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ON A UNIFORM NUMERICAL METHOD FOR A NONLOCAL PROBLEM

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Abstract

A numerical method for a singularly perturbed linear nonlocal problem is considered. A classical difference scheme on a special non-equidistant mesh, which is dense in the boundary layers, is applied. The second order convergence uniform in the perturbation parameter is proved. A numerical example is given.

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1. Introduction

Let us consider the following singularly perturbed nonlocal problem

$$-\epsilon^2 u'' + b(x)u = f(x),\ x \in I = [0,1],$$

$$u(0) = 0,$$

$$u(1) = \sum_{i=1}^{m} c_i u(s_i) + d,\ 0 < s_1 < s_2 < \dots < s_m < 1,$$

$$\sum_{i=1}^{m} | c_i |< 1,$$

where $\epsilon \in (0,\epsilon_0), \epsilon_0 \ll 1$, is a small perturbation parameter. The functions b and f are given and we assume

$$b, f \in C^k(I),\ \ b(x) \geq \beta^2 > 0,\ \ x \in I$$

with some $k \in \mathcal{N}$.

Problem (1)-(2) can be met as a model equation for some physical phenomene, see [2], [8] for instance. The numerical treatment of problem (1) - (2) was consider in [3], where finite elements method on an equidistant mesh was applied and second order uniform convergence was obtained. The same problem with $m = 1$ was studied in [6], also.

Our aim is to solve (1)-(2) numerically by using a classical difference scheme on a special non-equidistant mesh, which is dense in the layers of the solution to (1)-(2), located at $x = 0$ and $x = 1$. The same approach applied to the numerical solution of the two-point boundary value problem can be found in [1], [5], [7], [9], [10], [11]. In this paper we shall use a mesh generating function from [11] and give a scheme which has the second order convergence uniform in ϵ. Numerical examples which demonstrate the effectiveness of the method are presented.

Throughout the paper M denotes any positive constant that may take different values in different formulas, which are always independent of ϵ and of the discretization mesh.

2. The numerical method

Let I_h be the discretization mesh with the points:

$$x_i = \lambda(t_i),\ t_i = ih,\ \ i = 0,1,...,n,\ \ h = \frac{1}{n},$$

$$n = 2n_0,\ \ n_0 \in \mathcal{N},\ \ n_0 \geq 2,$$

where

$$\lambda(t) = \begin{cases} \frac{a\epsilon t}{(0.5+p\sqrt{a\epsilon}-t)^p}, & t \in [0,0.5], \\ 1-\lambda(1-t), & t \in [0.5,1]. \end{cases}$$

Here $a > 0$ and $p > 0$ are some constants, independent of ϵ. We have $\lambda^{(i)}(t) > 0,\ i = 1,2,\ t \in [0,0.5]$ and $\lambda(t) = O(\epsilon)$ as long as $0.5 - t \geq \delta$, some

$\delta > 0$. The density is increased when a is decreased. The same is true for p as long as $a2^p\epsilon > 1$. More details about these mesh generating functions can be found in [10], [11].

Since $\lambda(t)$ is a monotone increasing function, for each $s_i \in (0,1), i = 1,2,...,m$, we obtain a unique $t_{si} \in (0,1)$, such that $s_i = \lambda(t_{si})$. Now, we shall discretize the problem (1)-2) on the mesh $I_s = I_h \cup \{s_1,...,s_m\}$. Let us denote the points of the mesh I_s by z_i, $i = 0,1,...,N$, $n \le N \le n+m$, in such a way that

$$O = z_0 < z_1 < ... < z_{N-1} < z_N = 1.$$

By using the same scheme as in [11], we obtain

$$w_0 = 0,$$

$$a_1(i)w_{i-1} + a_0(i)w_i + a_2(i)w_{i+1} + b_i w_i = f(z_i), \; i = 1,...,N-1,$$

$$w_n = \sum_{i=1}^{N-1} C_i w_i + d,$$

where $w_h = [w_0, w_1, ..., w_N]^T \in \Re^N$, $(w_i = w_{hi})$, is a mesh function on I_s,

$$a_1(i) = \frac{-2\epsilon^2}{h_i(h_i + h_{i+1})}, \; a_2(i) + \frac{-2\epsilon^2}{h_{i+1}(h_i + h_{i+1})}, \; a_0(i) = \frac{2\epsilon^2}{h_i h_{i+1}},$$

$$C_i = \begin{cases} c_i, & \text{if } z_i = s_i \\ 0, & else \end{cases}$$

$$h_i = z_i - z_{i-1} \quad \text{and} \quad b_i = b(z_i), \; i = 1,2,...,N.$$

Theorem 1. *Let (3) hold with $k = 2$ and let the discrete problem (4) be given on the mesh I_s. Then the problem (1)-(2) has a unique solution u, problem (4) has a unique solution w_h, and it holds that*

$$\|w_h - u_h\|_\infty \le Mh^2,$$

where $u_h = [u(z_0), u(z_1), ..., u(z_N)]^T \in \Re^N$, and constant M is independent of ϵ and n.

Proof. The existence of u and the following estimates

$$| u^{(i)}(x) | \le \begin{cases} M(1 + \epsilon^{-i} \exp(-\beta x/\epsilon)), 0 \le x \le 0.5, \\ M(1 + \epsilon^{-i} \exp(-\beta(1-x)/\epsilon)), \; 0.5 \le x \le 1, \end{cases} \quad i = 0,1,...,k$$

follows from the inverse monotonicity of (1)-(2) under assumptions (3), see [3]. Let us write the discretization in the matrix norm:

$$Aw_h = f_h,$$

where $f_h = [f(z_0), ..., f(z_N)]^T \in \Re^N$, and matrix A has the form

$$\begin{bmatrix} 1 & & & & & & \\ a_1(1) & a_0(1)+b_1 & a_2(1) & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & a_1(s) & a_0(s)+b_s & a_2(s) & & \\ & 0 & \ddots & \ddots & & & \\ & & & & a_1(N-1) & a_0(N-1)+b_{N-1} & a_2(N-1) \\ 0 & C_1 & C_2 & \dots & C_{N-2} & C_{N-1} & 1 \end{bmatrix}$$

It is easy to see that

$$a_0(i)+b_i > 0,\ a_1(i) < 0,\ a_2(i) < 0,\quad i = 1,2,...,N-1,$$

and

$$| a_0(i)+b_i | - | a_1(i) | - | a_2(i) | \geq \beta^2,\quad i = 1,2,...,N-1.$$

In the last row we have

$$\sum_{i=1}^{N-1} | C_i | = \sum_{i=1}^{m} | c_i | < 1,$$

and the matrix A is strictly diagonally dominant. Let

$$\mu = \min\{\beta^2, 1, 1 - \sum_{i=1}^{m} | c_i |\},$$

then

(2.1) $$\|A^{-1}\|_\infty \leq \frac{1}{\mu}.$$

Now, we shall be concerned with the consistency error r_h :

(2.2) $$r_h = Au_h - Aw_h.$$

Under the assumptions of Theorem 1, for the consistency error in [11], it is proved that

$$\|r_h\|_\infty \le Mh^2.$$

Now, from (5) and (6) we have

$$\|u_h - w_h\|_\infty = \|A^{-1}r_h\|_\infty \le \frac{1}{\mu}\|r_h\|_\infty \le Mh^2. \quad \square$$

3. Numerical example

We shall use the following test example

$$-\epsilon^2 u'' + u + \cos^2 \pi x + 2(\epsilon\pi)^2 \cos 2\pi x = 0, \quad x \in I,$$

$$u(0) = 0, \ u(1) = \sum_{i=1}^{m} c_i u_i(s_i) + d,$$

where for given c_i, $i = 1, 2, ..., m$, the number d is determined so that $u(1) = 0$. This is possible because the exact solution of our problem in this case is known:

$$u(x) = \frac{\exp(-x/(\epsilon)) + \exp((x-1)/\epsilon)}{1 + \exp(-1/\epsilon)} - \cos \pi x.$$

In the tables we present the error

$$E_N = \|u_h - w_h\|_\infty,$$

(where w_h is the same as in Theorem 1), and the experimental order of convergence

$$Ord = \frac{\log E_N - \log E_{N_2}}{\log N - \log N_2},$$

where N_2 depends on $2n$ and m in the same way as N does on n and m. Different values of ϵ and n are considered with $m = 5$.

i	s_i	c_i
1	0.9999	0.03
2	0.1000	0.20
3	0.2000	0.50
4	0.5000	0.09
5	0.0001	0.05

and $a = 1, 2,\ \ p = 1, 2$.

Table 1. $a = 1,\ p = 1.$

$N(n)\backslash\epsilon$	2^{-5}	2^{-10}	2^{-15}	2^{-20}	2^{-30}	2^{-50}
8(4) -	4.949(-2) -	3.509(-1) -	4.296(-2) -	3.492(-1) -	3679(-1)	E_N Ord
12(8)	3.112(-1) 1.144	5.601(-2) 4.526	5.047(-2) 0.398	5.552(-2) 4.548	5.574(-2) 4.654	E_N Ord
20(16)	8.968(-3) 2.436	1.719(-2) 2.312	1.450(-2) 2.442	1.723(-2) 2.280	1.724(-2) 2.300	E_N Ord
36(32)	3.650(-3) 1.530	4.269(-3) 2.370	3.511(-3) 2.413	4.277(-3) 2.371	4.277(-3) 2.371	E_N Ord
68(64)	9.380(-4) 2.136	1.068(-3) 2.179	9.216(-4) 2.103	1.072(-3) 2.175	1.072(-3) 2.175	E_N Ord
132(128)	4.615(-4) 1.069	2.669(-4) 2.090	2.648(-4) 1.880	2.678(-4) 2.091	2.678(-4) 2.091	E_N Ord
260(256)	1.369(-4) 1.793	6.672(-5) 2.045	6.621(-5) 2.045	6.670(-5) 2.044	6.670(-5) 2.044	E_N Ord
516(512)	3.434(-5) 2.017	1.668(-5) 2.023	1.656(-5) 2.021	1.675(-5) 2.023	1.675(-5) 2.023	E_N Ord
1028(1024)	8.620(-6) 2.005	4.170(-6) 2.011	4.149(-6) 2.009	4.186(-6) 2.011	4.186(-6) 2.011	E_N Ord

Table 2. $a = 2, \ p = 3.$

$N(n)\backslash\epsilon$	2^{-5}	2^{-10}	2^{-15}	2^{-20}	2^{-30}	
9(4)	4.074(-2)	2.344(-1)	9.175(-3)	6.886(-4)	5.907(-7)	E_N
-	-	-	-	-		Ord
13(8)	2.482(-2)	4.975(-2)	2.443(-2)	2.909(-3)	4.154(-3)	E_N
	1.348	4.215	2.663	43.919	2.409	Ord
21(16)	1.040(-2)	2.782(-2)	1.774(-2)	2.563(-2)	2.393(-2)	E_N
	1.813	1.212	0.668	4.537	3.651	Ord
37(32)	2.585(-3)	6.252(-3)	5.692(-3)	6.915(-3)	7.393(-3)	E_N
	2.459	2.635	2.007	2.313	2.074	Ord
69(64)	6.518(-4)	1.430(-3)	1.629(-3)	1.939(-3)	1.997(-3)	E_N
	2.211	2.367	2.008	2.040	2.100	Ord
133(128)	1.637(-4)	3.510(-4)	4.273(-4)	4.849(-4)	4.998(-4)	E_N
	2.105	2.141	2.039	2.112	2.111	Ord
261(256)	4.138(-5)	8.443(-5)	1.099(-4)	1.212(-4)	1.254(-4)	E_N
	2.040	2.114	2.014	2.056	2.051	Ord
517(512)	1.041(-5)	2.111(-5)	2.756(-5)	3.031(-5)	3.135(-5)	E_N
	2.019	2.028	2.024	2.028	2.028	Ord
1029(1024)	2.608(-6)	5.276(-6)	6.921(-6)	7.577(-6)	7.841(-6)	E_N
	2.011	2.014	2.007	2.014	2.014	Ord

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REZIME

O UNIFORMNOM NUMERIČKOM POSTUPKU ZA NELOKALNI PROBLEM

Posmatra se numerički postupak za singularni linearni nelokalni problem. Primenjena je klasična diferencna šema na specijalnoj neekvidistrantnoj mreži, koja je gusta u graničnim slojevima. Dokazan je drugi red uniformne konvergencije. Numerički primer je dat.

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