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ON RICCI–RECURRENT SEMI–DECOMPOSABLE RIEMANNIAN SPACES

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Abstract

The paper treats the problem of Ricci-recurrent semi–decomposable (pseudo) Riemannian spaces; this is generalization of the results for Ricci–recurrent decomposable spaces, due to Patterson. Since the 1st classification gives the properties of just one component, several additional analytic conditions are given for the other component. If the space is decomposable, then the all additional conditions are satisfied.

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1. Introduction

A Riemannian space is said to be semi–decomposable if there exists a coordinate system in which the metric of the space can be expressed in this way:

(1) $$as^2 = g_{ij}dx^i dx^j = g_{ab}dx^a dx^b + \sigma a_{\alpha\beta}dx^\alpha dx^\beta$$

g_{ab} and σ depend only on coordinates x^a $(a = 1, \ldots, q)$ and $a_{\alpha\beta}$ depend only on coordinates x^α $(\alpha = q+1, \ldots, n)$. Indices i, j take all the values from 1 to n.

Directly from expression (1), one can conclude the a locally semi-decomposable space V_n consists of two mutually orthogonal subspaces V_q and V_{n-q}; the space V_q is totally geodesic and the space V_{n-q} is umbilical.

The kind of problem and the technique of proofs are not based on the fact that the metric of the space V_n is strictly positively definite. The following results will be valid for the space with an indefinite metric (a pseudo-Riemannian space).

Let us consider a semi-decomposable Riemannian space and its coordinate system which enables expression (1). In such a coordinate system, only these Christoffel symbols are not equal to zero:

$$(2)\qquad \Gamma^a_{bc} = \Gamma^a_{(1)bc}, \Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{(2)\beta\gamma}, \Gamma^\alpha_{a\beta} = \frac{1}{2\sigma}\sigma_{,a}\delta^\alpha_\beta, \Gamma^a_{\beta\gamma} = -\frac{1}{2}g^{ab}\sigma_{,b}a_{\beta\gamma}$$

$\Gamma^a_{(1)bc}$ denotes a Christoffel symbol of the subspace V_q, with respect to metric g_{ab}; $\Gamma^\alpha_{(2)\beta\gamma}$ is a Christoffel symbol with respect to metric $a_{\alpha\beta}$; a comma in front of a index denotes a covariant derivative with respect to metric (1) (in the case of a scalar function σ, it is equal to its partial derivative).

Now, we shall calculate the components of the Rieman-Christoffel curvature tensor for the space V_n. From (2), we can get that just the following components of the curvature tensor

$$(3)\qquad \begin{cases} (a)\ R_{abcd} &= R_{(1)abcd} \\ (b)\ R_{\alpha\beta\gamma\delta} &= \sigma R_{(2)\alpha\beta\gamma\delta} + \frac{1}{4}\Delta_1\sigma(a_{\alpha\gamma}a_{\beta\delta} - a_{\alpha\delta}a_{\beta\gamma}) \\ (c)\ R_{\alpha ab\beta} &= \sigma T_{ab}a_{\alpha\beta} \end{cases}$$

are not equal to zero, where

$$(4)\qquad \begin{cases} \Delta_1\sigma = g^{ab}\sigma_{,a}\sigma_{,b} \\ T_{ab} = -\frac{1}{2\sigma}[\sigma_{,a,b} - \frac{1}{2\sigma}\sigma, a\sigma, b] \end{cases}$$

The space V_n is said to be decomposable if in (1), $\sigma = 1$. Then, there locally holds $V_n = V_q \times V_{n-q}$.

A space is said to be Ricci-recurrent if its Ricci tensor is recurrent, i.e. if it satisfies the relation

$$(5)\qquad R_{ij,k} = æ_k R_{ij}$$

where $(æ_k)$ is a vector field and $(æ_k \neq 0 \neq R_{ij}$. We choose for our consideration such a non-trivially Ricci-recurrent space. We also suppose that

R, the scalar curvature of our Ricci–recurrent space V_n, does not vanish. Then, according to [4], we know that $(æ_k)$ will be a gradient vector field and, punctually, $æ_k = \frac{\partial \ln |R|}{\partial x^k}$.

Generally, the class of Ricci–recurrent spaces is much wider than the class of spaces with a recurrent curvature tensor (recurrent spaces ([3], [5])). Also, the class of semi–decomposable spaces is much wider that the class of decomposable spaces. The intention of this paper is to investigate semi-decomposable Ricci–recurrent spaces and to prove a classification–type theorem, analogous in some sense to the one which has been proved in paper [3]. If, in comparison to that theorem, we loose some nice properties, it is the sequel to the wideness of the class of Ricci–recurrent spaces. Also, we are going to generalize the result of Patterson ([2]) for Ricci–recurrent decomposable spaces:

If a Riemannian Ricci–recurrent space is locally a product of Riemannian spaces V_q and V_{n-q}, then one of them is an Einstein space with a vanishing scalar curvature and the other one is Ricci–recurrent. Conversely, the product of an Einstein space with a vanishing scalar curvature and a Ricci–recurrent Riemannian space is a Ricci–recurrent space.

The theorem which we are going to prove will be a generalization of this result.

Using (3), we can easily calculate the components of the Ricci tensor for a semi–decomposable space:

$$(6) \qquad \begin{cases} (i) & R_{ab} = g^{cd}R_{cabd} + g^{\alpha\delta}R_{\alpha ab\delta} = R_{(1)ab} + (n-q)T_{ab} \\ (ii) & R_{\alpha\beta} = g^{\gamma\delta}R_{\gamma\alpha\beta\delta} + g^{ab}R_{a\alpha\beta b} = \\ & \quad = R_{(2)\alpha\beta} + \frac{\Delta_1\sigma}{4\sigma}(q+1-n)a_{\alpha\beta} + \sigma T_{ab}g^{ab}a_{\alpha\beta} \end{cases}$$

From the metric form (1), it is straightforward that all the other components of the Ricci tensor vanish. $R_{(1)ab}$ denotes the Ricci tensor of the metric g_{ab}, and $R_{(2)\alpha\beta}$ denotes the Ricci tensor of the metric $a_{\alpha\beta}$.

Suppose, now, that our semi–decomposable space V_n is Ricci–recurrent, with a non–vanishing scalar curvature.

Differentiating relations (6), we can obtain the following relations for their covariant derivatives:

$$
(7)\quad \begin{cases} (i) & R_{ab,k} = R_{(1)ab\cdot k} + (n-q)T_{ab\cdot k} \\ (ii) & R_{ab,\alpha} = 0 \\ (iii) & R_{\alpha\beta,k} = [\frac{\partial}{\partial x^k}(\frac{\Delta_1\sigma}{4\sigma})r + \sigma_{,k}T_{ab}g^{ab} + \sigma T_{ab\cdot k}g^{ab}]a_{\alpha\beta} \\ (iv) & R_{\alpha\beta;\zeta} = R_{(2)\alpha\beta;\zeta} \end{cases}
$$

The symbol $_{,k}$ denotes a covariant derivative with respect to the metric (1); the symbol k denotes a covariant derivative with respect to the metric g_{ab}; α denotes a covariant derivative with respect to the metric $a_{\alpha\beta}$; r stands for the integer $q+1-n$.

From relations (6) and (7) and by the fact of the Ricci–recurence of the space, we can get the following relations:

$$
(8)\quad \begin{cases} (i) & R_{(1)ab\cdot k} = æ_k R_{(1)ab} + æ_k(n-q)T_{ab} - (n-q)T_{ab\cdot k} \\ (ii) & R_{ab,\alpha} = æ_\alpha R_{ab} = 0 \\ (iii) & æ_k R_{(2)\alpha\beta} = [\frac{\partial}{\partial x^k}(\frac{\Delta_1\sigma}{4\sigma})r + \sigma_k T_{ab}g^{ab} + \sigma T_{ab\cdot k}g^{ab} - \\ & -æ_k(\frac{\Delta_1\sigma}{4\sigma}r + \sigma T_{ab}g^{ab})]\cdot a_{\alpha\beta} \\ (iv) & R_{(2)\alpha\beta;\zeta} = æ_\zeta R_{(2)\alpha\beta} + æ_\zeta(\frac{\Delta_1\sigma}{4\sigma} + \sigma T_{ab}g^{ab})a_{\alpha\beta} \end{cases}
$$

We are going to prove the following

Theorem 1. *If a semi–decomposable Riemannian space V_n with a non–vanishing scalar curvature R is non–trivially Ricci–recurrent, then the space V_{n-q} is an Einstein space.*

It is straightforward from (8) (iii) that the Ricci tensor of the space V_{n-q} is proportional to its metric tensor $a_{\alpha\beta}$ if $æ_k \neq 0$. But, this condition cannot be satisfield. Here, we shall make an analysis on this case.

From (8) (iii), we have two possibilities:
I $æ_\alpha = 0$ or
II $R_{ab} = 0$.
We shall investigate the behaviour of Ricci tensors and scalar curvatures of the spaces V_{n-q} and V_q in both of these cases.

I Since $æ_\alpha = 0$, for any α, then $æ_k \neq 0$ for at least one value of k ($k = 1, \dots, q$) and, from (8) (iii), V_{n-q} will be an Einstein space. Besides this, from (8) (iv), it will be Ricci parallel. Then, there holds

$$
\frac{\partial R_{(2)}}{\partial x^\zeta} = R_{(2);\zeta} = (R_{(2)\alpha\beta}a^{\alpha\beta})_{;\zeta} = a^{\alpha\beta}R_{\alpha\beta;\zeta} = 0
$$

for the scalar curvature of the space V_{n-q}. Consequently, V_{n-q} is an Einstein space with a constant scalar curvature.

II If $æ_\alpha \neq 0$ for at least one value of α, then $R_{ab} = 0$ and

$$R_{(1)ab} = -(n-q)T_{ab} \tag{9}$$

for any value of a, b $(a, b = 1, \ldots, q)$ and the components of the Ricci tensor of the space V_q depend only on function σ.

Suppose, now $æ_k = 0$ for any k. Then, both sides of (8) (iii) vanish and we cannot make any conclusion about V_{n-q}. But $æ_k = \frac{1}{R}\frac{\partial R}{\partial x^k}$. We can calculate then scalar curvature R of V_n:

$$R = R_{ab}g^{ab} + \frac{1}{\sigma}R_{\alpha\beta}a^{\alpha\beta} = \frac{1}{\sigma}R_{(2)} + (n-q)[\frac{1}{\sigma}(\frac{\Delta_1\sigma}{4\sigma} + \sigma T_{ab}g^{ab}] \tag{10}$$

Differentiating (10) partially and taking into account $\frac{\partial R}{\partial x^k}$, we get

$$-\frac{\sigma,k}{\sigma^2}R_{(2)} + (n-q)\frac{\partial}{\partial x^k}[\frac{1}{\sigma}(\frac{\Delta_1\sigma}{4\sigma}r + \sigma T_{ab}g^{ab})] = 0 \tag{11}$$

Since $R_{(2)}$ is an object of inner geometry of V_{n-q}, it depends entirely on variables x^α. In expression (11) every member depends on function σ, and, consequently, on variables x^a. By these facts, $R_{(2)}$ will also depend on variables x^a. The multiplier $\sigma_{,k}$ cannot vanish, except in the case when the space V_n reduces to a decomposable space. From (10) and (11),

$$\frac{\partial R}{\partial x^\zeta} = \frac{1}{\sigma}\frac{\partial R_{(2)}}{\partial x^\zeta} = R æ_\zeta = 0.$$

for any value of ζ. Since $æ_k \neq 0$ for any k by our assumption, then the scalar curvature of the space V_n is an absolute constant and space V_n will be Ricci–parallel i.e. trivially Ricci-recurrent.

Since we have supposed that the space V_n is non–trivially Ricci–recurrent, then at least one $æ_k$ does not vanish and V_{n-q} is an Einstein space.

We have proved Theorem 1. Also, we have proved

Theorem 2. *If a semi–decomposable Riemannian space V_n with a non–vanishing scalar curvature is non–trivially Ricci–recurrent, then*

I V_{n-q} is an Einstein space with a constant scalar curvature and we have no information about V_q; or

II V_{n-q} is an Einstein space and the Ricci tensor of V_q depends only on function σ.

Our theorems 1 and 2 give, in some sense, a generalization of Patterson's results, but, it is too far from a classification-type theorem, as in [3]. We had too fewle assumptions and this resulted in too little information about V_q. Since the space V_{n-q} is an Einstein space with a constant scalar curvature or just an Einstein space (which are both generalizations of an Einstein space with a vanishing scalar curvature), it is reasonable to expect that V_q should be a Ricci–recurrent space.
Besides, in Patterson's theorem V_q and V_{n-q} have equal treatment. They cannot have equal treatment here, but as an Einstein space, V_{n-q} is also a Ricci–recurrent space, then we have to investigate when V_q will be an Einstein space, or a space with a constant scalar curvature or some other generalization of an Einstein space with a vanishing scalar curvature. With this intention, we shah have to include additional conditions, partial diferentional equations including the function σ, recurrence vector $æ_i$, metric tensor and, possibly, scalar curvatures of V_q and V_{n-q}.

In Theorem 2, we have two possible cases (I and II). We shall keep this division in our further investigation.

I $æ_\alpha = 0$ for all α. Then V_{n-q} is an Einstein space with a constant scalar curvature.

According to (8), (iii) will be Ricci–recurrent if and only if $T_{ab\cdot k} = æ_k T_{ab}$, which can be expressed analytically.

$$(12) \quad 2[-\sigma^2\sigma_{,a,b,k} + \sigma\sigma_{,k}\sigma_{,a}\sigma_{,b} + \sigma\sigma_{,a,k}\sigma_{,b} + \sigma\sigma_{,a}\sigma_{,b,k} - 2\sigma_{,a}\sigma_{,b}\sigma_{,k}] =$$

$$[-2\sigma^2\sigma_{,a,b} + \sigma\sigma_{,a}\sigma_{,b}]æ_k$$

Since the symbol $,a$ denotes a covariant derivative, (12) is a partial differential equation of the third order and there appear Christoffel symbols of the metric g_{ab} and their partial derivatives.

Suppose, now, that V_q is a space with a constant scalar curvature. Then, contracting (8) (i) by g^{ab}, we get

$$0 = æ_k R_{(1)} + æ_k(n-q)T_{ab}g^{ab} - (n-q)T_{ab\cdot k}g^{ab}$$

where, according to our assumption, $R_{(1)}$ is a constant. Then,

$$T_{ab\cdot k}g^{ab} = \frac{æ_k R_{(1)}}{(n-q)} + æ_k T_{ab}g^{ab}$$

Involving a differentional operator

$$\Delta_2\sigma = g^{ab}\sigma_{,a,b}, \tag{13}$$

we know that

$$T_{ab}g^{ab} = \frac{-2\sigma\Delta_2\sigma + \Delta_1\sigma}{4\sigma^2}$$

and

$$T_{ab\cdot k}g^{ab} = (\frac{-2\sigma\Delta_2\sigma + \Delta_1\sigma}{4\sigma^2}), kg^{ab}$$

Then, if we want V_q to have a constant scalar curvature, function σ has to satisfy

$$\frac{\partial}{\partial x^k}(\frac{-2\sigma\Delta_2\sigma + \Delta_1\sigma}{4\sigma^2})g^{ab} = \frac{æ_k R(1)}{(n-q)} + æ_k\frac{-2\sigma\Delta_2\sigma + \Delta_1\sigma}{4\sigma^2}, \tag{14}$$

where $R_{(1)}$ is that constant curvature.

Finally (for case I), we shall investigate the possibility for space V_q to be an Einstein space. The condition for this will be much less informative than conditions (12) and (14) because we have no relation for the tensor $R_{(1)ab}$, but just for its covariant derivative. Suppose that $R_{(1)ab} = fg_{ab}$, for some non–vanishing function f. Then,

$$T_{ab\cdot k} = æ_k T_{ab} - v_k g_{ab},$$

where v_k is a vector field which is a gradient if and only if $æ_k\frac{\partial f}{\partial x^j} = æ_j\frac{\partial f}{\partial x^k}$.

This condition can be expressed analytically

$$(15) \quad 2[-\sigma^2\sigma_{,a,b,k} + \sigma\sigma_{,k}\sigma_{,a}\sigma_{,b} + \sigma\sigma_{,a,k}\sigma_{,b} + \sigma\sigma_{,a}\sigma_{,b,k} - 2\sigma_{,a}\sigma_{,b}\sigma_{,k}] =$$

$$= \cdot[-2\sigma^2\sigma_{,a,b} + \sigma\sigma_{,a}\sigma_{,b}]æ_k + 4\sigma^3 v_k g_{ab}$$

The components of the vector field v_k can be expressed this way

$$v_k = \frac{\partial f}{\partial x^k} - fæ_k$$

II In this case, V_{n-q} is an Einstein space and the tensor depends only on function σ. Moreover,

$$R_{(1)ab} = -(n-q)T_{ab}$$

It is obvious that the space will be a Ricci–recurrent space if and only if

$$T_{ab\cdot k} = æ_k T_{ab}$$

i.e. if condition (12) is fulfilled.

Suppose, now, that V_q is a space with constant scalar curvature. Then,

$$R_{(1)} = -(n-q)T_{ab}g^{ab} = (n-q)\frac{2\sigma\Delta_2\sigma - \Delta_1\sigma}{4\sigma^2}$$

V_q will be a space with the constant scalar curvature $R_{(1)}$ if and only if

(16) $$2\sigma\Delta_2\sigma - \Delta_1\sigma = \frac{\sigma^2 R_{(1)}}{n-q}$$

V_q will be a space with a vanishing scalar curvature if $\Delta_2\sigma = \frac{\Delta_1\sigma}{2\sigma}$.

Suppose, now, that V_q is an Einstein space. Then $T_{ab} = fg^{ab}$, where f is an arbitrary function. But,

$$T_{ab} = \frac{-2\sigma\sigma_{,a,b} + \sigma_{,a}\sigma_{,b}}{4\sigma^2}$$

Let us consider the vector field

(17) $$\lambda_{,a} = -\frac{\sigma_{,a}}{2\sigma}$$

Then,

$$\lambda_{,a,b} = \frac{-2\sigma\sigma_{,a,b} + 2\sigma_{,a}\sigma_{,b}}{4\sigma^2}$$

and

(18) $$\lambda_{,a,b} - \lambda_{,a}\lambda_{,b} = T_{ab} = fg_{ab}$$

Condition (18) means that the vector field is a concircular vector filed.

Here is one more group of questions to be answered: if we transform the space V_q conformally, can we get an Einstein space, or a space with a constant scalar curvature, or a Ricci–flat space (which is an Einstein space with a vanishing scalar curvature)?

Suppose that

$$g_{ab} = \sigma\bar{g}_{ab}$$

is a conformal transformation of the space V_q to a space with the metric tensor $\bar{g}_{ab}$; σ is the function from (1). If this transformation is conformal, then σ must be a strictly positive C^∞ function, and

$$\bar{g}_{ab} = \frac{1}{\sigma} g^{ab} \tag{19}$$

According to [1], we have

$$\bar{R}_{ab} = R_{(1)ab} + (q-2)\lambda_{ab} + g_{ab}[\Delta_2\lambda + (q-1)\Delta_1\lambda], \tag{20}$$

where

$$\lambda = \frac{1}{\sqrt{\sigma}},\ \lambda_{,a} = -\frac{\sigma_{,a}}{2\sigma},\ \lambda_{,a,b} = \frac{-2\sigma\sigma_{,a,b} + 2\sigma_{,a}\sigma_{,b}}{4\sigma^2} \tag{21}$$

$$\lambda_{ab} = \lambda_{,a,b} - \lambda_{,a}\lambda_{,b} = T_{ab},\ \Delta_1\lambda = \frac{\Delta_1\sigma}{4\sigma^2},\ \Delta_2\lambda = \frac{-\sigma\Delta_2\sigma + \Delta_1\sigma}{4\sigma^2}$$

and

$$\bar{R} = q(q-1)\frac{\Delta_1\sigma}{4\sigma} - \frac{5q-2n-4}{4}\Delta_2\sigma \tag{22}$$

If we want space (with the metric tensor) to be an Einstein space or a Ricci–flat space, then it is clear from (20) and (21) that should be proportional to , and, consequently, the gradient of function is a concircular vector field, which happens if and only if is an Einstein space.

If the space has a constant scalar curvature, then, from (22)

$$\Delta_2\sigma = \frac{q(q-1)}{5q-2n-4}\frac{\Delta_1\sigma}{\sigma} - \frac{\bar{R}}{5q-2n-4} \tag{23}$$

Now,we can extend our Theorem 2 to two classification–type theorems:

Theorem 3. *If a semi–decomposable Riemannian space V_n with a non–vanishing scalar curvature is non–trivially Ricci–recurrent and $æ_\alpha = 0$, then V_{n-q} is an Einstein space with a constant scalar curvature and V_q is*

a) a Ricci–recurrent space if condition (12) is satisfied

b) a space with a constant scalar curvature if condition (14) is satisfied

c) an Einstein space if condition (15) is satisfied.

Theorem 4. *If a semi–decomposable Riemannian space V_q with a non–vanishing scalar curvature is non–trivially Ricci–recurrent and if $R_{ab} = 0$, then V_{n-q} is an Einstein space and V_q is* *a) a Ricci–recurrent space if condition (12) is satisfied*

b) a space with a constant scalar curvature if (16) is satisfied

c) an Einstein space if σ is a function with positive values and if vector filed (17) is concircular

d) in conformal correspondence by function σ to an Einstein space or to a Ricci–flat space if and only if it is an Einstein space and, then, this correspondence is a concircular one

e) in conformal correspondence by function σ to a space with a constant scalar curvature if condition (23) is satisfied.

Theorems 3 and 4 do not give a full classification, like the theorem from [3]. Function σ, the metric tensor and scalar curvature can not satisfy any of the analytycal conditions which have been mentioned above and V_q can have other properties, expressed analytically in a different way. But no geometrical properties towards curvature and the Ricci tensor are immanent to space V_q, under the assumption of the Ricci–recurrence of V_n. We have made a classification which is a logical generalization of latterson's result. The autor has to notify that all the special conditions (except for those which are related to conformal mapping) are satisfied in the case of a decomposable (pseudo) Riemannian space.

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REZIME

O RICCI - REKURENTNIM POLUDEKOMPONOVANIM RIMANOVIM PROSTORIMA

U radu se razmatra problem Riči–rekurentnih poludekomponovanih Rimanovih prostora; uopšten je rezultat koji se odnosi na Riči–rekurentne dekomponovane prostore, a koji je dobio Paterson. Pošto klasifikacija daje osobine samo jedne komponente ovakvog prostora, uvedeni su dodatni analitički uslovi koji daju osobine druge komponente. Dekomponovani prostor takodje zadovoljava sve ove naknadne analitičke uslove.

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