Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 22, 1 (1992), 11-22

Review of Research
Faculty of Science
Mathematics Series

FIXED POINT THEOREMS FOR MULTIVALUED NON-SELF MAPPINGS

Chumki Samanta
S. N. G. College
87, Biplabi Phanindra Mohan Bandyopadhyay Saranee
P. O. – Bhadrakali, Dt – Hooghly
PIN – 712 232
INDIA

S. K. Samanta
Department of Mathematics
VISVA–BHARATI
Santiniketan – 731 235
INDIA

Abstract

In this paper the authors considered the multivalued non - self mappings and proved a fixed point theorem which as its corollary gives its single - valued analogous result of S.K. Samanta,[8].

AMS Mathematics Subject Classification (1991): 47H10
Key words and phrases: multivalued mappings, fixed point.

Introduction

Let X be a normed linear space and K be a nonempty, closed, bounded, convex subset of X. Let $\Psi : K \to P(X)$ (where $P(X)$ denotes the family of nonempty subsets of X) be a multivalued mapping. Then a point $x \in K$ is called a fixed point of Ψ provided that $x \in \Psi(x)$. Such type of mappings are said to be multivalued non-self mappings whereas the mappings $\Psi : K \to P(K)$ are said to be multivalued self-mappings. The study of fixed point problems of multivalued mappings was initiated by Kakutani [4] in 1941. Later several authors have worked on this setting. For instance, S. B. Nadler Jr. [6] has generalized the Banach contraction principle theorem to multivalued mappings. Subsequently N. A. Assad and W. A. Kirk [1] have worked with contractive type mappings and J. T. Markin [5] pursued the theory to nonexpansive mappings in multivalued setting. In this paper we have considered the multivalued non-self mappings and proved a fixed point theorem which as its corollary gives its single-valued analogous result of S. K. Samanta [8]. Let

$CB(X)$ = the collection of all nonempty bounded subsets of X,
2^X = the collection of all nonempty compact subsets of X,
2^{-X} = the collection of all nonempty compact convex subsets of X.

Then according to [6], $(CB(X), H)$ is a metric space with respect to the Hausdorff metric H defined by $H(A,B) = \inf\{\alpha > 0 : A \subset N(\alpha, B), B \subset N(\alpha, A)\}$, where $A, B \in CB(X)$ and $N(\alpha, A) = \{x \in X : (\exists y \in A)\|x-y\| < \alpha\}$. A multivalued mapping $\Psi : K \to 2^X$ is said to be M-type non-self mapping if for $x, y \in K$,

$$H(\Psi(x), \Psi(y)) \leq a[D(x, \Psi(x)) + D(y, \Psi(y))] + b\|x-y\|,$$

where $0 \leq 2a + b \leq 1$ and $D(x, A) = \inf\{d(x,a) : a \in A\}$, $x \in X$, $A \in CB(X)$. In [8] S. K. Samanta has proved fixed point theorems for single-valued M-type non-self mappings over a Hilbert space and satisfying a boundary condition as $T(\operatorname{Bdr} K) \subset K$, ($\operatorname{Bdr} K$ denotes the boundary of K). In section 1 we have extended this result for multivalued mappings. To prove our theorem we have used an approximation mapping $R : X \to K$ defined by $R(x) = y$ where $\|x-y\| = D(x, K)$, $x \in K$ and K is nonempty closed convex subset of a Hilbert space X. Clearly R/K is an identity mapping on K and it is known from [2] that R is nonexpansive on X. In section 2 we have given a scheme of iteration for the determination of fixed points of M-type mappings in multivalued setting.

1.

Theorem 1. *Let K be a nonempty, closed, bounded and convex subset of a real Hilbert space X, and $\Psi : K \to 2^{-X}$ be a M-type mapping. If $\Psi(x) \subset K$ for $x \in \operatorname{Bdr} K$, then Ψ has a fixed point in K.*

We shall first prove the following Lemmas:

Lemma 1. *Let K be a nonempty, closed, bounded and convex subset of a real Hilbert space X, and $\Psi : K \to 2^X$ be a M-type mapping. If $\Psi(x) \subset K$ for $x \in \operatorname{Bdr} K$, then for any point $x_0 \in K$, there exist sequences $\{x_n\}$ and $\{y_n\}$ where $x_{n+1} = R(y_n)$ and $y_n \in \Psi(x_n)$ such that $\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0$.*

Proof. Let $x_0 \in K$. Then two cases arise.

Case 1: $\Psi(x_0) \bigcap K \neq \emptyset$ and *Case 2:* $\Psi(x_0) \bigcap K = \emptyset$

In *Case 1*, choose $y_0 \in \Psi(x_0) \bigcap K$. Then $y_0 = R(y_0) = x_1$ (say). Now,

$$H(\Psi(x_1), \Psi(x_0)) \leq a[D(x_1, \Psi(x_1)) + D(x_0, \Psi(x_0))] + b\|x_1 - x_0\|$$

Since $x_1 = y_0 \in \Psi(x_0)$, there exists $y_1 \in \Psi(x_1)$ such that

$$\begin{aligned} \|y_1 - x_1\| &\leq H(\Psi(x_1), \Psi(x_0)) \\ &\leq a[D(x_1, \Psi(x_1)) + D(x_0, \Psi(x_0))] + b\|x_1 - x_0\| \\ &\leq a(\|x_1 - y_1\| + \|x_0 - y_0\|) + b\|y_0 - x_0\| \end{aligned}$$

So,

(1) $$\|x_1 - y_1\| \leq \frac{a+b}{1-a}\|x_0 - y_0\|$$

Therefore

(2) $$\|x_1 - y_1\| \leq \|x_0 - y_0\|$$

In *Case 2*, choose $y_0 \in \Psi(x_0)$ and let $R(y_0) = x_1$. Then $x_1 \in \operatorname{Bdr} K$ and hence $\Psi(x_1) \subset K$. Now,

$$H(\Psi(x_1), \Psi(x_0)) \leq a[D(x_1, \Psi(x_1)) + D(x_0, \Psi(x_0))] + b\|x_1 - x_0\|$$

Since $y_0 \in \Psi(x_0)$ there exists $y_1 \in \Psi(x_1)$ such that

$$\|y_1 - y_0\| \leq H(\Psi(x_1), \Psi(x_0)) \leq a[D(x_1, \Psi(x_1)) + D(x_0, \Psi(x_0))] + b\|x_1 - x_0\|$$

Thus $\|y_1 - y_0\| \leq a(\|x_1 - y_1\| + \|x_0 - y_0\|) + b\|x_1 - x_0\|$. Since $y_1 \in \Psi(x_1) \subset K$, $y_1 = R(y_1)$, we have $\|x_1 - y_1\| = \|R(y_0) - R(y_1)\| \leq \|y_0 - y_1\| \leq H(\Psi(x_0), \Psi(x_1))$. So

$$\|x_1 - y_1\| \leq H(\Psi(x_0), \Psi(x_1)) \leq a(\|x_0 - y_0\| + \|x_1 - y_1\|) + b\|x_0 - x_1\|$$

Since $\|x_1 - x_0\| = \|R(y_0) - R(x_0)\| \leq \|y_0 - x_0\|$, we have $\|x_1 - y_1\| \leq a(\|x_1 - y_1\| + \|x_0 - y_0\|) + b\|x_0 - y_0\|$. So

$$\|x_1 - y_1\| \leq \frac{a+b}{1-a}\|x_0 - y_0\| \tag{3}$$

Therefore

$$\|x_1 - y_1\| \leq \|x_0 - y_0\| \tag{4}$$

Thus in any case,

$$\left\{ \begin{array}{lll} \|x_1 - y_1\| & \leq & H(\Psi(x_1), \Psi(x_0)) \\ \|x_1 - y_1\| & \leq & \frac{a+b}{1-a}\|x_0 - y_0\| \\ \|x_1 - y_1\| & \leq & \|x_0 - y_0\| \end{array} \right. \tag{5}$$

Repeating this argument, we can construct the sequences $\{x_i\}$ and $\{y_i\}$, with $x_{i+1} = R(y_i)$, $y_i \in \Psi(x_i)$ such that for any $i \geq 0$,

$$\|x_{i+1} - y_{i+1}\| \leq H(\Psi(x_i), \Psi(x_{i+1})) \tag{6}$$

$$\|x_{i+1} - y_{i+1}\| \leq \frac{a+b}{1-a}\|x_i - y_i\| \tag{7}$$

$$\|x_{i+1} - y_{i+1}\| \leq \|x_i - y_i\| \tag{8}$$

The rest of the proof is similar to that of the single-valued case given in [8] □

Lemma 2. (Lemma 1 of K. Goebel, W. A. Kirk and T. N. Shimi in [3]) *Let B_R denote the closed spherical ball in X centered at the origin with radius $R \geq 0$. Let $\epsilon > 0$ and $d > 0$. If x_1, x_2, $x_3 \in B_R$ satisfy $d + \epsilon \geq \|x_1 - x_2\| \geq d' > 0$ and $d + \epsilon \geq \|x_2 - x_3\| \geq d' > 0$ and if*

$$\|x_2\| \geq \left(1 - \frac{1}{2}\delta\left(\frac{d'}{R}\right)\right) R$$

then

$$\|x_1 - x_3\| \le \eta\left(1 - \frac{1}{2}\delta\left(\frac{d'}{R}\right)\right)(d+\epsilon)$$

where δ *denotes the modulus of convexity of* X *and* η *denotes the inverse of* δ. □

Lemma 3. *Let* $\Psi : K \to 2^{-X}$ *be a M-type mapping satisfying* $\Psi(x) \subset K$ *for* $x \in \operatorname{Bdr} K$. *For* $x \in X$ *let* $l(x) = \inf_{y \in \Psi(x)} \|x - y\|$. *Then*

$$\inf_{x \in K}\{l(x)\} = 0 \tag{9}$$

Proof. Let $d = \inf_{x\in K}\{l(x)\}$. If possible, let $d > 0$. Choose $\epsilon' > 0$ such that $d' = d - \epsilon' > 0$. Take $\epsilon > 0$ arbitrary. Then there is $x_0 \in K$ such that $d \le \|x_0 - y_0\| \le d + \epsilon$, where $y_0 \in \Psi(x_0)$. If $x_{i+1} = R(y_i)$ where $y_i \in \Psi(x_i)$ such that $\|x_{i+1} - y_{i+1}\| \le \|x_i - y_i\|$ for $i \ge 0$, put $M' = \limsup \|x_i\|$. Since $d > 0$ we have $M' > 0$. Choose $M > M'$ satisfying $(1 - \frac{1}{2}\delta(d'/M))M < M'$. Then for any positive integer i_0 there is an index $i \ge i_0$ such that $\|x_{i-1}\| \le M$, $\|x_i\| \le M$, $\|x_{i+1}\| \le M$, $r_{i-1} < \epsilon'$ (where $r_i = \|x_{i+1} - y_i\|$) and

$$\|x_i\| > (1 - \frac{1}{2}\delta(d'/M))M \tag{10}$$

Proceeding as in the case of single-valued mappings given in [8] we get,

$$\begin{cases} \|x_{i-1} - x_i\| & \le & d + \epsilon \\ \|x_i - x_{i+1}\| & \le & d + \epsilon \end{cases} \tag{11}$$

From (11) applying Lemma 2 we deduce that

$$\|x_{i+1} - x_{i-1}\| \le \eta(1 - \frac{1}{2}\delta(d'/M))(d+\epsilon) \tag{12}$$

Let $m = (x_i + x_{i+1})/2$. Since x_i, $x_{i+1} \in K$ and K is convex, then $m \in K$. Now,

$$H(\Psi(x_{i-1}), \Psi(m)) \le a[D(x_{i-1}, \Psi(x_{i-1})) + D(m, \Psi(m))] + b\|x_{i-1} - m\|$$

Since $y_{i-1} \in \Psi(x_{i-1})$, there is $m_1 \in \Psi(m)$ such that

$$\begin{aligned} \|y_{i-1} - m_1\| &\le H(\Psi(x_{i-1}), \Psi(m)) \\ &\le a(\|x_{i-1} - y_{i-1}\| + D(m, \Psi(m))) + b\|x_{i-1} - m\| \end{aligned} \tag{13}$$

Similarly there is $m_2 \in \Psi(m)$ such that

$$\begin{aligned}(14) \qquad \|y_i - m_2\| &\le H(\Psi(x_i), \Psi(m)) \\ &\le a(\|x_i - y_i\| + D(m, \Psi(m))) + b\|x_i - m\|\end{aligned}$$

Let $m' = (m_1 + m_2)/2$. Since m_1, $m_2 \in \Psi(m)$ and $\Psi(m)$ is convex, then $m' \in \Psi(m)$. Thus by using (13) and (14), we get,

$$\begin{aligned}\|m - m'\| - a\,D(m, \Psi(m)) &\le \frac{a}{2}(\|x_{i-1} - y_{i-1}\| + \|x_i - y_i\|) \\ &+ \frac{b}{2}(\|x_{i-1} - \frac{x_i + x_{i+1}}{2}\| + \|x_i - \frac{x_i + x_{i+1}}{2}\|) \\ &+ \frac{r_i + r_{i-1}}{2}\end{aligned}$$

Using (11) and (12) we get,

$$\begin{aligned}(1-a)d &\le (1-a)\|m - m'\| \\ &\le (a + \frac{b}{4} + \frac{b}{4})(d + \epsilon) + \frac{1}{2}(r_i + r_{i-1}) \\ &+ \frac{b}{4}\eta(1 - \frac{1}{2}\delta(d'/M))(d + \epsilon)\end{aligned}$$

Arguing as in [8] and using Lemma 1, we arrive at a contradiction. This completes the proof of Lemma 3. □

Proof of the Theorem 1 By Lemma 3, there exist sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_n \|x_n - y_n\| = 0$, where $y_n \in \Psi(x_n)$. Since K is a bounded, closed, convex subset of a Hilbert space and since $\{x_n\}$ is a sequence in K, there is a subsequence $\{x_{n_i}\}$ (say) of $\{x_n\}$ converging weakly to some $\xi \in K$. Now,

$$(15) \quad H(\Psi(x_{n_i}), \Psi(\xi)) \le a[D(x_{n_i}, \Psi(x_{n_i})) + D(\xi, \Psi(\xi))] + b\|x_{n_i} - \xi\|$$

Since $y_{n_i} \in \Psi(x_{n_i})$, there is $\xi_i \in \Psi(\xi)$ such that we get from (15),

$$\begin{aligned}\|y_{n_i} - \xi_i\| &\le H(\Psi(x_{n_i}), \Psi(\xi)) \\ &\le a(\|x_{n_i} - y_{n_i}\| + \|\xi - \xi_i\|) + b\|x_{n_i} - \xi\| \\ &\le a(\|x_{n_i} - y_{n_i}\| + \|\xi - x_{n_i}\| + \|x_{n_i} - y_{n_i}\| + \|y_{n_i} - \xi_i\|) \\ &\quad + b\|x_{n_i} - \xi\|\end{aligned}$$

So,

$$\|y_{n_i} - \xi_i\| \leq \frac{2a}{1-a}\|x_{n_i} - y_{n_i}\| + \frac{a+b}{1-a}\|x_{n_i} - \xi\| \tag{16}$$

Since $\Psi(\xi)$ is compact, $\{\xi_i\}$ has a strongly convergent subsequence which, without loss of generality, may be taken to be the whole sequence $\{\xi_i\}$ and let $\eta = \lim_i \xi_i$. If possible let $\eta \neq \xi$. Now by (16)

$$\begin{aligned} \|x_{n_i} - \eta\| &\leq \|x_{n_i} - y_{n_i}\| + \|y_{n_i} - \xi_i\| + \|\xi_i - \eta\| \\ &\leq \|x_{n_i} - y_{n_i}\| + \frac{2a}{1-a}\|x_{n_i} - y_{n_i}\| + \frac{a+b}{1-a}\|x_{n_i} - \xi\| + \|\xi_i - \eta\| \end{aligned}$$

So,

$$\|x_{n_i} - \eta\| \leq \frac{1+a}{1-a}\|x_{n_i} - y_{n_i}\| + \frac{a+b}{1-a}\|x_{n_i} - \xi\| + \|\xi_i - \eta\|$$

Thus

$$\liminf \|x_{n_i} - \eta\| \leq \liminf \|x_{n_i} - \xi\| \tag{17}$$

(which follows from $\lim_i \|x_{n_i} - y_{n_i}\| = 0$, $\lim_i \|\xi_i - \eta\| = 0$, $\frac{a+b}{1-a} \leq 1$).

Since X is a Hilbert space it satisfies Opial's condition (See [7]), and hence

$$\liminf \|x_{n_i} - \xi\| < \liminf \|x_{n_i} - \eta\|$$

which contradicts (17). Hence $\xi = \eta \in \Psi(\xi)$; i.e. ξ is a fixed point of Ψ. □

Corollary 1. [8] *Let K be a nonempty, closed, convex subset of a real Hilbert space X and $T : K \to K$ be such that for x, $y \in K$, $\|T(x) - T(y)\| \leq a(\|x - T(x)\| + \|y - T(y)\|) + b\|x - y\|$, where $0 \leq 2a + b \leq 1$. If $T : \operatorname{Bdr} K \to K$, then T has a fixed point in K.* □

However Theorem 1 cannot be generalized for the mapping satisfying $H(\Psi(x), \Psi(y)) \leq \max[D(x, \Psi(x)), D(y, \Psi(y)), \|x - y\|]$ even if it is a single-valued and self-mapping. This is shown by the following example.

Example 1. Let X be the real line and $K = [0, 1]$. Let $T : K \to K$ be such that

$$T(x) = \begin{cases} \frac{1}{2} + \frac{x}{2}, & 0 \leq x < 1 \\ \frac{1}{2}, & x = 1 \end{cases}$$

Then for all x, $y \in K$ it can be easily seen that

$$\|T(x) - T(y)\| \leq \max\{\|x - y\|, \|x - T(x)\|, \|y - T(y)\|\}$$

But, T has no fixed point.

2.

In this section we take K to be a closed, convex subset of a uniformly convex Banach space X and $\Psi : K \to 2^{-X}$ to be a mapping such that $\Psi(x) \subset K$, for $x \in \text{Bdr}\, K$. We also suppose that the mapping Ψ to be M-type mapping over K i.e. $H(\Psi(x), \Psi(y)) \leq a[D(x, \Psi(x)) + D(y, \Psi(y))] + b\|x - y\|, \forall x, y \in K$. If z, $z' \in K$, then (z, z') denotes the open line segment $\{\alpha z + (1-\alpha)z', 0 < \alpha < 1\}$ joining z and z'. Define another multivalued mapping Ψ' over K by the rule:

$$\Psi'(x) = \begin{cases} \bigcup\{(x,y) \bigcap \text{Bdr}\, K : y \in \Psi(x)\} & \text{if } \Psi(x) \bigcap K = \emptyset \\ \{\Psi(x) \bigcap K\} \bigcup G(x) & \text{otherwise} \end{cases}$$

where $G(x) = \bigcup\{(x,y) \bigcap \text{Bdr}\, K : y \in \Psi(x) \bigcap K\}$. Further, we assume that Ψ has a fixed point, say, p. Now construct sequences $\{x_n\}$, $\{y_n\}$ and $\{w_n\}$ as follows:

Take any point in $x \in K$ and call it x_0. Choose $y_0 \in \Psi(x_0)$ such that $\|y_0 - p\| \leq H(\Psi(x_0), \Psi(p))$. Put $w_0 = y_0$ if $y_0 \in K$. Otherwise, choose $w_0 = (1 - t_0)x_0 + t_0 y_0$ with $0 < t_0 < 1$, such that $w_0 \in \Psi'(x_0)$. Put $x_1 = (x_0 + w_0)/2$. Continuing in this manner, we construct sequences $\{x_n\}$, $\{y_n\}$ and $\{w_n\}$ such that $\|y_n - p\| \leq H(\Psi(x_n), \Psi(p))$, $w_n = y_n$ if $y_n \in K$, otherwise $w_n = (1 - t_n)x_n + t_n y_n \in \Psi'(x_n)$ for some t_n with $0 < t_n < 1$ and $x_{n+1} = (x_n + w_n)/2$, $n = 0, 1, 2, \ldots$ We shall call $\{x_n\}$ to be an iterative orbit of the mapping Ψ. In the Theorem 2 below, an iterative orbit $\{x_n\}$ of Ψ will be used to determine a fixed point of Ψ when it exists.

Theorem 2. *Let K be a closed, convex subset of a uniformly convex Banach space X and $\Psi : K \to 2^{-X}$ be a M-type mapping, such that*

(*i*) $\Psi(x) \subset K, \forall x \in \text{Bdr}\, K$
(*ii*) *$\Psi'(K)$ is compact*
(*iii*) *Ψ has a fixed point*

then for each $x \in K$, there is an iterative orbit $\{x_n\}$ of Ψ (as constructed above), which has a subsequence converging to a fixed point of Ψ, provided that $\{t_n\}$ has a limit point other than 0.

Proof. Let $x_0 = x \in K$ and p be a fixed point of Ψ. By construction, then, we get $y_0 \in \Psi(x_0)$ such that

$$\|y_0 - p\| \leq H(\Psi(x_0), \Psi(p))$$

$$\begin{aligned} &\leq a[D(x_0, \Psi(x_0)) + D(p, \Psi(p))] + b\|x_0 - p\| \\ &\leq a\|x_0 - y_0\| + b\|x_0 - p\| \\ &\leq a\|x_0 - p\| + a\|y_0 - p\| + b\|x_0 - p\| \end{aligned}$$

i.e. $\|y_0 - p\| \leq \frac{a+b}{1-a}\|x_0 - p\|$. So,

$$\|y_0 - p\| \leq \|x_0 - p\| \tag{18}$$

Take $w_0 \in \Psi(x_0)$ such that

$$w_0 = \begin{cases} y_0, & \text{if } y_0 \in \Psi(x_0) \bigcap K \\ (1-t_0)x_0 + t_0 y_0, 0 < t_0 < 1, & \text{if } y_0 \notin \Psi(x_0) \bigcap K \end{cases}$$

In both the cases, using (18), we get

$$\|w_0 - p\| \leq \|x_0 - p\| \tag{19}$$

Put $x_1 = \frac{1}{2}(x_0 + w_0)$. For $n = n_0$, assume that we have got $x_{n_0} \in K$, $y_{n_0} \in \Psi(x_{n_0})$ and $w_{n_0} \in \Psi'(x_{n_0})$ such that

1) $\|y_{n_0} - p\| \leq \|x_{n_0} - p\|$,
2) $w_{n_0} = \begin{cases} y_{n_0}, & \text{if } y_{n_0} \in \Psi(x_{n_0}) \bigcap K \\ (1-t_{n_0})x_{n_0} + t_{n_0} y_{n_0}, 0 < t_0 < 1, & \text{if } y_{n_0} \notin \Psi(x_{n_0}) \bigcap K \end{cases}$,
3) $\|w_{n_0} - p\| \leq \|x_{n_0} - p\|$.

Now, put $x_{n_0+1} = \frac{1}{2}(x_{n_0} + w_{n_0})$. By construction, $y_{n_0+1} \in \Psi(x_{n_0+1})$ is so choosen that

$$\begin{aligned} \|y_{n_0+1} - p\| &\leq H(\Psi(x_{n_0+1}), \Psi(p)) \\ &\leq a[D(x_{n_0+1}, \Psi(x_{n_0+1})) + D(p, \Psi(p))] + \|x_{n_0+1} - p\| \end{aligned}$$

Using triangle inequality of the norm, we get, $\|y_{n_0+1} - p\| \leq \frac{a+b}{1-a}\|x_{n_0+1} - p\| \leq \|x_{n_0+1} - p\|$. Now, $w_{n_0+1} \in \Psi'(x_{n_0+1})$ is such that $w_{n_0+1} = y_{n_0+1}$, if $y_{n_0+1} \in \Psi(x_{n_0+1}) \bigcap K$ and $w_{n_0+1} = (1 - t_{n_0+1})x_{n_0+1} + t_{n_0+1} y_{n_0+1}$, otherwise. Using 3) we have $\|w_{n_0+1} - p\| \leq \|x_{n_0+1} - p\|$. Thus by mathematical induction, we get sequences $\{x_n\}$, $\{y_n\}$ and $\{w_n\}$ satisfying for all $n \geq 0$,

4) $y_n \in \Psi(x_n)$ and $\|y_n - p\| \leq \|x_n - p\|$,
5) $w_n = y_n$ or $w_n = (1 - t_n)x_n + t_n y_n, (0 < t_n < 1)$
   if $y_n \in \Psi(x_n) \bigcap K$ or $y_n \notin \Psi(x_n) \bigcap K$, respectively
6) $\|w_n - p\| \leq \|x_n - p\|$,
7) $x_{n+1} = (x_n + w_n)/2$

Now, if possible, let there exist an $\epsilon > 0$ and a subsequence $\{n_i\}$ (which without loss of generality may be taken to be strictly increasing) of $\{n\}$ such that

(20) $$\|w_{n_i} - x_{n_i}\| > \epsilon, \text{ for all } i$$

Again, $\|x_{n+1} - p\| = \|(x_n + w_n)/2 - p\| \leq \frac{1}{2}(\|x_n - p\| + \|w_n - p\|) \leq \|x_n - p\|$ (using 6)). Thus $\{\|x_n - p\|\}$ is a decreasing sequence of nonnegative terms and therefore convergent. From (20), $\|(w_{n_i} - p) - (x_{n_i} - p)\| > \epsilon$. By uniform convexity of X, there is a positive $\delta < 1$ such that

$$\|\frac{1}{2}(x_{n_i} + w_{n_i}) - \frac{1}{2}(p + p)\| \leq \delta \max\{\|x_{n_i} - p\|, \|w_{n_i} - p\|\}$$

i.e.

$$\begin{aligned} \|x_{n_i+1} - p\| &\leq \delta\|x_{n_i} - p\| \\ &\leq \delta\|x_{n_{i-1}+1} - p\| \\ &\leq \delta^2\|x_{n_{i-1}} - p\| \leq \ldots \leq \delta^i\|x_0 - p\| \end{aligned}$$

So, $\|x_{n_i} - p\| \to 0$ as $i \to \infty$. Since $\{\|x_n - p\|\}$ is decreasing, it therefore converges to 0. Now, using 7), $\|x_n - w_n\| \leq \|x_n - p\| + \|w_n - p\| \leq 2\|x_n - p\| \to 0$ as $n \to \infty$, a contradiction to (20). Thus we have,

(21) $$\lim_{n\to\infty} \|x_n - w_n\| = 0$$

Let $\{t_m\}$ be a convergent subsequence of $\{t_n\}$ converging to t (say), with $0 < t < 1$. Using compactness of $\Psi'(K)$, take a convergent subsequence $\{w_{m_i}\}$ of $\{w_m\}$. Then $u = \lim_{i\to\infty} w_{m_i} \in \Psi'(K)$. By 5), if $y_{m_i} \in \Psi(x_{m_i}) \bigcap K$, then $\|w_{m_i} - y_{m_i}\| = \|((1 - t_{m_i})x_{m_i} + t_{m_i}y_{m_i} - y_{m_i}\| = (1 - t_{m_i})\|x_{m_i} - y_{m_i}\| \leq (1 - t_{m_i})(\|x_{m_i} - w_{m_i}\| + \|w_{m_i} - y_{m_i}\|)$. By 6), we get $\|w_{m_i} - y_{m_i}\| \leq \frac{1-t_{m_i}}{t_{m_i}}\|x_{m_i} - w_{m_i}\| \to 0$ as $i \to 0$ (since $\lim_{i\to\infty} t_{m_i} > 0$). On the other hand, if $y_{m_i} \in \Psi(x_{m_i}) \bigcap K$, then $w_{m_i} = y_{m_i}$ so that $\|w_{m_i} - y_{m_i}\| = 0$. Again,

(22) $$\|x_{m_i} - u\| \leq \|x_{m_i} - w_{m_i}\| + \|w_{m_i} - u\|$$

Since $u = \lim_{i\to\infty} w_{m_i}$, we have therefore by (21) and (22), $\lim_{i\to\infty} \|x_{m_i} - u\| = 0$. Next, we shall show that $u \in \Psi(u)$. Now,

$$D(u, \Psi(u)) \leq \|u - x_{m_i}\| + \|x_{m_i} - w_{m_i}\| + \|w_{m_i} - y_{m_i}\| + D(y_{m_i}, \Psi(u))$$

$$\leq \|u - x_{m_i}\| + \|x_{m_i} - w_{m_i}\| + \|w_{m_i} - y_{m_i}\| + H(\Psi(x_{m_i}), \Psi(u))$$

$$\leq \|u - x_{m_i}\| + \|x_{m_i} - w_{m_i}\| + \|w_{m_i} - y_{m_i}\|$$

$$+a[D(x_{m_i}, \Psi(x_{m_i})) + D(u, \Psi(u))] + b\|x_{m_i} - u\|$$

i.e. $D(u, \Psi(u)) \leq \frac{1+b}{1-a}\|x_{m_i} - u\| + \frac{1+a}{1-a}(\|x_{m_i} - w_{m_i}\| + \|w_{m_i} - y_{m_i}\|) \to 0$ as $i \to \infty$. Hence, u is a fixed point of Ψ and $\lim_{i\to\infty} x_{m_i} = u$. This completes the proof. □

The authors are thankful to a referee for suggestions in rewriting the paper in the present form.

References

[1] Assad, N. A., Kirk, W. A.: Fixed point theorems for set-valued mappings of contractive type, Pacific J. Math. 43(1972), 553–562

[2] Browder, F. E., Petryshyn, W. V.: Construction of fixed points of nonlinear mappings in Hilbert spaces, J. Math. Anal. Appl. 20(1967), 197–228

[3] Goebel, K., Kirk, W. A., Shimi, T. N.: A fixed point theorem in a uniformly convex Banach space, Boll. Un. Mat. Ital. (4), 7(1973), 67–75

[4] Kakutani, S.: A generalization of Brower's fixed point theorem, Duke Math. J. 8(1941), 457–459

[5] Markin, J. T.: A fixed point theorem for set-valued mappings, Bull. Amer. Math. Soc. 74(1968), 639–640

[6] Nadler, S. B.: Multivalued contraction mappings, Pacific J. Math. 30(2), (1969), 475–488

[7] Opial, Z.: Weak convergence of the sequence of succesive approximation for nonexpansive mappings, Bull. Amer. Math. Soc. 73, 591–597

[8] Samanta, S. K.: Fixed point theorems for nonself mappings, Indian J. Pure. Appl. Math. 15(3), 1984, 221–232

REZIME

TEOREME O NEPOKRETNOJ TAČKI ZA VIŠEZNAČNA PRESLIKAVANJA KOJA NE MORAJU PRESLIKAVATI DEFINICIONI DOMEN U SAMOG SEBE

U ovom radu autori posmatraju višeznačna preslikavanja koja ne moraju preslikavati definicioni domen u samog sebe. Dokazana je teorema o nepoktretnoj tački koja kao posledicu daje rezultat analogan rezultatu iz rada S.K. Samanta, [8], za jednoznačna preslikavanja.

Received by the editors October 12, 1989