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SOLVING A NONLOCAL SINGULARLY PERTURBED PROBLEM BY SPLINES IN TENSION

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Abstract

A numerical method for a singularly perturbed linear nonlocal problem is constructed. An exponential spline difference scheme on non-equidistant mesh is applied. The estimate of the form $\min(H, \sqrt{\epsilon})$, where ϵ is the perturbation parameter and H is the maximal mesh step width, is obtained. Numerical experiments which demonstrate the effectiveness of the method are presented.

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1. Introduction

Let us consider the following singular perturbed nonlocal problem

$$Ly = -\epsilon y'' + b(x)y = f(x), \quad x \in I = [0,1],$$

$$y(0) = 0$$

$$y(1) = \sum_{i=1}^{m} c_i y(s_i) + d, \tag{1}$$

$$0 < s_1 < s_2 < \dots < s_m < 1, \quad d, c_i \in \mathbf{R}, \quad i = 1(1)m,$$

where $\epsilon \in (0, \epsilon_0), \epsilon_0 \ll 1$, is a small perturbation parameter. The functions b and f are given and we assume that

$$b, f \in C^2(I), b(x) \geq \beta^2 > 0, \quad x \in I, \tag{2}$$

$$-\infty < \sum_{i=1}^{m} c_i \omega_0(s_i) < 1, \tag{3}$$

where

$$\omega_0(x) = \frac{\exp(\alpha_0 x) - \exp(-\alpha_0 x)}{\exp(\alpha_0) - \exp(-\alpha_0)}, \quad \alpha_0 = \frac{\beta}{\epsilon}.$$

General "boundary value problems" with nonlocal conditions are defined in [7]. Nonlocal conditions which connect the values of the unknown solution at the boundary with values in the interior are defined and studied in [2]. The problems with integral nonlocal conditions can be met in studying heat transsfer problems ([5], [6]). Discretization of these problems leads to the nonlocal conditions of the form (1). The same mathematical models arise in problems of semiconductors ([3], [15]), in problems of hydromechanics ([14]) and some others physical phenomena.

Numerical treatment of problem (1)-(2) was considered in [3], where finite elements on an equdistant mesh was applied and the second order uniform convergence was obtained. The same problem with $m = 1$ was also studied in [12]. The second order uniform convergence was obtained. In [9], by using results from [10] the fourth order uniform convergence is obtained. The basic idea in the papers mentioned is to consider the problem (1) as two problems with Dirichlet's boundary conditions ([4]) which are examined in [1], [8], [10-13], [15-24].

Our aim is to slove (1)-(3) numerically by using a tension spline difference scheme on a non-equidistant mesh. It is known that under conditions (2) and (3) problem (1) has a unique solution which exhibits some exponential properties in the boundary layers (at $x = 0$ and at $x = 1$). Because of that in [17], [18], [10], [20] the solution of problem (1) with Dirichlet's boundary conditions is sought in the form of the exponential spline. The corresponding difference scheme converge with respeet to the maximal mesh step width H uniformly in ϵ. Some of them converge with respect to ϵ, i.e. they have an optimal error estimate in the sense of [8]. The scheme consider in [10] is on a nonuniform mesh and the estimate has the form $min\{H, \sqrt{\epsilon}\}$ on the regular mesh. In this paper we use that scheme for solving problem (1).

The nonuniform mesh enables us to use the points s_i, $i = 1(1)m$, as mesh points. In that way dificulties due to nonlocal conditions are eliminated. We solve only one system of linear equations instead two as in [9] and [4]. The order of the convergence with respect to H obtained in [9] is better (the maximal for tridiagonal sschemes [7]), but here a convergence with respect to ϵ is achieved.

Throughout the paper M denotes any positive constant that may take different values in different formulas, but are always independent of ϵ and of discretization mesh.

2. The Numerical Method

Suppose the mesh Δ,

$$\Delta = \{x_i : 0 = x_0 < x_1 < ... < x_n < x_{n+1} = 1\},$$

be formed so that $s_i \in \Delta$, $i = 1(1)m$, and $H/h \leq M$, where $H = \max\{h_i : i = 0(1)n\}$, $h = \min\{h_i : i = 0(1)n\}, h_i = x_{i+1} - x_i$, $i = 0(1)n$. Then, using the scheme from [19] we obtain the following system of $n+1$ equations with $n+1$ unknowns $u_j, j = 1(1)n+1$,

$$\text{(4)} \qquad \begin{cases} Ru_j = Qf_j, \quad j = 1, 2, ..., n, \\ u_0 = 0, \quad u_{n+1} = \sum_{i=1}^{m} c_i u_i + d, \end{cases}$$

where

$$Ru_j = r_j^- u_{j-1} + r_j^c u_j + r_j^+ u_{j+1}, \quad Qf_j = q_j^- f_{j-1} + q_j^c f_j + q_j^+ u_{j+1},$$

$$r_j^- = \frac{-1}{h_{j-1}} - \left(\frac{\rho_j}{sh\mu_{j-1}} - \frac{1}{h_{j-1}}\right)\frac{p_{j-1}}{\epsilon p_j^2}, \quad r_j^+ = \frac{-1}{h_j} - \left(\frac{\rho_j}{sh\mu_j} - \frac{1}{h_j}\right)\frac{p_{j+1}}{\epsilon p_j^2},$$

$$r_j^c = \frac{1}{h_j} + \frac{1}{h_{j-1}} + \left(\rho_j cth\mu_j - \frac{1}{h_j} + \rho_j cth\mu_{j-1} - \frac{1}{h_{j-1}}\right)\frac{p_j}{\epsilon p_j^2},$$

$$q_j^- = \frac{-1}{\epsilon p_j^2}\left(\frac{\rho_j}{sh\mu_{j-1}} - \frac{1}{h_{j-1}}\right), \quad q_j^+ = \frac{-1}{\epsilon p_j^2}\left(\frac{\rho_j}{sh\mu_j} - \frac{1}{h_j}\right),$$

$$q_j^c = \frac{1}{\epsilon p_j^2}\left(\rho_j cth\mu_j - \frac{1}{h_j} + \rho_j cth\mu_{j-1} - \frac{1}{h_{j-1}}\right),$$

$$f_j = f(x_j), \quad p_j = p(x_j), \quad \mu_j = h_j\rho_j,$$

ρ_j are tension parameters.

When it is clear from the context, the subscripts in $r_j^-, r_j^c, r_j^+, q_j^-, q_j^c, q_j^+$ will be omitted.

Tension parameter

$$\rho_j, \quad j = 0(1)n,$$

are chosen according to the following theorem.

Theorem 1. *Let $y \in C^4[0,1]$ be a solution of problem (1). Then*

$$y(x) = v(x) + w(x) + g(x),$$

where

$$v(x) = q_0 v_0(x), \quad w(x) = q_1 w_0(x),$$

$$v_0(x) = \exp(-x\sqrt{p(0)/\epsilon}), \quad w_0(x) = \exp(-(1-x)\sqrt{p(1)/\epsilon}),$$

where q_0 and q_1 are bounded functions of ϵ independent of x and

$$| q^{(i)}(x) | \leq M(1 + \epsilon^{1-i)/2}), \quad i = 0,1,2,3,4.$$

If $p'(0) = p'(1) = 0$ then

$$| q^{(i)}(x) | \leq M(1 + \epsilon^{2-i)/2}), \quad i = 0,1,2,3,4.$$

Proof. The solution of problem (1) can be writen in the form ([4])

$$y(x) = t(x) + \lambda z(x),$$

where $z(x)$ and $t(x)$ are the solutions of the boundary value problems

$$(5) \qquad \begin{cases} -\epsilon z'' + p(x)z = 0, \\ z(0) = 0, \; z(1) = 1, \end{cases}$$

and

$$(6) \qquad \begin{cases} -\epsilon t'' + p(x)t = f(x), \\ t(0) = 0, \; t(1) = 0, \end{cases}$$

respectively. Parameter λ is determined so that $y(x)$ satisfies the boundary condition at $x = 1$,

$$\lambda = (\sum_{i=1}^{m} c_i t(s_i) + d)/(1 - \sum_{i=1}^{m} c_i z(s_i)),$$

$$1-\sum_{i=1}^{m} c_i z_i \neq 0.$$

Since, according to [8] and [23]

$$z(x) = v_z(x) + w_z(x) + g_z(x),$$

$$t(x) = v_t(x) + w_t(x) + g_t(x),$$

where

$$v_z(x) = q_2 v_0(x),\ w_z(x) = q_3 w_0(x),\ v_t(x) = q_4 v_0(x),\ w_t(x) = q_5 w_0(x),$$

(q_i, $i = 2,3,4,5$; are bounded functions of ϵ independent of x) and

$$\mid g_z^{(i)}(x) \mid,\ \mid g_t^{(i)}(x) \mid \leq M(1+\epsilon^{(1-i)/2}),\quad i = 0,1,2,3,4.$$

If $p'(0) = p'(1) = 1$, then

$$\mid g_z^{(i)}(x) \mid,\ \mid g_t^{(i)}(x) \mid \leq M(1+\epsilon^{(2-i)/2}),\quad i = 0,1,2,3,4.$$

Thus, the statement follows from the fact that $\mid \lambda \mid \leq M$. Using that

$$\rho_j = \begin{cases} \sqrt{p(0)/\epsilon} & \text{for } 0 \leq j \leq [n/2], \\ \sqrt{p(1)/\epsilon} & \text{for } [n/2] < j \leq n, \end{cases}$$

the boundary layer functions $v(x)$ and $w(x)$ become the basic functions for the corresponding spline in tension [17], [19].

3. The Convergence of the Method

If we apply scheme (4) to solving problem (5) we obtain the system

$$\text{(7)}\qquad \begin{cases} R\bar{z}_j = Q(Lz)_j, \\ \bar{z}_0 = 0,\quad \bar{z}_{n+1} = 1. \end{cases}$$

The same procedure for problem (6) leads to the system

$$\text{(8)}\qquad \begin{cases} R\bar{t}_j = Q(Lt)_j, \\ \bar{t}_0 = 0,\quad \bar{t}_{n+1} = 0. \end{cases}$$

The solutions $\bar{z}_j, j = 0(1)n+1$, and $\bar{t}_j$, $j = 0(1)n+1$, are the approximative values for z_j, $j = 0(1)n+1$, and t_j, $j = 0(1)n+1$, respectively.

For the error estimate we use the expresion

$$(9) \qquad | z_j - \bar{z}_j | \leq \|A^{-1}\| \max\{| \tau_j(z) | : j = 1,...,n\}.$$

where A is a matrix of system (8) and $\tau_j(z)$ is the local truncation error of the operator R for the function $z(x)$,

$$\tau_j(z) = R(z_j - \bar{z}_j) = Rz_j - Q(Lz)_j.$$

Since (see Theorem 1),

$$(10) \qquad \tau_j(z) = \tau_j(v) + \tau_j(w) + \tau_j(g),$$

we shall separately estimate the errors $\tau_j(v), \tau_j(w)$ and $\tau_j(g)$.

Let $H \leq \sqrt{\epsilon}$. After some standard formal Taylor developments up to the fourth derivatives we have

$$| \tau_j(g) + \tau_j(w) | \leq Mh_j^3/(\epsilon\sqrt{\epsilon}) \quad \text{for} \quad 0 \leq j \leq [n/2].$$

Since, according to the choice of tension parameters, $\tau_j(v) = 0$ we have that

$$(11) \qquad \tau_j(z) | \leq Mh_j^3/(\epsilon\sqrt{\epsilon}) \quad \text{for} \quad 0 \leq j \leq [n/2] \quad \text{when} \quad H \leq \sqrt{\epsilon}.$$

When $H \geq \sqrt{\epsilon}$, expanding up to the first derivative we obtain

$$| \tau_j(g) + \tau_j(w) | \leq M.$$

Since $\tau_j(v) = 0$ we have

$$(12) \qquad | \tau_j(z) | \leq M \quad \text{when} \quad H \geq \sqrt{\epsilon} \quad \text{for} \quad 0 \leq j \leq [n/2].$$

In the similar way we obtain that the above estimates are valid for $[n/2] \leq j \leq n$. Then $\tau_j(w) = 0$.

The norm of the matrix of system (7) is estimated in [19].

$$\|A^{-1}\|_\infty \leq 1/(\min\{| r^c | + | r^- | + | r^+ | : j = 1(1)n\},$$

$$(13) \qquad \|A^{-1}\|_\infty \leq \begin{cases} M\epsilon/h, & \text{for} \quad h \leq \sqrt{\epsilon}, \\ M\sqrt{\epsilon}, & \text{otherwise} . \end{cases}$$

Thus, from (8) - (13) we have that

$$(14)\qquad | z_j - \bar{z}_j | \le \begin{cases} MH^2/\sqrt{\epsilon}, & \text{for } H \le \sqrt{\epsilon}, \\ M\sqrt{\epsilon}, & \text{otherwise} . \end{cases}$$

Similarly, we obtain that

$$(15)\qquad | t_j - \bar{t}_j | \le \begin{cases} MH^2/\sqrt{\epsilon}, & \text{for } H \le \sqrt{\epsilon}, \\ M\sqrt{\epsilon}, & \text{otherwise} . \end{cases}$$

Under supposition

$$1 - \sum_{i=1}^{m} x_i \bar{z}_i \ne 0$$

we have that $u_j = \bar{y}_j + \lambda_h \bar{z}_j$, where

$$\lambda_h = (\sum_{i=1}^{m} c_i \bar{t}_i + d)/(1 - \sum_{i=1}^{m} c_i \bar{z}_i).$$

Since $\bar{t}_j$, $j = 0(1)n+1$, and $\bar{z}_j$, $j = 0(1)n+1$, are the unique solutions of systems (7) and (8), respectively and λ_h is a unique solution of the equation

$$u_{n+1} = \bar{t}_{n+1} + \lambda_h \bar{z}_{n+1} = \sum_{i=1}^{m} c_i(\bar{t}_i + \lambda_h \bar{z}_i) + d,$$

we have that u_j, $j = 1(1)n+1$, is a unique solution of system (4). Further,

$$\lambda - \lambda_h = K^{-1}(\sum_{i=1}^{m}(t_i - \bar{t}_i)(c_i + \sum_{i=1}^{m} c_i z_i) + \sum_{i=1}^{m}(z_i - \bar{z}_i)(dc_i + \sum_{i=1}^{m} c_i t_i))$$

where

$$K = (1 - \sum_{i=1}^{m} c_i z_i)(1 - \sum_{i=1}^{m} c_i \bar{z}_i) \ne 0.$$

Since (theorem 1)

$$| c_i + \sum_{i=1}^{m} c_i z_i | \le M, | K | \ge M \quad \text{and} \quad | c_i + \sum_{i=1}^{m} c_i t_i | \le M,$$

from (14) and (15) we can see that the estimate of the form (15) holds for $| \lambda - \lambda_h |$, i.e.

$$| \lambda - \lambda_h | \le \begin{cases} MH^2/\sqrt{\epsilon}, & \text{for } H \le \sqrt{\epsilon}, \\ M\sqrt{\epsilon}, & \text{otherwise} . \end{cases}$$

Thus the following theorem is proved.

Theorem 2. *Let conditions (2) and (3) be satisfied. Let u_j be the approximation to $y(x_j)$ obtained by solving system (4). Then for sufficiently small H the estimate*

$$| y(x_j) - u_j | \leq MH \min\{H, \sqrt{\epsilon}\}/(H + \sqrt{\epsilon}), \;\; j = 1(1)n+1,$$

holds.

Namely, it is proved in [14] that from condition (3) one can obtain that problem (7) has a unique solution and that

$$1 - \sum_{i=1}^{m} c_i z_i \neq 0.$$

Thus, for sufficiently small H the condition

$$1 - \sum_{i=1}^{m} c_i \bar{z}_i \neq 0$$

will be satisfied.

Systems (7) and (8) are auxiliaary systems for the proof and for practical purpose it is not necessary to solve them.

Theorem 3. *Let the suppositions of Theorem 2 be satisfied. Let $p'(0) = p'(1) = 0$. Then*

$$| y(x_j) - u_j | \leq MH \min\{H, \sqrt{\epsilon}\}, \;\; j = 0(1)n+1.$$

Proof. Proof is very similar to the one of the previous theorem. The derivatives of the function $g(x)$ in this case have a weaker dependence on ϵ which one can see from Theorem 1.

4. The Numerical Examples

In this section we present results of some numerical experiments using the scheme described in previous section. Our example is often used in the literature, see for instance [8], [9], [11], [12], to compare different codes.

We shall consider discretization on discretization mesh $\Delta = I_h \cup \{s_1, s_2, ..., s_m\}$ where

$$I_h = \{x_i = \lambda(t_i) : i = 0, 1, ..., N\},\ t_i = ih,\ h = \frac{1}{N},\ N = 2n_0, n_0 \in \mathbf{N},$$

with mesh generating function [9], [10], [23].

$$\lambda(t) = \begin{cases} \omega(t) = \frac{a\sqrt{\epsilon}t}{q-t}, & t \in [0, \alpha], \\ \omega(\alpha) + \omega'(\alpha)(t - \alpha), & t \in [\alpha, 0.5], \\ 1 - \lambda(1 - t), & t \in [0.5, 1], \end{cases}$$

and

$$q \in (0, 0.5), \quad a\epsilon_0 \leq q.$$

The parameter α is

$$\alpha = \frac{q - \sqrt{ag\sqrt{\epsilon}(1 - 2q + 2a\sqrt{\epsilon})}}{1 + 2a\sqrt{\epsilon}}.$$

From now on we denote the points of the mesh Δ as x_i, $i = 0, 1, ..., n + 1$, $N \leq n \leq N + m$.

The reason for using this mesh is our aim to obtain more mesh points in the region of boundary layers, whose width is $O(\sqrt{\epsilon})$. Other properties of the mesh generating function $\lambda(t)$, which are not important for our analysis in this paper, are given in [9], [10], [23], [24].

We consider the following example

$$-\epsilon y'' + y + \cos^2 \pi x + 2\epsilon\pi^2 \cos 2\pi x = 0, \ x \in [0, 1],$$

$$u(0) = 0, \ u(1) = \sum_{i=1}^{m} c_i u_i(s_i) + d,$$

where for given c_i, $i = 1, 2, ..., m$, the number d is determined so that $u(1) = 0$. This is possible because the exact solution of our problem is in this case known:

$$y(x) = \frac{\exp(-x\sqrt{\epsilon}) + \exp((x - 1)\sqrt{\epsilon})}{1 + \exp(-1/\sqrt{\epsilon})} - \cos^2 \pi x.$$

We denote by E_n the maximum of $| y(x_j) - u_j |$, $j = 0(1)n + 1$. Here $[u_0, u_1, ..., u_{n+1}]^T$ is the corresponding numerical solution of system (4).

The tables present the error E_n and the experimental order of convergence

$$Ord = \frac{\log E_n - \log E_{n_2}}{\log n - \log n_2},$$

where n_2 depends on $2N$ and m in the same way as n does on N and m. Different values of ϵ and N are considered with $m = 4$.

i	S_i	G_i
1	0.9000	0.03
2	0.1000	0.20
3	0.2000	5.00
4	0.5000	0.09

and $a = 1,\ q = 0.4$.

Table 2 gives number n_ϵ of the mesh points in $[0, \epsilon) \cup (1 - \epsilon, 1]$ for $\epsilon = 2^{-20},\ m = 4,\ a = 1,\ q = 0.4$.

Table 1.

$n(N) \setminus \epsilon$	2^{-4}	2^{-10}	2^{-20}	2^{-32}	2^{-64}	
7(4)	4.197(-1)	5.504(-2)	2.15(-3)	3.398(-5)	5.172(-10)	E_n
	-	-	-	-	-	Ord
11(8)	7.200(-2)	4.713(-2)	2.151(-3)	3.398(-5)	5.172(-10)	E_n
	3.901	3.432	0.000	0.000	0.000	Ord
19(16)	1.423(-2)	2.101(-2)	1.987(-3)	3.340(-5)	5.115(-10)	E_n
	2.966	1.479	0.145	0.027	0.020	Ord
35(32)	3.097(-3)	6.454(-3)	1.255(-3)	2.130(-5)	3.311(-10)	E_n
	2.496	1.932	0.753	0.736	0.712	Ord
67(64)	7.018(-4)	1.753(-3)	6.048(-4)	1.129(-5)	1.759(-10)	E_n
	2.287	2.007	1.124	0.978	0.974	Ord
131(128)	1.605(-4)	4.992(-4)	2.963(-4)	5.722(-6)	8.931(-11)	E_n
	2.201	1.873	1.064	1.013	1.011	Ord
259(256)	4.006(-5)	1.473(-4)	1.392(-4)	2.869(-6)	4.482(-11)	E_n
	2.036	1.790	1.109	1.013	1.011	Ord
515(512)	9.983(-6)	3.713(-5)	6.026(-5)	1.434(-6)	2.243(-11)	E_n
	2.022	2.005	1.218	1.010	1.007	Ord
1027(1024)	2.486(-6)	9.470(-6)	2.197(-5)	7.146(-7)	1.122(-11)	E_n
	2.014	1.980	1.462	1.009	1.002	Ord
2051(2048)	6.208(-7)	2.408(-6)	6.483(-6)	3.550(-7)	5.611(-12)	E_n
	2.006	1.980	1.764	1.011	1.000	Ord

Table 2.

N	n	n_ϵ
4	7	1
8	11	3
16	19	7
32	35	13
64	67	25
128	131	51
256	259	103
512	515	205
1024	1027	409
2048	2051	819

Table 1 presents the results for numerical solution obtained by our method. These results confirm our estimate obtained in Theorem 2. It is easy to see that the convergence order drops from two to one when ϵ decreases. This is because the order has been calculated with respect to H. Furthermore, it can be seen that the approximation error decreases together with ϵ, which indicates the convergence with respect to the small parameter. This is especially clear from the upper part of Table 1, which corresponds to greater values of H.

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REZIME

REŠAVANJE NELOKALNOG SINGULARNOG PERTURBACIONOG PROBLEMA PRIMENOM SPLAJNOVA U TENZIJI

Konstruisan je numerički metod za singularno perturbacioni problem. Diferencna šema je izvedena pomoću eksponencijalnog splajna na neekvidistantnoj mreži. Dobijena je ocena greške oblika $min(H, \sqrt{\epsilon}$, gde je ϵ perturbacioni parametar a H maksimalni korak mreže. Numerički primeri ilustruju efikasnost metoda.

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