Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 21, 2 (1991), 175-183

Review of Research
Faculty of Science
Mathematics Series

STABILITY THEORY FOR VOLTERRA EQUATIONS

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Abstract

Consider the integrodifferential equation

$$(1)\qquad \frac{d}{dt}[x(t)-\int_{t_0}^{t} D(t,s)x(s)ds] = A(t)x(t)+\int_{t_0}^{t} C(t,s)x(s)ds,$$

where A, C, D are $n\times n$ matrices continuous for $t_0 \le s \le t < \infty$. The boundedness and stability properties of (1) are studied.

AMS Mathematics Subject Classifications (1980): 45D05, 34K20
Key words and phrases: Integrodifferential equations, stability, Liapunov function.

In papers [1,2,3,4,5] the authors have used the Liapunov functional to give sufficient conditions for the stability and boundedness of a system of Volterra integrodifferential equations

$$x'(t) = g(t,x(t)) + \int_{t_0}^{t} K(t,s,x(s))ds.$$

In this paper we shall give conditions for which solutions of equation (1) have boundedness and stability properties. We consider the problem based on the modified Liapunov method contained in the construction of some function and the stability of some inequality.

1. Introduction

In this paper, we shall concentrate on a system of Volterra integrodifferential equation

$$(2)\ \frac{d}{dt}[x(t) - \int_{t_0}^{t} D(t,s)x(s)ds] = A(t)x(t) + \int_{t_0}^{t} C(t,s)x(s)ds, x(t_0) = x_0$$

in which A, C, D are $n \times n$ matrices continuous for $t_0 \leq s \leq t < \infty$ and $x :< t_0, \infty) \longrightarrow R^n$.

The following notation will be used throughout the paper:
R -denotes the set of real numbers, and R^n the set of real n-tuples,
J-denotes the interval $< t_0, \infty), (t_0 \geq 0)$, and $\|\cdot\|$ denotes the matrix norm, and $|\cdot|$ the vector norm. Troughout this work, $x(\cdot, t_0, x_0)$ will denote the unique solution of (1), satisfyng the initial condition $x(t_0, t_0, x_0) = x_0$ and continued to $t = +\infty$. By a solution of (1) with the initial condition $x(t_0) = x_0$ we mean a function $x : J \longrightarrow R^n$ such that $x(t_0) = x_0$, $x(t) - \int_{t_0}^{t} D(t,s)x(s)ds$ is continuously differentiable for $t \in J$ and $x(t)$ satisfies (2) for all $t \in J$.

Definition 1. *The trivial solution of (2) is stable if for a given $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon, t_0)$ such that $|x_0| < \delta$, implies that every solution $x(\cdot, t_0, x_0)$ of (2) is defined for $t \in J$ and satisfies*

$$|x(t, t_0, x_0)| < \varepsilon \quad for \quad t \in J.$$

Definition 2. *The solution of (2) is asymptotically stable if it is stable and if for $t_0 \geq 0$ there is an $\eta > 0$ such that $|x_0| < \eta$ implies $|x(t, t_0, x_0)| \longrightarrow 0$ as $t \longrightarrow \infty$.*

Definition 3. *The zero solution of*

$$(3) \qquad |Z(t, y(t))| = |y(t) - \int_{t_0}^{t} D(t,s)y(s)ds| \leq f(t), y(t_0) = x_0$$

for $t \in J$; $f : J \longrightarrow R$ is continuous and nonnegative, is said to be

a) f-stable if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $t \in J$

$$[|x_0| \leq \delta \wedge f(t) \leq \delta] \Rightarrow |y(t, t_0, x_0)| \leq \varepsilon,$$

b) asimptotically f-stable if it is f-stable and

$$\lim_{t \longrightarrow \infty} |y(t, t_0, x_0)| = 0$$

for every $|x_0| \leq \delta_1$ *and every* $f(t) \longrightarrow 0$ *as* $t \longrightarrow \infty$*,*

c) f-bounded if for every bounded function $f : J \longrightarrow R$ *there exists a bounded solution* $y(t, t_0, x_0)$ *of (3).*

2. Stability and boundedness

Let

$$|Z(t, y(t))| = |y(t) - \int_{t_0}^{t} D(t, s)y(s)ds| \leq f(t), \text{ for all } t \in J.$$

Let $V(t) = V(t, x(\cdot), Z(t, x(\cdot)))$ be a scalar functional defined and continuous for $t \in J$ and $x \in S_\rho = \{x : J \longrightarrow R^n, |x| \leq \rho = const.\}$, x is continuous for $t \in J$. We denote by $\frac{dV}{dt}$ the derivative of $V(t, x(t), Z(t, x(t)))$ along any solution of (2).

Theorem 1. *Assume that*
1° *there exist continuous and strictly increasing functions* $w_i(\cdot), (i = 1, 2, 3), w_i : J \longrightarrow < 0, \infty)$ *such that* $w_i(0) = 0, w_i(r) > 0$ *for* $r > 0$ *and* $w_i(r) \longrightarrow \infty$ *as* $r \longrightarrow \infty, (i = 1, 2, 3)$,
2° *there exists a functional* $V(\cdot)$ *continuous for* $t \in J$ *satisfying the following properties:*

$$a) \quad w_1(|Z(t, x(t))|) \leq V(t) \leq \alpha w_2(|Z(t, x(t)|),$$

$$b) \quad \frac{dV}{dt} \leq -\mu w_2(|Z(t, x(t))|),$$

for every solution $x(t) = x(t, t_0, x_0)$ *for (2) and some constants* $\mu > 0, \alpha > 0$,
3° *the zero solution of (3) is f-stable,*
then the solution $x(t) = 0$ *of (2) is asymptotically stable.*

Proof. Let $0 < \varepsilon < \rho$ and $t_0 \geq 0$ be given. The f-stability of the trivial solution of (3) implies that for a given $\varepsilon > 0$ there exists a $\delta_1 > 0$ such that for every solution of integral inequalities

$$|Z(t, x(t))| = |x(t) - \int_{t_0}^{t} D(t, s)x(s)ds| \leq f(t), x(t_0) = x_0$$

we have

$$|x(t,t_0,x_0) \le \varepsilon \text{ for } t \in J,$$

provided that $|x_0| \le \delta_1$ and $f(t) \le \delta_1$ for $t \in J$. Choose $\delta_2 = \delta_2(\delta_1) > 0$ such that $0 < \delta_2(\delta_1) \le \delta_1 < \varepsilon$ and $\alpha w_2(s) < w_1(\delta_1)$ for $0 \le s \le \delta_2(\delta_1)$. Then, using 2° for $|x_0| \le \delta_2$ we obtain

$$w_1(|Z(t,x(t))|) \le V(t) \le V(t_0) \le \alpha w_2(|x_0|) < w_1(\delta_1).$$

Hence $|Z(t,x(t))| = f(t) < \delta_1$. From this and f-stability, it follows easily that $|x(t,t_0,x_0)| \le \varepsilon$ for $t \in J$ and $|x_0| \le \delta_1$.

Next, integrating both sides of b) from t_0 to t we obtain

$$\int_{t_0}^{t} w_2(|Z(s,x(s))|)ds \le -\mu^{-1}[V(t) - V(t_0)] \le \mu^{-1}V(t_0) \le \alpha\mu^{-1}w_2(\delta_1),$$

so that 2° a) implies

$$\int_{t_0}^{t} V(s)ds \le \alpha \int_{t_0}^{t} w_2(|Z(s,x(s))|)ds \le \alpha^2\mu^{-1}w_2(\delta_1).$$

On the other hand, as $V(t)$ is decreasing, we have

$$\int_{t_0}^{t} V(s)ds \ge V(t)(t-t_0),$$

so that, by 2° a), it follows that

$$w_1(|Z(t,x(t))|) \le V(t) \le (t-t_0)^{-1} \int_{t_0}^{t} V(s)ds \le (t-t_0)^{-1}\mu^{-1}\alpha^2 w_2(\delta_1)$$

and hence

$$|Z(t,x(t))| \le w_1^{-1}([\mu(t-t_0)]^{-1}\alpha^2 w_2(\delta_1).$$

From this we have the asymptotic stability. The proof is now completed.

Theorem 2. *Assume that*
1° *condition* 1° *of Theorem 1 is satisfied,*
2° *there exists a functional* $V(t)$ *satisfying the following properties:*

$$a) \quad w_1(|Z(t,x(t))|) \le V(t) \le \alpha w_2(|Z(t,x(t))|) + \alpha_1 \int_{t_0}^{t} w_3(|Z(s,x(s))|)ds$$

$$b) \quad \frac{dV}{dt} \le -\mu w_3(|Z(t,x(t))|) + v(t),$$

for every solution $x(t) = x(t, t_0, x_0)$ of (2) and some constants $\mu > 0, \alpha > 0, \alpha_1 > 0$ and continuous, nonnegative function $v : J \longrightarrow R$ such that $v(t) \leq N = constant$ for all $t \in J$, $\int_{t_0}^{\infty} v(s)ds \leq B = constant$,
3° the zero solution of (3) is f-bounded,
then the solutions of (2) are bounded.

Proof. Let $H > 0$ and $|x_0| \leq H$. We show that $|Z(t, x(t))| \leq constant$ for all $t \in J$.

Let $V(t) \leq V(t_0)$ for all $t \in J$. Then by a), we have

$$w_1(|Z(t, x(t))|) \leq V(t) \leq V(t_0) \leq w_2(|x_0|) \leq w_2(H).$$

Thus, $|Z(t, x(t))| \leq w_1^{-1}(w_2(H))$, and $x(t, t_0, x_0)$ is bounded.

Let $V(t) > V(t_0)$ for some $t \geq t_0$. We choose $t \geq t_0$ so that $V(t) = \max_{t_0 \leq s \leq t} V(s)$. Then by b)

$$\int_{t_0}^{t} V'(s)ds \leq -\mu \int_{t_0}^{t} w_3(|Z(s, x(s))|)ds + \int_{t_0}^{t} v(s)ds.$$

Thus

$$\mu \int_{t_0}^{t} w_3(|Z(s, x(s))|)ds \leq V(t_0) + B.$$

By a) $V(t_0) \leq \alpha w_2(|x_0|)$. Thus,

$$\int_{t_0}^{t} w_3(|Z(s, x(s))|)ds \leq \mu^{-1}(\alpha w_2(|x_0|) + B).$$

Also,

$$\begin{aligned} V(t) &\leq \alpha w_2(|Z(t, x(t))|) + \alpha_1 \int_{t_0}^{t} w_3(|Z(s, x(s))|)ds \leq \\ &\leq \alpha w_2(|Z(t, x(t))|) + \alpha_1 \mu^{-1}(\alpha w_2(|x_0|) + B). \end{aligned}$$

Since $w_3(r) \longrightarrow \infty$ as $r \longrightarrow \infty$, then there is $L > 0$ such that

$$w_3(L) = \frac{N}{\mu}.$$

By b), if $|Z(t,x(t))| > L$ for $t \in J$, then $\frac{dV}{dt} < 0$.
Thus any maximum of $V(t)$ must occur when $|Z(t,x(t))| \le L$.
Hence, whenever $V(t)$ is maximum, we have

$$w_1(|Z(t,x(t))|) \le V(t) \le \alpha w_2(|Z(t,x(t))|) + \\ +\alpha_1\mu^{-1}(\alpha w_2(|x_0|) + B) \le \\ \le \alpha w_2(L) + \alpha_1\mu^{-1}(\alpha w_2(H) + B).$$

Consequently,

$$|Z(t,x(t))| \le W_1^{-1}[\alpha w_2(L) + \alpha_1\mu^{-1}(\alpha w_2(H) + B)].$$

Hence by f-boundedness we have

$$|x(t)| \le constant \text{ for all } t \in J.$$

This completes the proof.

The next results extend Theorem 2.

Theorem 3. *Assume that*
1° *the condition* 1° *of Theorem 1 is satisfied*
2° *there exists a real-valued continuous function* $\Phi(t,s)$ *for* $t_0 \le s \le t < \infty$ *with* $\Phi \ge 0, \frac{\partial \Phi}{\partial t} \le 0, \frac{\partial \Phi}{\partial s} \ge 0$,

$$\Phi(t_0,t_0) \le \Phi_0, \int_{t_0}^t \Phi(t,s)ds \le B$$

for some constants B *and* Φ_0,
3° *there exists a functional* $V(t)$ *continuous for* $t \in J$ *and satisfying the following properties:*

$$a) \quad w_1(|Z(t,x(t))|) \le V(t) \le \alpha w_2(|Z(t,x(t))|) + \\ \alpha_1 \int_{t_0}^t \Phi(t,s) w_3(|Z(s,x(s))|)ds,$$

$$b) \quad \frac{dV}{dt} \le -\mu w_3(|Z(t,x(t))|) + K$$

for every solution $x(\cdot)$ *of (2) and some constants* $\mu > 0, \alpha > 0, \alpha_1 > 0$, *and* $K > 0$,
4° *the zero solution of (3) is f-bounded,*
then the solutions of (2) are bounded.

Proof. We shall proceed as in the proof of Theorem 2. If $V(t) \leq V(t_0)$ for all $t \geq t_0$, it is trivial. Let $V(t) > V(t_0)$ for some $t \geq t_0$, we choose $t \geq t_0$ so that $V(t) = \sup_{t_0 \leq s \leq t} V(s)$.

We multiply both sides b) by $\Phi(t,s)$ and integrate from $s = t_0$ to $s = t$ to obtain

$$\int_{t_0}^{t} V'(s)\Phi(t,s)ds \leq -\mu \int_{t_0}^{t} \Phi(t,s)w_3(|Z(s,x(s))|)ds + KB.$$

An integration by parts of the left integral yields

$$\mu \int_{t_0}^{t} \Phi(t,s)w_3(|Z(s,x(s))|)ds \leq -V(t)\Phi(t,t) + V(t_0)\Phi(t,t_0) +$$

$$\int_{t_0}^{t} V(s)\Phi_s(t,s)ds + KB.$$

Since $\Phi_s(t,s) \geq 0$, then

$$\mu \int_{t_0}^{t} \Phi(t,s)w_3(|Z(s,x(s))|)ds \leq -V(t)\Phi(t,t) +$$

$$+V(t_0)\Phi(t,t_0) + V(t)(\Phi(t,t) - \Phi(t,t_0)) + KB \leq V(t_0)\Phi_0 + KB.$$

By a) $V(t_0) \leq \alpha w_2(|x_0|)$.

Thus,

$$\mu \int_{t_0}^{t} \Phi(t,s)w_3(|Z(s,x(s))|)ds \leq \alpha\Phi_0 w_2(|x_0|) + KB.$$

Hence by a) we obtain

$$V(t) \leq \alpha w_2(|Z(t,x(t))|) + \alpha_1\mu^{-1}[\alpha\Phi_0 w_2(H) + KB].$$

The rest of the proof is the same as that of Theorem 2.

We shall illustrate Theorem 2 by considering the scalar equation

$$\text{(4)} \qquad \frac{d}{dt}[x(t) - \int_0^t D(t,s)x(s)ds] = A(t)x(t) + \int_0^t C(t,s)x(s)ds$$

where $A(t)$ is a continuous function for $0 \leq t < \infty$ and $D(t,s), C(t,s)$ are continuous for $0 \leq s \leq t < \infty$.

Suppose there exists $m > 0, m_1 > 0$ and $M > 0$ such that

$$x^2 \leq mZ^2(t,x) \text{ if } |x| \leq M, t \in J$$

and

$$|A(t)D(t,s)| \leq m_1|C(t,s)| \text{ for } 0 \leq s \leq t < \infty.$$

We define

$$V(t,x(\cdot),Z(t,x(\cdot))) = \frac{1}{2}Z^2(t,x(\cdot)) + k\int_0^t\int_t^\infty |C(u,s)|duZ^2(s,x(s))ds,$$

$(0 < k = constant)$ so that along with a solution $x(\cdot)$ of (4) w have

$$V'_{(4)}(t,x(\cdot),Z(t,x(\cdot))) = Z(t,x(\cdot))Z'(t,x(\cdot))+$$

$$+k\int_t^\infty |C(u,t)|duZ^2(t,x(t))ds - k\int_0^t |C(t,s)|Z^2(s,x(s))ds \leq$$

$$\leq [A(t) + k\int_t^\infty |C(u,t)|du + \frac{1}{2}\int_0^t |A(t)D(t,s) + C(t,s)|ds]Z^2(t,x(t))+$$

$$+(\frac{1}{2}mm_1 - k)\int_0^t |C(t,s)|Z^2(s,x(s))ds =$$

$$= a(t,k)Z^2(t,x(t)) + (\frac{1}{2}mm_1 - k)\int_0^t |C(t,s)|Z^2(s,x(s))ds,$$

where

$$a(t,k) = A(t) + k\int_t^\infty |C(u,t)|du + \frac{1}{2}\int_0^t |A(t)D(t,s) + C(t,s)|ds.$$

If $a(t,k) \leq -a < 0$ for $t \in J$ and

$$\int_0^t |C(t,s)|Z^2(s,x(s))ds \leq b < \infty \text{ for all } t \in J, |x| \leq M,$$

$$\int_0^\infty (\int_0^t |C(t,s)|Z^2(s,x(s))ds)dt \leq c < \infty, (\frac{1}{2}mm_1 - k) \geq 0,$$

the solution $x(t) = x(t,t_0,x_0)$ satisfies 2° of Theorem 2.

References

[1] Burton, T.A.: Stability Theory for Volterra Equations, J.Differential Equations 32(1979), 101-118.

[2] Burton, T.A.: Uniform Stabilities for Volterra Equations, J.Differential Equations 36(1980), 40-53.

[3] Burton, T.A.: An Integrodifferential Equation, Proc. Amer. Math. Soc. 79,2(1980), 393-399.

[4] Seifert, G.: Liapunov-Razumikhin Conditions for Asymptotic Stability in Functional Differential Equations of Volterra Type, J.Differential Equations 16(1974), 289-297.

[5] Seifert, G.: Liapunov-Razumikhin Conditions for Stability and Boundedness of Functional Differential Equations of Volterra Type, 14(1973), 424-430.

REZIME

TEORIJA STABILNOSTI ZA VOLTERRA JEDNAČINE

Posmatra se integrodiferencijalna jednačina

$$\frac{d}{dt}[x(t) - \int_{t_0}^{t} D(t,s)x(s)ds] = A(t)x(t) + \int_{t_0}^{t} C(t,s)x(s)ds,$$

gde su A, C, D $n \times n$ matrice neprekidne za $t_0 \leq s \leq t < \infty$. Izučavane su ograničenost i stabilnost te jednačine.

Received by the editors August 28, 1988