)^{1/p}\left(\int_0^t \frac{ds}{(t-s)^{\alpha q}}\right)^{1/q} \le \|a\|_{L^p}\frac{T^{(1-\alpha q)/q}}{1-\alpha q}. \tag{8}$$

Now, if $y \in \Gamma(x)$, $x \in \Omega$ and $t_1, t_2 \in [0,T]$, $t_2 > t_1$, then

$$\|y(t_2)-y(t_1)\| \le \|A^{-\alpha}(e^{A(t_2-t_1)}-I)e^{At_1}\| \times$$

$$\|(I-e^{AT})^{-1}\int_0^T A^{\alpha}e^{A(T-s)}f(s)ds\| +$$

$$+\|\int_0^{t_1} A^{-\alpha}(e^{A(t_2-t_1)}-I)A^{\alpha}E^{A(t_1-s)}f(s)ds\| +$$

$$+\|\int_{t_1}^{t_2} e^{A(t_2-s)}f(s)ds\| = J_1 + J_2 + J_3,$$

where $f \in S^1_{F(.,x(.))}$. Using (4)-(8) we have the following estimations:

$$J_1 \le \frac{M(1-\alpha)}{\alpha}(t_2-t_1)^{\alpha}Ce^{\gamma t_1}K\int_0^Y \frac{a(s)}{(T-s)^{\alpha}}ds \le \tag{9}$$

$$C_1\frac{M(1-\alpha)}{\alpha}\|a\|_{L_p}\frac{T^{(1-\alpha q)/q}}{1-\alpha q}(t_2-t_1)^{\alpha}$$

$$J_2 \le C_2\frac{M(1-\alpha)}{\alpha}(t_2-t_1)^{\alpha}\int_0^{t_1}\frac{a(s)}{(t_1-s)^{\alpha}}ds \tag{10}$$

$$\le C_2\frac{M(1-\alpha)}{\alpha}\frac{T^{(1-\alpha q)/q}}{1-\alpha q}\|a\|_{L_p}(t_2-t_1)^{\alpha}$$

$$J_3 \le \int_{t_1}^{t_2} Ce^{\gamma(t_2-s)}a(s)ds \le \left(\int_{t_1}^{t_2}(a(s))^p ds\right)^{1/p}\left(C\int_{t_1}^{t_2} e^{q\gamma(t_2-s)}ds\right)^{1/q} \tag{11}$$

$$\le C_3\|a\|_{L_p}(t_2-t_1)^{1/q}.$$

These estimations show that

$$\|y(t_2)-y(t_1)\| \le \tilde{C}_1(t_2-t_1)^{\alpha} + \tilde{C}_2(t_2-t_1)^{1/q}$$

estabilishing the demanded equicontinuity of $\Gamma(\Omega)$.□

Now, consider the measure of noncompactness $\psi : 2^{\mathcal{C}} \to \mathbf{R}_+$

$$\psi(\Omega) = \sup_{t\in[0,T]} \chi(\Omega(t))$$

(see $\chi 5)$).

Theorem 2.4. *The multioperator* Γ *is a* (k,ψ)*-contraction.*

Proof. If $\Omega \subset C$ is unbounded, then $\psi(\Omega) = +\infty$ and the inequality $\psi(\Gamma(\Omega)) \le k\psi(\Omega)$ evidently holds. Let $\Omega \subset C$ be nonempty and bounded. Since for every $x \in \Omega$ we have $S^1_{F(.,x(.))} \subset S^1_{F(.,\Omega(.))}$ on each interval $[0,t] \subseteq [0,T]$, the integral

$$\int_0^t e^{A(t-s)} F(s,\Omega(s))ds$$

is defined for all $t \in [0,T]$ and

$$\{\int_0^t e^{A(t-s)} f(s)ds : f \in S^1_{F(.,x(.))},\ x \in \Omega\}$$

$$\subset \int_0^t e^{A(t-s)} F(s,\Omega(s))ds.$$

By virtue of condition F3) the multifunction $S \mapsto e^{A(t-s)}F(s\Omega(s))$, $0 \le s \le t$, is integrally bounded and using Lemma 1.1 and condition F4) we have

$$\chi(e^{A(t-s)}F(s,\Omega(s)) \le \|e^{A(t-s)}\|^{(\chi)}\chi(F(s,\Omega(s))) \le$$

$$\le k\delta e^{-\delta(t-s)}\chi(\Omega(s)) \le k\delta e^{-\delta(t-s)}\psi(\Omega)$$

for almost all $s \in [0,t]$.

For each $t \in [0,T]$ we have

$$\Gamma(\Omega)(t) \subset e^{At}[I - e^{AT}]^{-1} \int_0^T e^{A(T-s)} F(s,\Omega(s))ds +$$

$$+ \int_0^t e^{A(t-s)} F(s,\Omega(s))ds,$$

and now using the properties $\chi 1), \chi 2)$ and Lemmas 1.1, 1.2 and 1.3, we obtain the following estimation:

$$\chi(\Gamma(\Omega)(t)) \le \|e^{At}\|^{(\chi)}\|(I - e^{AT})^{-1}\|^{(\chi)}\times$$

$$\chi(\int_0^T e^{A(t-s)}F(s,\Omega(s))ds) + \chi(\int_0^t e^{A(t-s)}F(s,\Omega(s))ds) \leq$$
$$\leq e^{-\delta t}(1-e^{-\delta T})^{-1}k\delta\psi(\Omega)\int_0^T e^{-\delta(T-s)}ds +$$
$$+k\delta\psi(\Omega)\int_0^t e^{-\delta(t-s)}ds =$$
$$k\delta\psi(\Omega)[e^{-\delta t}(1-e^{-\delta T})^{-1}\frac{1}{\delta}(1-e^{-\delta T}) + \frac{1}{\delta}(1-e^{-\delta t})] = k\psi(\Omega).$$

Therefore,

$$\psi(\Gamma(\Omega)) = \sup_{t\in[0,T]} \chi(\Gamma(\Omega)(t)) \leq k\psi(\Omega),$$

proving the theorem . □

Corollary 2.5. *For every bounded set $\Omega \subset \mathcal{E}$, the multioperator Γ is a $(k,\chi_{\mathcal{E}})$-contraction on the set $\Omega \cap \overline{co}\Gamma(\Omega)$ where $\chi_{\mathcal{E}}$ is the Hausdorff measure of noncompactness in $\mathcal{E}$.*

Proof. It follows immediately from the property $\chi5)$ and Theorem 2.3. □

3. The existence of periodic solutions

The properties described above of the integral multioperator allow us to use the theory of the fixed points index to search for periodic solutions of problem (1). Indeed, the following general principle is valid.

Theorem 3.1. *Let $\Omega \subset C$ be an open bounded set; $K = \overline{co}\Gamma(\Omega)$; $x \notin \Gamma(x)$ for all $x \in \partial\Omega_K$ and $ind_K(\Gamma,\Omega_K) \neq 0$ where $\Omega_K = \Omega \cap K$. Then inclusion (1) has at least one T-periodic solution in Ω_K.*

To ensure the conditions of this theorem, we may use a priori bounds of T-periodic solutions of (1). Consider, for example, the following statement.

Theorem 3.2. *Let the multimap F satisfy conditions $F1), F2), F4)$ and $F'3)$, there exists a function $a \in L^p_+([0,T]), p > 1$ such that*

$$\|F(t,e)\| \leq a(t)$$

for all $e \in E$ and almost all $t \in [0,T]$. Then, inclusion (1) has at least one $T-$ periodic solution.

Proof. From condition $F'3$) it follows that the set $\Gamma(C)$ is bounded. Let $\Omega \subset C$ be an open ball containing the set $\Gamma(C)$. Then, from the properties of the fixed points index we have that $ind_K(\Gamma, \Omega_K) = 1$, where $K = \overline{co}\Gamma(C)$. □

The homotopy invariance property of the fixed points index allows us to apply the general principle to families of differential inclusions continuously depending on a parameter. Let the multimap $F : [0,T] \times E \times [0,1] \to Kv(E)$ satisfy the following properties: s continuously

1Fλ) $F(\cdot, e, \lambda) : [0,T] \to Kv(E)$ admits a measurable selector for all $(e, \lambda) \in E \times [0,1]$;

2Fλ) $F(t, \cdot, \cdot) : E \times [0,1] \to Kv(E)$ is upper semicontinuous for almost all $t \in [0,T]$;

3Fλ) for every bounded $D \subset E$ there exists a function $a \in L^p_+([0,T])$, $p > 1$ such that $\|F(t, e, \lambda)\| \leq a(t)$ for all $(e, \lambda) \in E \times [0,1]$ and almost all $t \in [0,T]$;

4Fλ) for every bounded $D \subset E$ we have $\chi(F(\{t\} \times D \times [0,1])) \leq k\delta\chi(D), 0 \leq k < 1$ for a.e. $t \in [0,T]$.

As an example of such a family we may regard the multimap $\lambda F, \lambda \in [0,1]$, where F satisfies the conditions $F1) - F4)$.

Consider the family of differential inclusions

(1λ) $x'(t) \in Ax(t) + F(t, x(t), \lambda)$, where the multimap F satisfies the conditions $1F\lambda) - 4F\lambda)$.

Modifying our previous reasonings, we may show that the family of integral multioperators

$$\Gamma : C \times [0,1] \to Cv(C),$$

$$\Gamma(x, \lambda) = \{y \in C : y(t) = e^{At}[I - e^{At}]^{-1} \int_0^T e^{A(T-s)} f(s)ds +$$

$$\int_0^t e^{A(t-s)} f(s)de, \ f \in S^1_{F(\cdot, x(\cdot), \lambda)}\}$$

generated by the family (1λ) has the following properties.

1$\Gamma\lambda$) The multimap $\Gamma(x, \lambda)$ is closed.

$2\Gamma\lambda$) For every bounded $\Omega \subset C$, the set $\Gamma(\Omega \times [0,1])$ is bounded and equicontinuous.

$3\Gamma\lambda$) The family $\Gamma(x,\lambda)$ is (k,ψ)– contractive. It is (k,χ_C)– contractive on every set of the form $\Omega \cap \overline{co}\Gamma(\Omega \times [0,1])$, where $\Omega \subset C$ is bounded.

These properties allow us to justify the following statement as a direct sequence of the property of homotopic invariance of the fixed points index of the multimap $\Gamma(\cdot,\lambda)$.

Theorem 3.3. *Let $\Omega \subset C$ be an open bounded set whose boundary does not contain T-periodic solutions of the family (1λ), $\lambda \in [0,1]$. Let $ind_K(\Gamma(\cdot,0),\Omega_K) \neq 0$, where $K = \overline{co}\Gamma(\Omega \times [0,1])$. Then, the differential inclusion*

$$x'(t) \in Ax(t) + F(t,x(t)) \tag{12}$$

has as least one T-periodic solution in Ω_K. □

The following application of the antipodal theorem is the concretization of this principle.

Theorem 3.4. *Let $\Omega \subset C$ be a symmetric open - bounded neighbourhood of the origin whose boundary does not contain T- periodic solutions of the family (1λ). If*

$$F(t,-e,0) = -F(t,e,0)$$

for all $e \in E$ and almost all $t \in [0,T]$, then the differential inclusion (12) has at least one T-periodic solution in Ω.

Proof. The set $K_1 = \overline{co}(K \cup (-K))$, where $K = \overline{co}\Gamma(\bar{\Omega} \times [0,1])$ is symmetric with respect to the origin and equicontinuous. It is easy to see that the integral multioperator $\Gamma(\cdot,0)$ is odd, and hence $ind_{K_1}(\Gamma(\cdot,0),\Omega_{K_1}) \equiv 1(mod\, 2)$([14]). But, now, using the homotopy invariance property we have $ind_{K_1}(\Gamma(\cdot,1),\Omega_{K_1}) \equiv 1(mod\, 1)$. □

Corollary 2.5. *Let $\Omega \subset C$ be a symmetric open-bounded neighbourhood of the origin whose boundary does not contain T- periodic solutions of the family*

$$x'(t) \in Ax(t) + \lambda F(t,x(t)),$$

where $\lambda \in [0,1]$ and F satisfies the properties $F1) - F4)$. Then inclusion (1) has at least one T-periodic solution in Ω.

Note, in conclusion, that the operator method described above can be used in the problem of the existence of optimal periodic solutions for control systems in a Banach space. Describe briefly the scheme of this application.

Let E, E_1 be Banach spaces; a map $f : [0,T] \times E \times E_1 \to E$ is such that:

1) $f(\cdot, e, e_1) : [0,T] \to E$ is measurable for all $(e, e_1) \in E \times E_1$;

2) $f(t, \cdot, \cdot) : E \times E_1 \to E$ is continuous for almost all $t \in [0,T]$.

Let a multimap $U : [0,T] \times E \to K(E_1)$ characterize the varying admissible controls domain and have the following peoperties: $U(\cdot, e)$ is measurable for all $e \in E$ and $U(t, \cdot)$ is upper semicontinuous for almost all $t \in [0,T]$.

Consider a nonlinear control system with feedback

$$x'(t) = Ax(t) + f(t, x(t)), u(t)) \tag{13}$$

$$u(t) \in U(t, x(t)) \tag{14}$$

Its solution is the pair $(x(\cdot), u(t))$ consisting of the trajectory $x(\cdot) \in C$, where $x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)} f(s, x(s), u(s))ds$ is a mild solution of equation (13), and the control $u : [0,T] \to E$, where u is a measurable function satisfying inclusion (14) for almost all $t \in [0,T]$.

The multimap $F : [0,T] \times E \to K(E), F(t,e) = f(t, e, U(t,e))$ satisfies the conditions $F1$) and $F2$) (see, for example [5]). Assume also that the multimap F has convex values and satisfies the conditions $F3$) and $F4$). It should be noted that condition $F4$) is satisfied, if we assume that the multimap $U(t, \cdot)$ is completely continuous for almost all $t \in [0,T]$, i.e. the set $U(\{t\} \times D)$ is relatively compact for every bounded $D \subset E$ and the map $f(t, \cdot, e_1) : E \to E$ is a $k\delta-$ Lipschitzian for all $e_1 \in E_1$ and almost all $t \in [0,T]$ (see [9]),

Therefore the methods described can be applied to the differential inclusion

$$x'(t) \in Ax(t) + f(t, x(t), U(t, x(t))). \tag{15}$$

The existence of the control realizing the $T-$ periodic solution of (15) as the trajectory of the control systems (13), (14) follows from the Filippov implicit functions lemma (see, for example [5]).

The topological properties of the integral multioperator allow us to justify the following optimal control principle.

Let the set of all T−periodic solutions of the control system (13), (14) be nonempty and bounded. Then, there exists the control u_* such that the corresponding trajectory x_* minimize the given lower semicontinuous functional $j : C \to \mathbf{R}$.

It follows directly from the compactness of a fixed points set of a (k, χ_C)−contractive integral multioperator Γ.

References

[1] Akhmerov, R.R., Kamenskiĭ, M.I., Potapov, A.S., Sadovskiĭ, B.N.: Condensing operators (in Russian), Itogi Nauki Tehniki, Ser. Math. Anal. 18 (1980), 185-250.

[2] Akhmerov, R.R., Kamenskiĭ, M.I., Potapov, A.S., Rodkina, A.E., Sadovskiĭ, B.N.: Measures of noncompactness and condensing operators (in Russian). Nauka, Novosibirsk, 1986.

[3] Borisovich, Yu.G., Gel'man, B.D., Myshkis, A.D., Obukhovskiĭ, V.V.: Topological methods in the fixed point theory of multivalued maps (in Russian), Uspehi Mat. Nauk 35, 1(1980), 59- 126.

[4] Borisovich, Yu.G., Gel'man, B.D., Myshkis, A.D., Obukhovskiĭ, V.V.: Multivalued maps (in Russian), Itogi Nauki Tehniki, Ser.Mat. Anal. 19(1982), 127-230.

[5] Borisovich, Yu.G., Gel'man, B.D., Myshkis, A.D., Obukhovskiĭ, V.V.: Introduction to the theory of multivalued maps (in Russian), Izdat. Voronezh Gos. Univ., Voronezh, 1986.

[6] Castaing, C., Valadier, M.: Convex analysis and measurable multifunctions, Lect. Notes in Math., 580. Springer- Verlag, Berlin-Heidelberg-New York, 1977.

[7] Diestel, J.: Remarks on weak compactness in $L_1(\mu, X)$, Glasgow Math. J., 18, 1 (1977), 87-91.

[8] Ekeland, I., Temam, R.: Convex analysis and variational problems, North-Holland Publ. Company, Amsterdam, 1976.

[9] Gel'man, B.D., Obukhovskiĭ, V.V.: Some fixed point theorems for multivalued maps of condensing type (in Russian), to appear in News in global analysis, Izdat. Voronezh Univ., Voronezh.

[10] Kisielewics, M.: Multivalued differential equations in separable Banach spaces. J.Optimizat. Theory and Applic. 37, 2(1982), 231-249.

[11] Krasnosel'skii, M.A., Zabreiko, P.P., Pustyl'nik, E.I., Sobolevskii, P.E.: Integral operators in spaces of summable functions (in Russian), Nauka, Moscow, 1966.

[12] Ovukhovskiĭ, V.: On evolution differential inclusions in a Banach space, Fourth Conf. Diff. Equat. and Appl. , Rousse, 1989, 430.

[13] Obukhovskiĭ, V.V.: On the fixed points index for some classes of noncompact multivalued maps (in Russian) , to appear.

[14] Papageorgiou, N.S.: On multivalued evolution equations and differential inclusions in Banach spaces. Comment. Math. Univ. Sancti Pauli, 36, 1(1987), 21-39.

[15] Papageorgiou, N.S.: Boundary value problems for evolution inclusions, Comment. math. Univ. Carolinae, 29, 2(1988), 355-363.

[16] Papageorgiou, N.S.: On multivalued semilinear evolution equations, Boll. U.M.I., (7) 3-B (1989), 1-16.

[17] Tolstonogov, A.A.: Differential inclusions in a Banach space (in Russian), Nauka, Novosibirsk, 1986.

[18] Zecca, P., Zezza, P.: Nonlinear boundary valued problems in Banach spaces for multivalued differential equations on a non- compact interval, Nonlinear Anal.: Theory, Methods, Appl., 3, 3(1979), 347-352.

REZIME

O PERIODIČNIM REŠENJIMA DEFIRENCIJALNIH INKLUZIJA SA NEOGRANIČENIM OPERATORIMA U BANAHOVIM PROSTORIMA

U ovom radu se razmatra jedna operatorska metoda, korišćenjem teorije indeksa nepokretne tačke kondenzujućeg višeznačnog preslikavanja, za ispitivanje postojanja blago periodičnih rešenja diferencijalne inkluzije

$$x'(t) \in Ax(t) + F(t, x(t)),$$

gde je A linearni operator i F višeznačno preslikavanje u separabilnom Banahovom prostoru. Razmatrane su neke primene na egzistenciju optimalnih periodičnih rešenja kontrolnih sisitema u Banahovim prostorima.

Received by the editors January 7, 1990.