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# ON COINCIDENCE POINTS IN PROBABILISTIC METRIC SPACES WITH A CONVEX STRUCTURE

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**Abstract**

In this paper a theorem on the existence of a coincidence point in probabilistic metric spaces with a convex structure is proved. The theorem is a generalization of Theorem 3 from [6].

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## 1. Introduction

The notion of a probabilistic metric space is introduced by K.Menger in [9] and the first result about the existence of a fixed point of a mapping which is defined on a Menger space $(S, \mathcal{F}, min)$ is obtained by V.Sehgal and A.Barucha-Reid in [1]. Since then, many fixed point theorems for mappings which are defined on a Menger space $(S, \mathcal{F}, t)$ are obtained ([4], [5], [6], [13], [15], [16]). The theory of probabilistic metric spaces is now an important part of the stochastic analysis and in the books [3] and [10] extensive bibliographies on probabilistic metric spaces and their applications are given.

In [6] the notion of a probabilistic metric space with a convex structure $W$ is given and a theorem on coincidence point in such a space is proved. Here, we shall prove a generalization of Theorem 3 from [6].

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## 2. Preliminaries

A triplet $(S, \mathcal{F}, t)$ is a Menger space if and only if $S$ is a nonempty set, $\mathcal{F} : S \times S \to \Delta$, where $\Delta$ denotes the set of all distribution functions, and $t$ is a $T$-norm [10] so that the following conditions are satisfied ($\mathcal{F}(p,q) = F_{p,q}$ for every $p, q \in S$):

1. $F_{p,q}(x) = 1$, for every $x \in \mathbf{R}^+$ if only if $p = q$.
2. $F_{p,q}(0) = 0$, for every $p, q \in S$.
3. $F_{p,q} = F_{q,p}$, for every $p, q \in S$.
4. $F_{p,r}(u+v) \geq t(F_{p,q}(u), F_{q,r}(v))$, for every $p, q, r \in S$ and every $u, v \in \mathbf{R}^+$.

The $(\epsilon, \lambda)$-topology is introduced by the $(\epsilon, \lambda)$-neighbourhoods of $v \in S$ :

$$U_v(\epsilon, \lambda) = \{u; u \in S, F_{u,v}(\epsilon) > 1 - \lambda\},\ \epsilon > 0,\ \lambda \in (0,1).$$

In [8] some relations between a Menger space and a fuzzy metric space are obtained. It is proved that every Menger space $(S, \mathcal{F}, t)$ is a fuzzy metric space $(S, d, 0, R)$, where

$$R(a,b) = 1 - t(1-a, 1-b)$$

$$d(x,y)(s) = \begin{cases} 0, & s < \sup\{s; F_{x,y}(s) = 0\} \\ 1 - F_{x,y}(s), & s \geq \sup\{s; F_{x,y}(s) = 0\}. \end{cases}$$

**Definition 1.** *Let $(S, \mathcal{F}, t)$ be a Menger space. A mapping $W : S \times S \times [0,1] \to S$ is said to be a convex structure if for every $(u, x, y, \lambda) \in S \times S \times S \times (0,1)$ :*

$$F_{u,W(x,y,\lambda)}(2\epsilon) \geq t(F_{u,x}(\frac{\epsilon}{\lambda}), F_{u,y}(\frac{\epsilon}{1-\lambda})),$$

*for every $\epsilon \in \mathbf{R}^+$ and $W(x,y,0) = y$, $W(x,y,1) = x$.*

Every random normed spaces [12] is a probabilistic metric space $S$ with a convex structure $W(x,y,\lambda) = \lambda x + (1-\lambda)y$, $x, y \in S$, $\lambda \in (0,1)$.

**Example.** A nontrivial example of a Menger space with a convex structure is the following. Let $(M, d)$ be a separable metric space with a convex structure $W$ [14] such that for every $\delta \in [0,1]$ the mapping $(x, y) \longmapsto W(x, y, \delta)$ is continuous. Let $(\Omega, \Sigma, P)$ be a probability space. We shall prove that the Menger space $(S, \mathcal{F}, t_m)(t_m(a,b) = \max\{a+b-1, 0\},\ a, b \in [0,1])$ has a convex structure, where $S$ is the space of all measurable mappings from $\Omega$ into $M$ (the space of equivalence classes ) and for every $X, Y \in S$ :

$$F_{X,Y}(u) = P(\{\omega,\ \omega \in \Omega, d(X(\omega), Y(\omega)) < u\}),\ u \in \mathbf{R}^+.$$

Let $\bar{W} : S \times S \times [0,1] \to S$ be define by the relation: $\bar{W}(X, Y, \delta)(\omega) = W[X(\omega), Y(\omega), \delta]$, for every $\omega \in \Omega$, every $X, Y \in S$ and every $\delta \in [0,1]$.

From the measurability of the mappings $X$ and $Y$ and the continuity of the mapping $(x, y) \to W(x, y, \delta)$ it follows that $\bar{W}(X, Y, \delta) \in S$. It is easy to verify that

$$F_{U,\bar{W}(X,Y,\delta)}(2\epsilon) \geq F_{U,X}(\frac{\epsilon}{\delta}) + F_{U,Y}(\frac{\epsilon}{1-\delta}) - 1$$

for every $X, Y, U \in S$; $\delta \in (0,1)$, $\epsilon > 0$. Further, $\bar{W}(X, Y, 0) = Y$ and $\bar{W}(X, Y, 1) = X$ and so $(S, \mathcal{F}, t_m)$ is a probabilistic metric space with the convex structure $\bar{W}$.

In this paper we shall suppose that a convex structure $W$ on a Menger space $(S, \mathcal{F}, t)$ satisfies the following inequality:

$$F_{W(x,z,\lambda),W(y,z,\lambda)}(\lambda\epsilon) \geq F_{x,y}(\epsilon)$$

for every $(x, y, z, \lambda, \epsilon) \in S \times S \times S \times (0,1) \times \mathbf{R}^+$.

The convex structure $W(x, y, \lambda) = \lambda x + (1-\lambda)y$ in a random normed space satisfies this condition.

In Itoh's paper [7] a similar condition is introduced for metric spaces with a convex structure.

If $(S, \mathcal{F})$ is a probabilistic metric space with a convex structure $W, x_0 \in S$ and $T : S \to S$, then $T$ is said to be $(W, x_0)$-convex if $W(Tz, x_0, \lambda) = T(W(z, x_0, \lambda))$, for every $\lambda \in (0,1)$ and $z \in S$.

A nonempty set $K$ of $S$, where $(S, \mathcal{F})$ is a probabilistic metric space with a convex structure $W$, is said to be starshaped in respect to $x_0 \in K$ if and only if $W(x, x_0, \lambda) \in K$ for every $x \in K$ and every $\lambda \in (0,1)$ . Then $x_0$ is a starcenter of $K$.

Let $(S,\mathcal{F})$ be a probabilistic metric space and $A$ a nonempty subset of $S$. The function $D_A(.) : \mathbf{R}^+ \to [0,1]$ defined by [3]:

$$D_A(u) = \sup_{s<u} \inf_{p,q\in A} F_{p,q}(s), \quad u \in \mathbf{R}^+$$

is called ***the probabilistic diameter*** of the set $A$ and the set $A$ is ***probabilistic bounded*** if and only if

$$\sup_{u\in\mathbf{R}^+} D_A(u) = 1.$$

In [15] the function $\{\beta_A(.)\}$, as a generalization of the notion of the measure of noncompactness, is defined for a probabilistic bounded subset $A \subset S$ in the following way:

$$\beta_A(u) = \sup\{\epsilon; \epsilon > 0, \text{ there exists a finite subset}$$

$$A_f \text{ of } S \text{ such that } \tilde{F}_{A,A_f}(u) \geq \epsilon\}$$

where

$$\tilde{F}_{A,B}(u) = \sup_{s<u} \inf_{x\in A} \sup_{y\in B} F_{x,y}(s)$$

for every two probabilistic bounded subsets of $S$.

The function $\{\beta_A(.)\}$ has the following properties [15] :

1. $\beta_A \in \Delta$.

2. $\beta_A(u) \geq D_A(u)$, for every $u \in \mathbf{R}^+$.

3. $\emptyset \neq A \subset B \subset S \Rightarrow \beta_A(u) \geq \beta_B(u)$, for every $u \in \mathbf{R}^+$.

4. $\beta_{A\cup B}(u) = \min\{\beta_A(u), \beta_B(u)\}$, for every $u \in \mathbf{R}^+$.

5. $\beta_A(u) = \beta_{\bar{A}}(u)$($u \in \mathbf{R}^+$, where $\bar{A}$ is the closure of $A$).

6. $\beta_A = H \Leftrightarrow A$ is precompact, where

$$H(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0. \end{cases}$$

**Definition 2.** *Let $(S, \mathcal{F})$ be a probabilistic metric space, $B(S)$ the family of all probabilistic bounded subset of $S, M \subseteq S$ and $A_1$ and $A_2$ mappings from $S$ into $S$. If for every $K \subseteq M$ such that $A_1(K), A_2(K) \in B(S)$ the implication*

$$\beta_{A_2(K)}(u) \geq \beta_{A_1(K)}(u),\ u \in \mathbf{R}^+ \Rightarrow \bar{K} \text{ is compact}$$

*holds, $A_1$ is said to be a $(\beta, A_2)$- densifying mapping on $M$.*

If $X$ is a topological space, $M \subseteq X$ is an attractor for a mapping $F : X \to X$ if for every $x \in X$ :

$$M \cap \overline{(\cup_{n \in \mathbf{N}} F^n(x))} \neq \emptyset.$$

## 3. A theorem on coincidence points

The following theorem is a generalization of Theorem 3 from [6].

**Theorem 1.** *Let $(S, \mathcal{F}, t)$ be a complete Menger space with a convex structure $W$ and continuous $T$-norm $t, A$ and $B$ continuous, commutative mappings from $S$ into $S$ such that $AS$ is probabilistic bounded subset of $BS, x_0 \in S$ and $B$ be $(W, x_0)$-convex so that*

$$F_{Ax,Ay}(\epsilon) \geq F_{Bx,By}(\epsilon),\ \textit{for every}\ \ x, y \in S \ \textit{and every}\ \ \epsilon \in \mathbf{R}^+.$$

*If there exists a nonempty subset $M$ of $S$ such that $BM$ is an attractor for the mapping $A$ and $A$ is $(\beta, B)$- densifying on $M$ then there exists $x \in S$ such that $Ax = Bx$.*

*Proof.* Similarly as in [6] it follows that there exists a sequence $\{y_n\}_{n \in \mathbf{N}}$ from $M$ such that

$$(1) \qquad \lim_{n \to \infty} F_{By_n, Ay_n}(\epsilon) = 1, \text{ for every } \ \epsilon > 0$$

but we shall give the proof of (1) because of the completeness.

Let $\{k_n\}_{n \in \mathbf{N}}$ be a sequence from $(0,1)$ such that $\lim_{n \to \infty} k_n = 1$. For every $n \in \mathbf{N}$ and every $x \in S$ let

$$A_n x = W(Ax, x_0, k_n).$$

It is easy to verify that for $A_n, S = B$ and $T = B$ all the conditions of Theorem B from [5] are satisfied. Hence, for every $n \in \mathbf{N}$ there exists $x_n \in S$ such that

$$x_n = A_n x_n = B x_n.$$

Since $(S, \mathcal{F}, t)$ is a probabilistic metric space with a convex structure $W$ we obtain that

$$F_{x_n, Ax_n}(\epsilon) \geq F_{Ax_n, x_0}\left(\frac{\epsilon}{2(1-k_n)}\right) \tag{2}$$

for every $n \in \mathbf{N}$ and every$\epsilon > 0$. From

$$BM \cap (\overline{\cup_{m\in\mathbf{N}} A^m x_n}) \neq \emptyset, \text{ for every } \quad n \in \mathbf{N}$$

it follows that there exists, for every $n \in \mathbf{N}, y_n \in M$ such that $By_n \in \overline{\cup_{m\in\mathbf{N}} A^m x_n}$. Further, for every $n \in \mathbf{N}$ and every $k \in \mathbf{N}$

$$F_{A^k x_n, A^{k+1} x_n}(\epsilon) \geq F_{x_n, Ax_n}(\epsilon), \text{ for every } \quad \epsilon > 0. \tag{3}$$

Let $\lambda \in (0,1)$. We shall prove that for every $\epsilon > 0$ there exists $n_0(\epsilon, \lambda) \in \mathbf{N}$ such that $F_{By_n, Ay_n}(\epsilon) > 1 - \lambda$ for every $n \geq n_0(\epsilon, \lambda)$.

From (2) and (3) we obtain that for every $\epsilon > 0$ and $k \in \mathbf{N}$

$$\begin{aligned} F_{By_n, Ay_n}(\epsilon) \geq t(F_{By_n, A^k x_n}(\tfrac{\epsilon}{3}), t(F_{Ax_n, x_0}(\tfrac{\epsilon}{6(1-k_n)}), \\ F_{A^k x_n, By_n}(\tfrac{\epsilon}{3}))). \end{aligned} \tag{4}$$

Since $By_n \in \overline{\cup_{m\in\mathbf{N}} A^m x_n}$ we conclude that for every $n \in \mathbf{N}$ there exists $m_n \in \mathbf{N}$ such that

$$F_{By_n, A^{m_n} x_n}\left(\frac{\epsilon}{3}\right) > \eta, \text{ for every } n \in \mathbf{N} \tag{5}$$

where $\eta \in (0,1)$ is such that the following implication holds

$$x, y, z \geq \eta(\lambda) \Rightarrow t(x, t(y, z)) > 1 - \lambda. \tag{6}$$

The existence of such an element $\eta$ follows from the continuity of the mapping $t$. Since $AS$ is a probabilistic bounded set it follows that there exists $n_0(\epsilon, \eta(\lambda)) \in \mathbf{N}$ such that

$$F_{Ax_n, x_0}\left(\frac{\epsilon}{6(1-k_n)}\right) > \eta, \text{ for every } \quad n \geq n_0(\epsilon, \eta(\lambda)). \tag{7}$$

From (4), (5), (6) and (7) it follows that

$$F_{By_n,Ay_n}(\epsilon) \geq t(\eta, t(\eta,\eta)) > 1-\lambda, \text{ for every } \quad n \geq n_o(\epsilon, \eta(\lambda)).$$

Hence $\lim_{n\to\infty} F_{By_n,Ay_n}(\epsilon) = 1$, for every $\epsilon > 0$.

We shall prove that the set $\overline{\{y_n; n \in \mathbf{N}\}}$ is compact. Using the assumption that $A$ is $(\beta, B)$ - densifying it is enough to prove that

$$(8) \qquad \beta_{B[\{y_n;n\in\mathbf{N}\}]}(u) = \beta_{A[\{y_n;n\in\mathbf{N}\}]}(u)$$

for every $u > 0$.

In order to prove (8) we shall prove (9) and (10) where

$$(9) \qquad \beta_{B[\{y_n;n\in\mathbf{N}\}]}(u) \leq \beta_{A[\{y_n;n\in\mathbf{N}\}]}(u)$$

for every $u \in \mathbf{R}^+$ and

$$(10) \qquad \beta_{A[\{y_n;n\in\mathbf{N}\}]}(u) \geq \beta_{B[\{y_n;n\in\mathbf{N}\}]}(u)$$

for every $u \in \mathbf{R}^+$.

In order to prove (9) we shall prove that for every $u > 0$ and $s \in (0, u)$

$$(11) \qquad \beta_{B[\{y_n;n\in\mathbf{N}\}]}(u-s) \leq \beta_{A[\{y_n;n\in\mathbf{N}\}]}(u).$$

Since $\beta$ is left continuous (11) implies (9). We can suppose that

$$\beta_{B[\{y_n;n\in\mathbf{N}\}]}(u-s) > 0,$$

since in the oposite case (11) holds. If we prove that

$$(12) \qquad 0 < r < \beta_{B[\{y_n;n\in\mathbf{N}\}]}(u-s) \Rightarrow r \leq \beta_{A[\{y_n;n\in\mathbf{N}\}]}(u),$$

then (11) holds.

From the inequality $r < \beta_{B[\{y_n;n\in\mathbf{N}\}]}(u-s)$ it follows the existence of a finite subset $A_f \subseteq S$ such that

$$\tilde{F}_{B[\{y_n;n\in\mathbf{N}\}],A_f}(u-s) > r.$$

Hence, for every $n \in \mathbf{N}$ there exists $z_n \in A_f$ such that

$$F_{By_n,z_n}(u-s) > r.$$

Let $\delta_1 \in (0, r)$. From the continuity of $t$ and the relation $t(1, r) = r$ it follows the existence of $\delta_2 \in (0, 1)$ such that $1 \geq h > 1 - \delta_2$ implies $t(h, r) > r - \delta_1$. Let $n(s, \delta_2) \in \mathbf{N}$ be such that $F_{By_n, Ay_n}(\frac{s}{2}) > 1 - \delta_2$, for every $k > n(s, \delta_2)$. Then

$$F_{Ay_n, z_n}(u - \frac{s}{2}) \geq t(F_{Ay_n, By_n}(\frac{s}{2}), F_{By_n, z_n}(u - s)) \geq$$

$$t(F_{Ay_n, By_n}(\frac{s}{2}), r) > r - \delta_1$$

and so $\beta_{A[\{y_n; n \in \mathbf{N}\}]}(u) \geq r$ and (12) is proved. This proves (9). Similarly, we can prove (10) and so we have that for every $u > 0$ (8) holds. Since $A$ is $(\beta, B)$ densifying we obtain that the set $\overline{\{y_n; n \in \mathbf{N}\}}$ is compact. Let $\lim_{k \to \infty} y_{n_k} = y$. Then from $\lim_{n \to \infty} By_{n_k} - Ay_{n_k} = 0$ and the continuity of $B$ and $A$ it follows that $By = Ay = y$.

**Corollary 1.** *Let $(S, \mathcal{F}, t)$ be a Menger space with a convex structure $W$ and continuous $T$-norm $t$, $A$ and $B$ continuous, commutative mappings from $S$ into $S$ such that $AS$ is probabilistic bounded subset of $BS$, $x_0 \in S$ and $B$ $(W, x_0)$ - convex so that*

$$F_{Ax, Ay}(\epsilon) \geq F_{Bx, By}(\epsilon), \text{ for every } x, y \in S \text{ and every } \epsilon \in \mathbf{R}^+.$$

*If there exists a compact set $M \subseteq S$ such that $BM$ is an attractor for $A$ then there exists $x \in S$ such that $x = Ax = Bx$.*

*Proof.* It is obvious that $A$ is $(\beta, B)$ - densifying on $M$ since $M$ is a compact set and so all the conditions of the Theorem are satisfied.

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**REZIME**

O TAČKAMA KOINCIDENCIJE U VEROVATNOSNIM METRIČKIM PROSTORIMA SA KONVEKSNOM STRUKTUROM

U ovom radu je dokazana teorema o postojanju tačke koincidencije u verovatnosnim metričkim prostorima sa konveksnom strukturom. Teorema je uopštenje teoreme 3 iz rada [6].

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