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ON (n, f, g)-LOCALLY CONTRACTIONS IN PROBABILISTIC METRIC SPACES

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Abstract

T. Hicks introduced in [3] the notion of a C-contraction in a probabilistic metric space $(S, \mathcal{F}, T)$. If t-norm T is min Hicks proved a fixed point theorem for a C-contraction. Here we shall introduce the notion of an (n, f, g)-locally contraction, as a generalization of the notions of a C-contraction and of an f-contraction in the sense of V. Radu [5]. Two fixed point theorems for an (n, f, g)-locally contraction are proved.

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1. Introduction and Preliminaries

In [3] T. Hicks introduced the notion of a C-contraction $f : S \to S$, where $(S, \mathcal{F})$ is a probabilistic metric space.

Definition 1. *Let $(S, \mathcal{F})$ be a probabilistic metric space and $A : S \to S$. The mapping A is a C-contraction if there exists a $k \in (0,1)$ such that the following implication holds for every $r > 0$ and every $p, q \in S$:*

$$F_{p,q}(r) > 1 - r \Rightarrow F_{Ap,Aq}(kr) > 1 - kr.$$

If $(S,\mathcal{F},T)$ is a Menger space such that $T \geq T_m$, $T_m(a,b) = \max\{a+b-1,0\}$ V. Radu proved that $A : S \to S$ is a C-contraction if and only if A is a metric contraction on the metric space (S,d), where d is defined by

$$d(p,q) = \inf\{h;\ h > 0,\ F_{p,q}(h^+) > 1-h\}$$

$(p,q \in S)$ and d induces the (ε,λ)-topology in S.

In [9] it is proved that a contraction in the sense of Sehgal and Bharucha-Reid need not be a C-contraction and a C-contraction need not be a contraction in the sense of Sehgal and Bharucha-Reid.

A generalization of the notion of a C-contraction is given by V. Radu in [5]. Let $\mathcal{M}$ be the family of all functions $m : [0,\infty) \to [0,\infty)$ such that:

a) $m(t+s) \geq m(t) + m(s),\ t,s \geq 0$.

b) $m(t) = 0 \Leftrightarrow t = 0$.

c) m is continuous.

If T is an Archimedean t-norm with the additive generator f and $(S,\mathcal{F},T)$ is a Menger space then for every $p,q \in S$ and every $x,y \in \mathbf{R}$:

$$f \circ F_{p,q}(x+y) \leq f \circ F_{p,r}(x) + f \circ F_{r,q}(y).$$

Using this inequality it can be proved that for $m_1,\ m_2 \in \mathcal{M}$ by $(p,q \in S)$:

$$d_{m_1,m_2}(p,q) = \sup\{t;\ t \geq 0,\ m_1(t) \leq f \circ F_{p,q}(m_2(t))\}$$

a metric is defined, and d_{m_1,m_2} induces the (ε,λ)-topology in S.

The following implication holds:

$$d_{m_1,m_2}(p,q) < t \Leftrightarrow f \circ F_{p,q}(m_2(t)) < m_1(t).$$

Definition 2. *Let $(S,\mathcal{F},t)$ be a Menger space, $A : S \to S$, $m_1, m_2 \in \mathcal{M}$ and $f : [0,1] \to [0,\infty)$, $g : [0,\infty) \to [0,\infty)$. If for any $x \in S$ there exists $n(x) \in \mathbf{N}$ such that for any $v \in O_A(x;0,\infty) = \{A^n x;\ n \in \mathbf{N}\bigcup\{0\}\}$ the following implication holds:*

$$r > 0,\ f \circ F_{x,v}(m_2(r)) < m_1(r) \Rightarrow$$

$$f \circ F_{A^{n(x)}x,A^{n(x)}v}(m_2(g(r))) < m_1(g(r)), \qquad (a)$$

the mapping A is an (n,f,g)-locally contraction.

If $T = T_f$, where f is the additive generator of T_f and A is an (n, f, g)-locally contraction then the following condition is satisfied [5]:

For any $x \in S$ there exists $n(x) \in \mathbf{N}$ such that for any $v \in O_A(x; 0, \infty)$ the following implications holds:

$$d_{m_1,m_2}(x, v) < r \Rightarrow d_{m_1,m_2}(A^{n(x)}x, A^{n(x)}v) < g(r).$$

If, in addition, g is continuous from the right then

$$d_{m_1,m_2}(A^{n(x)}x, A^{n(x)}v) \leq g(d_{m_1,m_2}(x, v)). \tag{b}$$

If in Definition 2 $n(x) = 1$, for every $x \in S$ and (a) holds for every $x, v \in S$ and $g(r) = r$, $r \geq 0$, then an (n, f, g)-locally contraction is an f-contraction [4].

In [4] Kaleva and Seikkala introduced the notion of a fuzzy metric space. Some connections between probabilistic metric and fuzzy metric spaces are given in [4].

A **fuzzy number** is a mapping $x : \mathbf{R} \to [0, 1]$ which is **convex** iff the following implication holds:

$$s \leq t \leq r \Rightarrow x(t) \geq \min\{x(s), x(r)\}.$$

For $0 < \alpha \leq 1$ and a fuzzy number x its α-**level sets** $[x]_\alpha$ are defined by $[x]_\alpha = \{u;\ x(u) \geq \alpha\}$. A fuzzy number x is convex if and only if $[x]_\alpha$ is a convex set in $\mathbf{R}$ for every $\alpha \in (0, 1]$. If there exists an element $u \in \mathbf{R}$ such that $x(u) = 1$ then the fuzzy number x is called **normal**. If a fuzzy number x is uppersemicontinuous, convex and normal then $[x]_\alpha = [a^\alpha, b^\alpha]$ ($a^\alpha = -\infty$ and $b^\alpha = \infty$ are admissible) is a closed interval. A fuzzy number x is **nonnegative** if $x(u) = 0$ for all $u < 0$.

Let G be the set of all nonnegative uppersemicontinuous normal convex fuzzy numbers, X a nonempty set, $d : X \times X \to G$ and $L, R : [0, 1] \times [0, 1] \to [0, 1]$. Let for all $x, y \in X$ and $\alpha \in (0, 1]$:

$$[d(x, y)]_\alpha = [\lambda_\alpha(x, y), \varrho_\alpha(x, y)].$$

By $\bar{O}$ we shall denote the fuzzy number defined by $\bar{O}(u) = 1$, for $u = 0$ and $\bar{O}(u) = 0$ for $u \neq 0$.

The quadruple (X, d, L, R) is called a **fuzzy metric space** and d a **fuzzy metric** if L and R are symmetric mappings, nondecreasing in both arguments, $L(0, 0) = 0$, $R(1, 1) = 1$ and the following conditions are satisfied:

(i) $d(x,y) = 0$ if and only if $x = y$.

(ii) $d(x,y) = d(y,x)$, for all $x, y \in X$.

(iii) $d(x,y)(s+u) \geq L(d(x,z)(s), d(z,y)(u))$ for all $x, y, z \in X$, whenever $s \leq \lambda_1(x,z)$, $u \leq \lambda_1(z,y)$, $s+u \leq \lambda_1(x,y)$;
$d(x,y)(s+u) \leq R(d(x,z)(s), d(z,y)(u))$ for all $x, y, z \in X$, whenever $s \geq \lambda_1(x,z)$, $u \geq \lambda_1(z,y)$, $s+u \geq \lambda_1(x,y)$.

A metric space may be regarded as a fuzzy metric space, since the nonnegative numbers belong to G and L and R are given by $L(a,b) \equiv 0$, $R(a,b) = 0$ for $a = b = 0$, $R(a,b) = 1$ otherwise.

If $R = \max$ it is known that for every $\alpha \in (0,1]$ the triangle inequality

$$\varrho_\alpha(x,y) \leq \varrho_\alpha(x,z) + \varrho_\alpha(z,y) \ (x,y,z \in X)$$

holds.

We shall suppose that R is associative and satisfies the condition $R(a,0) = a$, for every $a \in [0,1]$. This implies that the mapping $t(a,b) = 1 - R(1-a, 1-b)$ $(a,b \in [0,1])$ is a T-norm.

In [4] it was proved that if $(S, \mathcal{F}, t)$ is a Menger space then (X, d, L, R) is a fuzzy metric space if we take $X = S$,

$$d(x,y)(u) = \begin{cases} 0, & u \leq \sup\{s;\ F_{x,y}(s) = 0\} = u_{x,y} \\ 1 - F_{x,y}(u), & u \geq u_{x,y}, \end{cases}$$

$R(a,b) = 1 - t(1-a, 1-b)$ $((a,b) \in [0,1] \times [0,1])$ and $L \equiv 0$.

The converse statement does not hold. But, under some additional conditions on d and R, a fuzzy metric space (X, d, L, R) is a Menger space. These conditions are the following: $\lim\limits_{u \to \infty} d(x,y)(u) = 0$, for all $x, y \in X$ and $R(a,1) = R(1,a) = 1$, for every $a \in [0,1]$.

Under these conditions a fuzzy metric space (X, d, L, R) is a Menger space $(X, \mathcal{F}, t)$, where $t(a,b) = 1 - R(1-a, 1-b)$, for every $a, b \in [0,1]$ and for every $x, y \in X$ and every $s \in \mathbf{R}$:

$$F_{x,y}(s) = \begin{cases} 0, & s \leq \lambda_1(x,y) \\ 1 - d(x,y)(s), & s \geq \lambda_1(x,y). \end{cases}$$

2. Fixed point theorems

Theorem 1. *Let $(S,\mathcal{F},T)$ be a complete Menger space such that $\sup_{a<1} T(a,a) = 1$, $A : S \to S$ be a continuous mapping, $m_1, m_2 \in \mathcal{M}$ and $f : [0,1] \to [0,\infty)$ be continuous , decreasing and $f(1) = 0$. If A is an (n,f,g)-locally contraction such that $\lim_{n\to\infty} g^n(r) = 0$ for every $r > 0$, then there exists $x \in S$ such that $x = Ax$.*

Proof. Let x_0 be an element from S and

$$x_n = A^{n(x_{n-1})}x_{n-1},\ n \in \mathrm{N}.$$

First, we shall prove that $\{x_n\}_{n\in\mathrm{N}}$ is a Cauchy sequence i.e. that for every $r > 0$ and $\lambda \in (0,1)$ there exists $n(r,s) \in \mathrm{N}$ such that

$$F_{x_{m+p},x_m}(r) > 1 - \lambda, \text{ for every } m \geq n(r,s) \text{ and every } p \in \mathrm{N}.$$

From $f(0) < \infty$ and $\lim_{t\to\infty} m_1(t) = \infty$ it follows that there exists $t > 0$ such that $f(0) < m_1(t)$. Since

$$F_{A^{n(x_{m+p-1})}\dots A^{n(x_m)}x_0,x_0}(m_2(t)) \geq 0$$

and f is decreasing we have that

$$f \circ F_{A^{n(x_{m+p-1})}\dots A^{n(x_m)}x_0,x_0}(m_2(t)) \leq f(0) < m_1(t). \tag{1}$$

Since A is an (n,f,g)-locally contraction from (1) it follows that:

$$f \circ F_{A^{n(x_0)}A^{n(x_{m+p-1})}\dots A^{n(x_m)}x_0,A^{n(x_0)}x_0}(m_2(g(t))) < m_1(g(t)). \tag{2}$$

From (2) we obtain that

$$f \circ F_{A^{n(x_{m+p-1})}\dots A^{n(x_m)}A^{n(x_1)}x_1,A^{n(x_1)}x_1}(m_2(g^2(t))) < m_1(g^2(t))$$

and similarly for every $p \in \mathrm{N}$ and every $m \in \mathrm{N}$

$$f \circ F_{A^{n(x_{m+p-1})}\dots A^{n(x_m)}x_{m-1},x_{m-1}}(m_2(g^{m-1}(t))) < m_1(g^{m-1}(t)).$$

Let $r > 0$ and $s \in (0,1)$. Since $\lim_{m\to\infty} g^m(t) = 0$ and $m_1(0) = m_2(0) = 0$ there exists $n_0(r,s) \in \mathrm{N}$ such that

$$m_2(g^{m-1}(t)) < r,\ \ m_1(g^{m-1}(t)) < f(1-s)$$

for every $m \geq n_0(r,s)$.

Hence for every $p \in \mathbf{N}$ and $m \geq n_0(r,s)$ we have that:

$$F_{A^{n(x_{m+p-1})}\dots A^{n(x_m)}x_{m-1},x_{m-1}}(m_2(g^{m-1}(t))) > 1-s$$

which implies that

$$F_{A^{n(x_{m+p-1})}\dots A^{n(x_m)}x_{m-1},x_{m-1}}(r) \geq$$

$$\geq F_{A^{n(x_{m+p-1})}\dots A^{n(x_m)}x_{m-1},x_{m-1}}(m_2(g^{m-1}(t))) > 1-s$$

for every $m \geq n_0(r,s)$ and every $p \in \mathbf{N}$.

Hence $\{x_m\}_{m\in\mathbf{N}}$ is a Cauchy sequence and since S is complete there exists $z = \lim\limits_{n\to\infty} x_n$. We shall prove that $\lim\limits_{n\to\infty} Ax_n = z$. In order to prove this we shall prove that for every $r > 0$ and $s \in (0,1)$ there exists $n_1(r,s) \in \mathbf{N}$ such that

(3) $$F_{Ax_m,x_m}(r) > 1-s, \text{ for every } m \geq n_1(r,s).$$

If t is such that $f(0) < m_1(t)$ we have that

$$f \circ F_{Ax_0,x_0}(m_2(t)) \leq f(0) < m_1(t)$$

and so

$$f \circ F_{A^{n(x_0)}Ax_0,A^{n(x_0)}x_0}(m_2(g(t)) = f \circ F_{Ax_1,x_1}(m_2(g(t))) < m_1(g(t)).$$

Hence, for every $m \in \mathbf{N}$:

(4) $$f \circ F_{Ax_m,x_m}(m_2(g^m(t))) < m_1(g^m(t)).$$

If $n_1(r,s)$ is such that $m_2(g^m(t)) < r$ and $m_1(g^m(t)) < f(1-s)$ for every $m \geq n_1(r,s)$ from (4) we have that

$$f \circ F_{Ax_m,x_m}(m_2(g^m(t))) < f(1-s)$$

and so

$$F_{Ax_m,x_m}(r) \geq F_{Ax_m,x_m}(m_2(g^m(t))) > 1-s$$

which proves (3).

Since A is continuous and $\sup\limits_{a<1} T(a,a) = 1$ from the inequality $(u > 0)$:

$$F_{Az,z}(u) \geq T(T(F_{Az,Ax_m}(\frac{u}{3}), F_{Ax_m,x_m}(\frac{u}{3})), F_{x_m,z}(\frac{u}{3}))$$

it follows that $Az = z$.

Theorem 2. *Let $(S,\mathcal{F},T)$ be a complete Menger space such that $\sup_{a<1} T(a,a) = 1$ and $A : S \to S$ be as in Theorem 1 so that (a) holds for every $(x,v) \in S \times S$. If $\lim_{n\to\infty} g^n(r) = 0$, for every $r > 0$ then there exists a unique fixed point z of A and $z = \lim_{n\to\infty} A^n x_0$, where x_0 is an arbitrary element from S.*

Proof. From Theorem 1 it follows that there exists z such that $z = Az, z = \lim_{m\to\infty} x_m$ and $\{x_m\}_{m\in\mathbb{N}}$ is defined as in Theorem 1. Let $t > 0$ be such that $f(0) < m_1(t)$. Then

$$f \circ F_{A^{n(z)}x_0,x_0}(m_2(t)) < m_1(t)$$

and so

$$f \circ F_{A^{n(z)}x_m,x_m}(m_2(g^m(t))) < m_1(g^m(t)).$$

If $n_0(r,s)$ is such that $m_2(g^m(t)) < r$ and $m_1(g^m(t)) < f(1-s)$ $(r > 0,\ s \in (0,1))$ for $m \geq n_0(r,s)$ then

$$F_{A^{n(z)}x_m,x_m}(r) > 1 - s, \text{ for every } m \geq n_0(r,s).$$

This implies that $\lim_{m\to\infty} A^{n(z)}x_m = z$. We shall prove that $\lim_{m\to\infty} A^{n(z)}x_m = A^{n(z)}z$.

From $\lim_{m\to\infty} x_m = z$ it follows that for every $t > 0$ there exists $n_0(t) \in \mathbb{N}$ such that
$f \circ F_{x_m,z}(m_2(t)) < m_1(t)$ for every $m \geq n_0(t)$. Then

$$f \circ F_{A^{n(z)}x_m,A^{n(z)}z}(m_2(g(t))) < m_1(g(t))$$

and if for $r \in (0,\infty)$ and $s \in (0,1)$ t is such that $m_2(g(t)) < r$ and $m_1(g(t)) < f(1-s)$ we obtain that

$$F_{A^{n(z)}x_m,A^{n(z)}z}(r) \geq F_{A^{n(z)}x_m,A^{n(z)}z}(m_2(g(t))) > 1 - s$$

for $m \geq n_0(t) = n_0(r,s)$. Hence $\lim_{m\to\infty} A^{n(z)}x_m = A^{n(z)}z$.

Let us prove the uniqueness of the fixed point of the mapping $A^{n(z)}$.

Let $v \in S$ be such that $A^{n(z)}v = v$. If $t > 0$ is such that $f(0) < m_1(t)$ we have that

$$f \circ F_{z,v}(m_2(t)) < m_1(t)$$

which implies that

$$f \circ F_{A^{n(z)}z,A^{n(z)}v}(m_2(g(t))) < m_1(g(t))$$

i.e:

$$f \circ F_{z,v}(m_2(g(t))) < m_1(g(t)).$$

In this way we have that

$$f \circ F_{z,v}(m_2(g^m(t))) < m_1(g^m(t))$$

and since $\lim_{m\to\infty} g^m(t) = 0$ it follows that $z = v$. Since $A^{n(z)}z = z$ it follows that $AA^{n(z)}z = Az = A^{n(z)}Az$ and so $Az = z$.

We shall prove that $z = \lim_{n\to\infty} A^n x_0$. If $t > 0$ is such that $f(0) < m_1(t)$ we have for $k \in \{0, 1, \ldots, n(z) - 1\}$:

$$f \circ F_{A^k x_0, z}(m_2(t)) < m_1(t).$$

This implies that

$$f \circ F_{A^{m \cdot n(z)+k} x_0, z}(m_2(g^m(t))) < m_1(g^m(t))$$

i.e if $n = m \cdot n(z) + k$:

$$f \circ F_{A^n x_0, z}(m_2(g^{[\frac{n}{n(z)}]}(t))) < m_1(g^{[\frac{n}{n(z)}]}(t)).$$

Hence if $n(r, s)$ is such that

$$m_2(g^m(t)) < r, \;\; m_1(g^m(t)) < f(1 - s)$$

for $m \geq n(r, s)$ it follows that

$$F_{A^n x_0, z}(r) \geq F_{A^n x_0, z}(m_2(g^{[\frac{n}{n(z)}]}(t))) > 1 - s$$

for $[\frac{n}{n(z)}] > n(r, s)$. Hence $\lim_{n\to\infty} A^n x_0 = z$.

Corollary 1. *[5] Let $(S, \mathcal{F}, T)$ be a complete Menger space such that $\sup_{a<1} T(a, a) = 1$ and $A : S \to S$ an f-contraction, where $f : [0, 1] \to [0, \infty)$ is continuous, decreasing and $f(1) = 0$. Then A has a unique fixed point z and $z = \lim_{n\to\infty} A^n x_0$, where x_0 is an arbitrary element of S.*

Corollary 2. *[2] Let $(S, \mathcal{F}, T)$ be a complete Menger space such that $\sup_{a<1} T(a, a) = 1$, $A : S \to S$ an (n, f, g)-locally contraction for $f(s) = 1 - s$ $(s \in [0, 1])$ and $\lim_{n\to\infty} g^n(r) = 0$ for every $r > 0$. If A is continuous then there exists $z \in S$ such that $Az = z$. If (a) holds for every $(x, v) \in S \times S$, then there exists a unique fixed point $z \in S$ of A and $z = \lim_{n\to\infty} A^n x_0$, where x_0 is an arbitrary element of S.*

Corollary 3. *Let (X,d,L,R) be a complete fuzzy metric space such that $\lim_{u\to\infty} d(x,y)(u)=0$, for all $x,y\in X$, $R(a,1)=R(1,a)=1$, for all $a\in[0,1]$ and $\lim_{a\to 0+} R(a,a)=0$. Let $A:X\to X$ be continuous in the (ε,λ)-topology, $m\in\mathcal{M}$ and $g:[0,\infty)\to[0,\infty)$ so that $\lim_{n\to\infty} g^n(u)=0$ for all $u>0$ and the following condition holds:*

For any $x\in X$ there exists $n(x)\in N$ such that for any $v\in O_A(x;0,\infty)$ the following implication holds:

$$(5)\qquad s>0,\ d(x,v)(s)<m(s)\Rightarrow 1-F_{A^{n(x)}x,A^{n(x)}v}(g(s))<m(g(s)).$$

Then there exists $x\in X$ such that $Ax=x$.

Proof. From (5) it follows that A is an (n,f,g)-locally contraction, where $f(s)=1-s$, $s\in[0,1]$, $m_2(s)=s$ and $m_1(s)=m(s)$, $s\geq 0$. In order to prove that (a) in the Definition 2 holds suppose that for some $r>0$, $x\in X$ and $v\in O_A(x;0,\infty)$:

$$1-F_{x,v}(r)<m(r).$$

If $r\geq\lambda_1(x,v)$ then $1-F_{x,v}(r)=d(x,v)(r)<m(r)$ and (5) implies that

$$1-F_{A^{n(x)}x,A^{n(x)}v}(g(r))<m(g(r)).$$

If $r\leq\lambda_1(x,v)$ then $F_{x,v}(r)=0$ and so $m(r)>1$. This implies that $d(x,v)(r)<m(r)$ and so

$$1-F_{A^{n(x)}x,A^{n(x)}v}(g(r))<m(g(r)).$$

Hence from Theorem 1 it follows that there exists $x\in X$ so that $Ax=x$.

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REZIME

O $(n, f, g)-$ LOKALNOJ KONTRAKCIJI U VEROVATNOSNIM METRIČKIM PROSTORIMA

T.Hicks je uveo u [3] pojam $C-$kontrakcije u verovatnosnom metričkom prostoru $(S, \mathcal{F}, T)$. Ako je $t-$norma T min Hicks je dokazao teoremu o nepokretnoj tački za $C-$kontrakciju.U ovom radu je uveden pojam $(n, f, g)-$ lokalne kontrakcije,kao uopštenje pojma $C-$ kontrakcije i $f-$ kontrakcije u smislu V.Radua [5].Dokazane su dve teoreme o nepokretnoj tački za $(n, f, g)-$ lokalnu kontrakciju.

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