Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 21, 2 (1991), 61-74

Review of Research
Faculty of Science
Matematics Series

COMPARATIVE CHARACTERISATION OF GENERALIZED LAGRANGE AND GENERALIZED HAMILTON SPACES I

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Abstract

This paper presents a short exposition of generalized Lagrange and Hamilton spaces. The purpose is to compare the construction of generalized Lagrange and Hamilton spaces and to show their duality. In this paper the generalization of metric and connection is given and coefficients for recurrent case are determined.

AMS Mathematics Subject Classification (1980): 53B40, 53C60
Key words and phrases: Generalized Hamilton spaces, generalized Lagrange spaces.

1. Introduction

It should be remarked here that the introduction of the notion of Lagrange geometry belongs to J. Kern [10], but the whole theory of Lagrange and Hamilton geometry was developed by Romanian geometers, led by R. Miron [13], [14], [15], [16], [17] and others [1], [2], [16], [17].

It is obvious that this geometry is more general than the Finslerian and recently more authors have introduced the generalized theory of connection in k–Hamilton and k–Lagrangian spaces [2], [3], [4], [5], [11], [18].

R. Miron, T. Kawaguchi and others also studied Lagrange, Hamilton and Finsler geometry in view of its application in theoretical physics [6], [7], [8], [12], [19], [20].

The duality of the Lagrange and Hamilton geometry has been perceived a long time ago and applied in mechanics in the study of the equation of motion of the arbitrary point in the space

$$\left(\frac{\partial}{\partial t}\frac{\partial L(x^i,y^a)}{\partial y^a}-\frac{\partial L(x^i,y^a)}{\partial x^i}=0\right),$$

where $L(x^i,y^a)$ is the Lagrangian, and dual to this, of the equation for total mechanical energy in which the Hamiltonian $H(q^i,p_a)$ figures.

From the above equation it can be seen that it represents the Euler-Lagrange equation for $L(x^i,y^a)$, while the solutions are the extreme curves of a corresponding variation calculation.

The paper studies the duality of generalized Lagrange and Hamilton spaces and on the basis of this duality, new results for generalized Lagrange spaces are given, by using the results obtained for generalized Hamilton spaces in paper [3].

As the generalization of Lagrange and Hamilton spaces will be introduced in this paper, the notion of such spaces is given by the following definition:

Definition 1.1. *The generalized recurrent*

Lagrange space	*Hamilton spaces*
$L^{n+m}=(E,M,N,G,\nabla,T,\lambda)$	$H^{n+m}=(E^*,M,\overset{*}{N},\overset{*}{G},\overset{*}{\nabla},\overset{*}{T},\overset{*}{\lambda})$

is the $n+m$ dimensional differentiable manifold for which the following conditions hold:

a) The coordinates of some point in

$$E \qquad\qquad E^*$$

are given by (2.1).

b) The allowable coordinate transformations are prescribed by (2.2) and (2.3) in which the functions

$$M^{a'}_a, M^a_{a'} \qquad\qquad M^a_{a'}, M^{a'}_a$$

are involved.

c) The adapted bases in

$$T(E) \qquad\qquad T(E^*)$$

are constructed by arbitrary, but fixed nonlinear connection

$$N \qquad\qquad \overset{*}{N}$$

for which (2.4) holds.

d) The metric tensor

$$G \qquad\qquad \overset{*}{G}$$

is determined by (3.1), where the non-diagonal blocks are non-zero matrices.

e) The generalized connection

$$\nabla \qquad\qquad \overset{*}{\nabla}$$

is determined by (4.1).

f) The arbitrary torsion tensor

$$T \qquad\qquad \overset{*}{T}$$

which has 8 kinds of components is given by (4.4).

g) The recurrent connections which depend on

$$\lambda \qquad\qquad \overset{*}{\lambda}$$

are determined by (4.5).

2. Adapted bases in L^{n+m} and H^{n+m}

Let M be an n-dimensional and $n+m$ dimensional differentiable manifold and let

$$\eta = (E, \pi, M) \qquad\qquad \eta^* = (E^*, \pi^*, M)$$

be vector bundles and dual vector bundels such that

$$\pi(E) = M \qquad\qquad \pi^*(E^*) = M.$$

The total spaces are

$$E \qquad\qquad E^*.$$

The differential structures

$$(U, \phi, R^{n+m}) \qquad\qquad (U, \phi^*, R^{n+m})$$

are vector charts of the vector bundles

$$\eta \qquad \text{and} \qquad \eta^*.$$

Hence the canonical coordinates on

$$\pi^{-1}(U) \qquad\qquad (\pi^*)^{-1}(U)$$

are

$$(2.1)\quad \begin{array}{ll} (x^1, \dots, x^n, y^1, \dots, y^m) = (x^i, y^a) & (q^1, \dots, q^n, p_1, \dots, p_m) = (q^i, p_a) \\ i, j, k, l = 1, 2, \dots, n & i, j, k, l = 1, \dots, n \\ a, b, c, d = 1, 2, \dots, m & a, b, c, d = 1, \dots, m. \end{array}$$

Transformation map on

$$E \qquad \text{and} \qquad E^*$$

are

$$(2.2)\quad \begin{array}{ll} \text{(a) } x^{i'} = x^{i'}(x^1, \dots, x^n) & \text{(a) } q^{i'} = q^{i'}(q^1, \dots, q^n) \\ \text{(b) } y^{a'} = M_a^{a'}(x^1, \dots, x^n) y^a = M_a^{a'}(x^i) y^a & \text{(b) } p_{a'} = M_{a'}^a(q^1, \dots, q^n) p_a \\ \operatorname{rank} \left[\frac{\partial x^{i'}}{\partial x^i}\right] = n, & \operatorname{rank} \left[\frac{\partial q^{i'}}{\partial q^i}\right] = n \\ \operatorname{rank} \left[\frac{\partial y^{a'}}{\partial y^a}\right] = \operatorname{rank} M_a^{a'} = m & \operatorname{rank} \left[\frac{\partial p_{a'}}{\partial p_a}\right] = m. \end{array}$$

The inverse transformations are:

$$(2.3)\quad \begin{array}{ll} \text{(a) } x^i = x^i(x^{1'}, \dots, x^{n'}) & \text{(a) } q^i = q^i(q^{1'}, \dots, q^{n'}) \\ \text{(b) } y^a = M_{a'}^a(x^{1'}, \dots, x^{n'}) y^{a'} & \text{(b) } p_a = M_a^{a'}(q^{1'}, \dots, q^{n'}) p_{a'}, \end{array}$$

where

$$M^a_{a'}M^{a'}_a = \delta^a_b \qquad\qquad M^a_{b'}M^{a'}_a = \delta^{a'}_{b'}.$$

The transformation laws show that

$$y^i \qquad\qquad p_i$$

can be considered as

contravariant $\qquad\qquad$ **covariant**

vectors.

They are Lionville vectors. $\qquad\qquad$ They are Lionville 1-forms.

The local natural bases of the tangent spaces

$$T_u(E), (u \in E) \qquad\qquad T_{u^*}(E^*), (u^* \in E^*)$$

are

$$\{\partial_i, \partial_a\} \qquad\qquad \{\partial_i, \partial^a\}$$

$$\partial_a = \frac{\partial}{\partial y^a} = M^{a'}_a(x^i)\partial_{a'} \qquad\qquad \partial^a = \frac{\partial}{\partial p_a} = M^a_{a'}(q^i)\partial^{a'}$$

$$\partial_i = \frac{\partial}{\partial x^i} = \frac{\partial x^{i'}}{\partial x^i}\partial_{i'} + (\partial_i M^{a'}_b(x))y^b\partial_{a'} \qquad\qquad \partial_i = \frac{\partial}{\partial q^i} = \frac{\partial q^{i'}}{\partial q^i}\partial_{i'} + \frac{\partial M^a_{a'}}{\partial q^i}p_a\partial^{a'}.$$

Their dual bases are

$$\{dx^i, dy^a\} \qquad\qquad \{dq^i, dp_a\}.$$

The nonlinear connection on

$$E \qquad\qquad E^*$$

is distribution

$$N : u \in E \to N_u \subset T_u(E) \qquad\qquad \overset{*}{N}: u^* \in E^* \to \overset{*}{N}_{u^*} \subset T^*_u(E^*)$$

which is supplementary to the distribution V, [11] i.e.

$$\text{(2.4)} \quad T_u(E) = N_u \oplus V_u \qquad\qquad T_{u^*}(E^*) = \overset{*}{N}_{u^*} \oplus \overset{*}{V}_{u^*}$$

$$\forall u \in E \qquad\qquad \forall u^* \in E^*.$$

They are locally determined by

$$\delta_i = \partial_i - N^a_i\partial_a \qquad\qquad \overset{*}{\delta}_i = \partial_i + \overset{*}{N}_{ai}\,\partial^a.$$

The local bases adapted to the decompositions in (2.4) are

$$\{\delta_i, \partial_a\} \qquad\qquad \{\overset{*}{\delta}_i, \partial^a\}.$$

The dual bases are

$$\{dx^i, \delta y^a\} \qquad\qquad \{dq^i, \overset{*}{\delta}\, p_a\}$$

where

$$\delta y^a = dy^a + N^a_j dx^j \qquad\qquad \overset{*}{\delta}\, p_a = dp_a - \overset{*}{N}_{aj}\, dq^j$$

and

$$< \delta_i, dx^j >= \delta^j_i, < \delta_i, \delta y^a >= 0\ ; \quad < dq^i, \overset{*}{\delta}_j >= \delta^i_j, < \overset{*}{\delta}\, p_a, \overset{*}{\delta}_j >= 0$$
$$< \partial_a, dx^j >= 0, < \partial_a, \delta y^b >= \delta^b_a\ ; \quad < dq^i, \partial^a >= 0, < \delta_{\overset{*}{p_a}}, \partial^b >= \delta^b_a$$

hold.

It is easy to prove that on

$$\{\delta_i, \partial_a\} \qquad\qquad \{\overset{*}{\delta}_i, \partial^a\}$$

$$\delta_{i'} = \delta_i \frac{\partial x^i}{\partial x^{i'}}, \partial_{a'} = \frac{\partial y^a}{\partial y^{a'}} \partial_a \qquad\qquad \overset{*}{\delta}_{i'} = \frac{\partial q^i}{\partial q^{i'}}\, \overset{*}{\delta}_i, \partial^{a'} = \frac{\partial p_a}{\partial p_{a'}} \partial^a$$

and on

$$\{dx^i, \delta y^a\} \qquad\qquad \{dq^i, \delta p_a\}\ .$$

$$dx^{i'} = \frac{\partial x^{i'}}{\partial x^i} dx^i, \delta y^{a'} = \frac{\partial y^{a'}}{\partial y^a} \delta y^a \qquad\qquad dq^{i'} = \frac{\partial q^{i'}}{\partial q^i} dq^i, \overset{*}{\delta}\, p_{a'} = \frac{\partial p_{a'}}{\partial p_a}\, \overset{*}{\delta}\, p_a.$$

The subspace of

$$T(E) \qquad\qquad T(E^*)$$

spaned by

$$\{\delta_i\} \qquad\qquad \{\overset{*}{\delta}_i\}$$

will be denoted by

$$T_H(E) \text{ or } E_H \qquad\qquad T_H(E^*) \text{ or } E^*_H$$

and the subspace spaned by

$$\{\partial_a\} \qquad\qquad \{\partial^a\}$$

will be denoted by

$$T_V(E) \text{ or } E_V \qquad\qquad T_V(E^*) \text{ or } E^*_V.$$

$$T(E) = T_H(E) \oplus T_V(E) = E_H \oplus E_V \quad T(E^*) = T_H(E^*) \oplus T_V(E^*) = E^*_H \oplus E^*_V$$
$$\dim T_H(E) = n, \dim T_V(E) = m \qquad \dim T_H(E^*) = n, \dim T_V(E^*) = m.$$

3. Covariant metric tensor in

$$T^*(E) \otimes T^*(E) \qquad \text{and} \qquad T^*(E^*) \otimes T^*(E^*)$$

Riemannian metrical structure on

$$E \qquad\qquad E^*$$

with respect to the base

$$\{dx^i, \delta y^a\} \qquad\qquad \{dq^i, \overset{*}{\delta}\, p_a\}$$

is given by

(3.1) $G =$ $\qquad\qquad$ $\overset{*}{G} =$

$$g_{ij} dx^i \otimes dx^j + g_{ib} dx^i \otimes \delta y^b + \qquad g_{ij} dq^i \otimes dq^j + g_i{}^b dq^i \otimes \overset{*}{\delta}\, p_b +$$
$$g_{aj} \delta y^a \otimes dx^j + g_{ab} \delta y^a \otimes \delta y^b \qquad g^a_{\ j} \overset{*}{\delta}\, p_a \otimes dq^j + g^{ab} \overset{*}{\delta}\, p_a \otimes \overset{*}{\delta}\, p_b.$$

Coordinates of metric tensor in the new coordinate system

$$(x^{i'}, y^{a'}) \qquad\qquad (q^{i'}, p_{a'})$$

are transforming in the following way

$$g_{i'j'} = g_{ij} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} \qquad g_{i'j'} = g_{ij} \frac{\partial q^i}{\partial q^{i'}} \frac{\partial q^j}{\partial q^{j'}}$$
$$g_{i'b'} = g_{ib} \frac{\partial x^i}{\partial x^{i'}} M^b_{b'} \qquad g_{i'}{}^{b'} = g_i{}^b \frac{\partial q^i}{\partial q^{i'}} M^{b'}_b$$
$$g_{a'j'} = g_{aj} M^a_{a'} \frac{\partial x^j}{\partial x^{j'}} \qquad g^{a'}_{\ i'} = g^a_{\ i} M^{a'}_a \frac{\partial q^i}{\partial q^{i'}}$$
$$g_{a'b'} = g_{ab} M^a_{a'} M^b_{b'} \qquad g^{a'b'} = g^{ab} M^{a'}_a M^{b'}_b.$$

We shall suppose that the metric tensor is symmetric, positive definite tensor field of rank $m+n$. ($g_i{}^b = g^b_{\ i} = g^b_i$)).

The change of the position of the indices (lowering and raising) by the metric tensor is given in the following way: ([3], [4]).

For the field

$$X = X^i\delta_i + X^a\partial_a \qquad \overset{*}{X} = X^i \overset{*}{\delta}_i + X_a\partial^a$$

we have

$$X_i = g_{ij}X^j + g_{ib}X^b \qquad X_i = g_{ij}X^j + g_i{}^aX_a$$
$$X_a = g_{aj}X^j + g_{ab}X^b \qquad X^a = g^a_iX^i + g^{ab}X_b$$

and for the field

$$\omega = \omega_i dx^i + \omega_a\delta y^a \qquad \overset{*}{\omega} = \omega_i dq^i + \omega^a \overset{*}{\delta} p_a$$

we get

$$\omega^i = g^{ij}\omega_j + g^{ia}\omega_a \qquad \omega^i = g^{ij}\omega_j + g^i{}_a\omega^a$$
$$\omega^a = g^{aj}\omega_j + g^{ab}\omega_b \qquad \omega_a = g_a{}^j\omega_j + g_{ab}\omega^b$$

where

$$g_{ij}g^{jk} + g_{ib}g^{bk} = \delta^k_i \qquad g_{ij}g^{jk} + g^a_ig^k_a = \delta^k_i$$
$$g_{aj}g^{jb} + g_{ac}g^{cb} = \delta^b_a \qquad g^a_jg^j_b + g^{ac}g_{cb} = \delta^a_b$$
$$g_{ij}g^{jb} + g_{ic}g^{cb} = 0 \qquad g_{ij}g^j_b + g_i{}^cg_{cb} = 0$$
$$g_{aj}g^{jk} + g_{ab}g^{bk} = 0 \qquad g^a_jg^{jk} + g^{ab}g_b{}^k = 0.$$

From these formulae we can see that

$$X_j, X_a, \omega^i, \omega^a \qquad X_i, X^a, \omega^i, \omega_a$$

are transforming in the following way:

$$X_{j'} = X_j\frac{\partial x^j}{\partial x^{j'}} \qquad X_{j'} = X_j\frac{\partial q^j}{\partial q^{j'}}$$
$$X_{a'} = X_aM^a_{a'} \qquad X^{a'} = X^aM^{a'}_a$$
$$\omega^{i'} = \omega^i\frac{\partial x^{i'}}{\partial x^i} \qquad \omega^{i'} = \omega^i\frac{\partial q^{i'}}{\partial q^i}$$
$$\omega^{a'} = \omega^aM^{a'}_a \qquad \omega_{a'} = \omega_aM^a_{a'}.$$

As the special case of Definition 1.1 the usual notion of Lagrange and Hamilton spaces is given by:

Definition 3.1.

A Lagrange space is a pair $(M, L(x^i, y^a))$ where M is n-dim differentiable manifold, $L(x^i, y^a)$ is a regular Lagrangian defined over $E = T(M)$.

A Hamilton space is a pair $(M, H(q^i, p_a))$ where M is n-dim differentiable manifold, $H(q^i, p_a)$ is a regular Hamiltonian defined over $E^* = T^*(M)$.

The Riemannian metrical structure on

E	E^*

is given by

$$G = g_{ij}(x^i, y^a)dx^i \otimes dx^j + \qquad \overset{*}{G} = g_{ij}(q^i, p_a)dq^i \otimes dq^j +$$

$$+ g_{ab}(x^i, y^a)\delta y^a \otimes \delta y^b \qquad + g^{ab}\, \overset{*}{\delta}\, p_a \otimes \overset{*}{\delta}\, p_b,$$

where

$$g_{ij}(x^i, y^a) = g_{ij}(x^i) \qquad g_{ij}(q^i, p_a) = g_{ij}(q^i)$$
$$g_{ab} = \tfrac{1}{2}\partial_a \partial_b L(x^i, y^a) \qquad g^{ab} = \tfrac{1}{2}\partial^a \partial^b H(q^i, p_a)$$

and $L(x^i, y^a)$ is a Lagrange function | $H(q^i, p_a)$ is a Hamilton function

The metric tensors defined by (2.1) are the generalization of the Lagrange and Hamilton metric, because they do not consist only of diagonal blocks.

4. Linear connection

The generalized connection $\nabla(\overset{*}{\nabla})$ has the following properties

$$\forall U \in T(E), \forall X \in T(E) \qquad \forall U \in T(E^*), \forall X \in T(E^*)$$
$$\Rightarrow \nabla_X U \in T(E) \qquad \Rightarrow \overset{*}{\nabla}_X U \in T(E^*)$$

which follows from

Definition 4.1.

$$(4.1) \quad \nabla_{\delta_i} \delta_j = F_j{}^k{}_i \delta_k + F_j{}^c{}_i \partial_c \qquad \overset{*}{\nabla}_{\overset{*}{\delta}_i} \overset{*}{\delta}_j = F_j{}^k{}_i\, \overset{*}{\delta}_k + F_{jci} \partial^c$$
$$\nabla_{\delta_j} \partial_a = F_a{}^k{}_j \delta_k + F_a{}^c{}_j \partial_c \qquad \overset{*}{\nabla}_{\overset{*}{\delta}_j} \partial^a = F^{ak}{}_j\, \overset{*}{\delta}_k + F^a_{cj} \partial^c$$
$$\nabla_{\partial_a} \delta_j = C_j{}^k{}_a \delta_k + C_j{}^c{}_a \partial_c \qquad \overset{*}{\nabla}_{\partial^a} \overset{*}{\delta}_j = C_j{}^{ka}\, \overset{*}{\delta}_k + C_{jc}{}^a \partial^c$$
$$\nabla_{\partial_a} \partial_b = C_b{}^k{}_a \delta_k + C_b{}^c{}_a \partial_c \qquad \overset{*}{\nabla}_{\partial^a} \partial^b = C^{bka}\, \overset{*}{\delta}_k + C^b{}_c{}^a \partial^c.$$

From (4.1) it is obvious that $\nabla_X(\overset{*}{\nabla}_X)$ applied on Y^H or Y^∇ will have both horizontal and vertical components. This connection is the generalization of the Miron's connection. Miron's d-connection is the linear connection $\nabla(\overset{*}{\nabla})$ with the following properties [13], [14]:

$$\forall U \in T_H(E), \forall X \in T(E) \Rightarrow \nabla_X U \in T_H(E), \quad \forall Z \in T_V(E), \forall X \in T(E) \Rightarrow \nabla_X Z \in T_V(E)$$

$$\forall U \in T_H(E^*), \forall X \in T(E^*) \Rightarrow \overset{*}{\nabla}_X U \in T_H(E^*) \quad \forall Z \in T_V(E^*), \forall X \in T(E^*) \Rightarrow \overset{*}{\nabla}_X Z \in T_V(E^*).$$

In the Lagrange and Hamilton spaces the d-connection is introduced in [1], [2], [9], [16].

From (4.1) the Miron's d-connection is obtained when the following relations are valid:

$$(4.2) \qquad F_{j\,i}^{\,c} = 0 \quad F_{a\,j}^{\,k} = 0 \quad C_{j\,a}^{\,c} = 0 \quad C_{b\,a}^{\,k} = 0 \qquad F_{jci} = 0 \quad F^{ak}_{\;\;j} = 0 \quad C_{jc}^{\;\;a} = 0 \quad C^{bka} = 0.$$

The torsion tensor is defined by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \qquad \overset{*}{T}(X,Y) = \overset{*}{\nabla}_X Y - \overset{*}{\nabla}_Y X - [X,Y]$$
(4.1)
where

$$X,Y \in T(E) \qquad X,Y \in T(E^*).$$

Theorem 4.1. *The torsion tensor for the connection $\nabla(\overset{*}{\nabla})$ has the form:*

$$T(X,Y) = T_{HHH} + T_{HHV} + T_{HVH} + T_{VHH} + T_{HVV} + T_{VHV} + T_{VVH} + T_{VVV}$$

$$\overset{*}{T}(X,Y) = T^{HHH} + T^{HHV} + T^{HVH} + T^{VHH} + T^{HVV} + T^{VHV} + T^{VVH} + T^{VVV},$$

where for instance

$$T_{VVH} = [T(H_V, Y_V)]_H \qquad T^{VVH} = [T(X^V, Y^V)]^H.$$

The components of the torsion tensor are

4.4 (a) $T_{HHH} = (F_{j\ i}^{\ k} - F_{i\ j}^{\ k})X^iY^j\delta_k$ (a) $T^{HHH} = (F_{j\ i}^{\ k} - F_{i\ j}^{\ k})X^iY^j \overset{*}{\delta}_k$

(b) $T_{HHV} = (F_{j\ i}^{\ a} - F_{i\ j}^{\ a} + \delta_i N_j^a - \delta_j N_i^a)X^iY^j\delta_a$ (b) $T^{HHV} = (F_{jci} - F_{icj} + \delta_i \overset{*}{N}_{cj} - \delta_j \overset{*}{N}_{ci})X^iY^j\partial^c$

(c) $T_{HVH} = (F_{b\ i}^{\ k} - C_{i\ b}^{\ k})X^iY^b\delta_k$ (c) $T^{HVH} = (F^{bk}_{\ \ i} - C_i^{\ kb})X^iY_b \overset{*}{\delta}_k$

(d) $T_{HVV} = (F_{b\ i}^{\ c} - C_{b\ i}^{\ c} - \partial_b N_i^c)X^iY^b\partial_c$ (d) $T^{HVV} = (F^{b}_{\ ci} - C_{ic}^{\ \ b} - \partial^b \overset{*}{N}_{ci})X^iY_b\partial^c$

(e) $T_{VHH} = -T_{HVH}$ (e) $T^{VHH} = -T^{HVH}$

(f) $T_{VHV} = -T_{HVV}$ (f) $T^{VHV} = -T^{HVV}$

(g) $T_{VVH} = (C_{b\ a}^{\ k} - C_{a\ b}^{\ k})X^aY^b\delta_k$ (g) $T^{VVH} = (C^{bka} - C^{akb})X_aY_b \overset{*}{\delta}_k$

(h) $T_{VVV} = (C_{b\ a}^{\ c} - C_{a\ b}^{\ c})X^aY^b\partial_c$ (h) $T^{VVV} = (C^{b\ a}_{\ c} - C^{a\ b}_{\ c})X_aY_b\partial^c$

see [5] see [3].

It is obvious from (4.4) that for the generalized connection, $T(X,Y)$ has 3 components more, then the torsion for Miron's d-connection. For the special case (4.2), the components of the torsion tensor given in (4.4), reduce to the components of the torsion for d-connection, in Lagrange and Hamilton space.

Another generalization is obtained, when the existence of such 1-form fields

$$\lambda(x^i, y^a) = \lambda_k dx^k + \lambda_c \delta y^c \qquad \overset{*}{\lambda}(q^i, p_a) = \overset{*}{\lambda}_k\, dq^k + \overset{*}{\lambda}{}^c \overset{*}{\delta} p_c$$

is supposed, so that the connection coefficients satisfy the following conditions

(4.5) (a) $g_{ij|k} = \lambda_k g_{ij}$ (a) $g_{ij|k} = \overset{*}{\lambda}_k\, g_{ij}$

(b) $g^{aj}_{\ \ |k} = \lambda_k g_{aj}$ (b) $g_{i\ |k}^{\ a} = \overset{*}{\lambda}_k\, g_i^{\ a}$

(c) $g_{ab|k} = \lambda_k g_{ab}$ (c) $g^{ab}_{\ \ |k} = \overset{*}{\lambda}_k\, g^{ab}$

(d) $g_{ij}|_c = \lambda_c g_{ij}$ (d) $g_{ij}|^c = \overset{*}{\lambda}{}^c\, g_{ij}$

(e) $g_{aj}|_c = \lambda_c g_{aj}$ (e) $g_i^{\ a}|^c = \overset{*}{\lambda}{}^c\, g_i^{\ a}$

(f) $g_{ab}|_c = \lambda_c g_{ab}$ (f) $g^{ab}|^c = \overset{*}{\lambda}{}^c\, g^{ab}$.

The connections obtained under former conditions are called the connections in the recurrent Lagrange (Hamilton) spaces.

The connection in the Lagrange and Hamilton spaces which satisfies the conditions (4.5), when

$$\lambda(x^i, y^a) = 0 \qquad \overset{*}{\lambda}(q^i, p_a) = 0$$

is called d-metrical connection [15], [17].

Introducing the notation

$$(a)\begin{bmatrix} F_{hik} \\ F_{hak} \end{bmatrix} = \begin{bmatrix} g_{ij} & g_{ib} \\ g_{aj} & g_{ab} \end{bmatrix} \begin{bmatrix} F_h{}^j{}_k \\ F_h{}^b{}_k \end{bmatrix} \qquad \begin{bmatrix} F_{ijk} \\ F_i{}^a{}_k \end{bmatrix} = \begin{bmatrix} g_{hj} & g^d_j \\ g^a_h & g^{da} \end{bmatrix} \begin{bmatrix} F_i{}^h{}_k \\ F_{idk} \end{bmatrix}$$

$$\begin{bmatrix} F_h{}^j{}_k \\ F_h{}^b{}_k \end{bmatrix} = \begin{bmatrix} g^{ij} & g^{aj} \\ g^{ib} & g^{ab} \end{bmatrix} \begin{bmatrix} F_{hik} \\ F_{hak} \end{bmatrix} \qquad \begin{bmatrix} F_i{}^h{}_k \\ F_{idk} \end{bmatrix} = \begin{bmatrix} g^{jh} & g_a{}^h \\ g^j{}_d & g_{ad} \end{bmatrix} \begin{bmatrix} F_{ijk} \\ F_i{}^a{}_k \end{bmatrix}$$

$$(b)\begin{bmatrix} F_{aij} \\ F_{abj} \end{bmatrix} = \begin{bmatrix} g_{ik} & g_{ic} \\ g_{bk} & g_{bc} \end{bmatrix} \begin{bmatrix} F_a{}^k{}_j \\ F_a{}^c{}_j \end{bmatrix} \qquad \begin{bmatrix} F^a{}_{ik} \\ F^{ab}{}_k \end{bmatrix} = \begin{bmatrix} g_{ih} & g_i{}^d \\ g^b{}_h & g^{bd} \end{bmatrix} \begin{bmatrix} F^{ah}{}_k \\ F^a{}_{dk} \end{bmatrix}$$

$$\begin{bmatrix} F_a{}^k{}_j \\ F_a{}^c{}_j \end{bmatrix} = \begin{bmatrix} g^{ik} & g^{kb} \\ g^{ic} & g^{bc} \end{bmatrix} \begin{bmatrix} F_{aij} \\ F_{abj} \end{bmatrix} \qquad \begin{bmatrix} F^{ah}{}_k \\ F^a{}_{dk} \end{bmatrix} = \begin{bmatrix} g^{ih} & g_b{}^h \\ g^i{}_d & g_{bd} \end{bmatrix} \begin{bmatrix} F^a{}_{ik} \\ F^{ab}{}_k \end{bmatrix}$$

$$(c)\begin{bmatrix} C_{jia} \\ C_{jca} \end{bmatrix} = \begin{bmatrix} g_{ik} & g_{ib} \\ g_{ck} & g_{bc} \end{bmatrix} \begin{bmatrix} C_j{}^k{}_a \\ C_j{}^b{}_a \end{bmatrix} \qquad \begin{bmatrix} C_{ij}{}^c \\ C_i{}^{ac} \end{bmatrix} = \begin{bmatrix} g_{hj} & g^d_j \\ g^a_h & g^{da} \end{bmatrix} \begin{bmatrix} C_i{}^{hc} \\ C_{id}{}^c \end{bmatrix}$$

$$\begin{bmatrix} C_j{}^k{}_a \\ C_j{}^b{}_a \end{bmatrix} = \begin{bmatrix} g^{ik} & g^{kc} \\ g^{bi} & g^{bc} \end{bmatrix} \begin{bmatrix} C_{jia} \\ C_{jca} \end{bmatrix} \qquad \begin{bmatrix} C_i{}^{hc} \\ C_{id}{}^c \end{bmatrix} = \begin{bmatrix} g^{jh} & g_a{}^h \\ g^j{}_d & g_{ad} \end{bmatrix} \begin{bmatrix} C_{ij}{}^c \\ C_i{}^{ac} \end{bmatrix}$$

$$(d)\begin{bmatrix} C_{bia} \\ C_{bca} \end{bmatrix} = \begin{bmatrix} g_{ik} & g_{id} \\ g_{ck} & g_{cd} \end{bmatrix} \begin{bmatrix} C_b{}^k{}_a \\ C_b{}^d{}_a \end{bmatrix} \qquad \begin{bmatrix} C^a{}_i{}^c \\ C^{abc} \end{bmatrix} = \begin{bmatrix} g_{ih} & g_i{}^d \\ g^b{}_h & g^{bd} \end{bmatrix} \begin{bmatrix} C^{ahc} \\ C^a{}_d{}^c \end{bmatrix}$$

$$\begin{bmatrix} C_b{}^k{}_a \\ C_b{}^d{}_a \end{bmatrix} = \begin{bmatrix} g^{ik} & g^{kc} \\ g^{di} & g^{dc} \end{bmatrix} \begin{bmatrix} C_{bia} \\ C_{bca} \end{bmatrix} \qquad \begin{bmatrix} C^{ahc} \\ C^a{}_d{}^c \end{bmatrix} = \begin{bmatrix} g^{ih} & g_b{}^h \\ g^i{}_d & g^{bd} \end{bmatrix} \begin{bmatrix} C^a{}_i{}^c \\ C^{abc} \end{bmatrix}$$

and from formulae (4.5) we obtain the recurrent connection in the following form:

(4.6) (a) $2F_{ijk} = (\delta_k g_{ij} + \delta_i g_{ih} - \delta_j g_{ki}) - -(\lambda_k g_{ij} + \lambda_i g_{jh} - \lambda_j g_{ki}) + \overline{F}_{kij} + \overline{F}_{ikj} - \overline{F}_{kji}$,

(a) $2F_{ijk} = (\overset{*}{\delta}_k\, g_{ij} + \overset{*}{\delta}_i\, g_{jk} - \overset{*}{\delta}_j\, g_{ki}) - (\overset{*}{\lambda}_k\, g_{ij} + \overset{*}{\lambda}_i\, g_{jk} - \overset{*}{\lambda}_j\, g_{ki}) + (\overline{F}_{kij} + \overline{F}_{ikj} - \overline{F}_{kji})$,

where

$$\overline{F}_{kij} = F_{kij} - F_{jik} = = g_{ih}T^h_{kj} + g_{ka}T_k{}^a{}_j \qquad \overline{F}_{kij} = F_{kij} - F_{jik} = = g_{ih}T_k{}^h{}_j + g_i{}^d T_{kdj}$$

and so on.

The complete list of connection coeffitients is given in [5] and [3].

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REZIME

KOMPARATIVNA KARAKTERIZACIJA GENERALISANIH LAGRANGEOVIH I GENERALISANIH HAMILTONOVIH PROSTORA I

Ovaj rad predstavlja kratak pregled generalisanih Lagrangeovih i Hamiltonovih prostora. Cilj je da se uporedi konstrukcija generalisanih Lagrangeovih i Hamiltonovih prostora i da se pokaže njihova dualnost. U ovom radu je data generalizacija metrike i koneksije i odredjeni su koeficijenti koneksije za rekurentni slučaj.

Received by the editors March 27, 1991