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CONVERTING CONTRACTIVE MAPPINGS

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Abstract

Let $\{T_n\}$ be a sequence of mappings from metric space (X,d) into itself which converges pointwise to the contraction mapping T. $T^{n+1} = T_{n+1}T^n$ is identified with $T_1 = T^1$ and the convergence of the sequence $\{T^n x\}$ is investigated for any point $x \in X$.

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1. Introduction

In [1], [2], and [3] the authors considered sequences of functions $\{T_n\}$, $T_n : X \to X$, where (X,d) is a metric space, which converge on X to a contraction mapping T with fixed point a. They supposed that the functions $\{T_n\}$ have a fixed point for each $n = 1,2,3,\ldots$ and proved the convergence of the sequence of these fixed points to the fixed point a of T.

In this article we shall consider such sequences of functions, define as in [4] $T^{n+1} = T_{n+1}T^n$ with $T_1 = T^1$ and study the convergence of the sequence $\{T^n x\}$, which occurs from the consecutive application of the mappings $T_n, n = 1,2,3,\ldots$ to any point $x \in X$.

2. Results in metric spaces

Theorem 1. *Let (X,d) be a metric space and for $n = 1,2,3,\ldots$, let T_n be a contraction mapping. Suppose that the sequence $\{T_n\}$ converges pointwise to a contraction mapping T with fixed point a, and that the sequence of the consecutives, $\{T^n\}$ converges for some point $x_1 \in X$ to a point $x_0 \in X$. Then $x_0 = a$.*

Proof. Let $\varepsilon > 0$ be given. Choose natural numbers m_0, n_0 such that for all $j \geq n_0$ it follows that

$$d(Tx_0, T_jx_0) < \frac{\varepsilon}{3} \tag{2.1}$$

and for all $j \geq m_0$

$$d(T^jx_1, x_0) < \frac{\varepsilon}{3} \tag{2.2}$$

Now if $j \geq \max\{n_0+1, m_0+1\}$ (2.1) and (2.2) give

$$d(Tx_0, x_0) \leq d(Tx_0, T_jx_0) + d(T_jx_0, T^jx_1) + d(T^jx_1, x_0) <$$

$$< \frac{\varepsilon}{3} + k_j\, d(x_0, T^{j-1}x_1) + \frac{\varepsilon}{3} < \varepsilon,$$

where k_j is the Lipschitz constant of T_j.

The next theorem deals with contraction mappings T_n, such that $d(T_nx, T_ny) \leq k_nd(x,y)$, restricted to the condition: $\sup\{k_n\} = k < 1$, [1], [2], [3].

Theorem 2. *Let (X,d) be a metric space and $\{T_n\}$ be a sequence of contractions with the same constant $k < 1$. Assume that $\{T_n\}$ converges pointwise to T. Then T is a contraction with constant k and $\{T^nx\}$ converges to the fixed point, say a, of T for every $x \in X$.*

Proof. The poitwise convergence of $\{T_n\}$ to T gives

$$d(Tx, Ty) \leq d(Tx, T_nx) + d(T_nx, T_ny) + d(T_ny, Ty) \leq$$

$$\leq \varepsilon + kd(x,y) + \varepsilon.$$

Since this is true for every ε we get

$$d(Tx, Ty) \leq kd(x, y)$$

which implies that T is a contraction with constant k. Now we give the proof of the Theorem in two steps.

Step I. We shall show first that for all $x \in X$ the set $\{T^n x\}$ is bounded. Suppose $x \in X$ is given. Choose $r > 0$ such that:

$$x \in K(a, r) \equiv \{x \in X : d(a, x) \leq r\}$$

Then there exists $m_0 \in \mathbf{N}$ such that $n \geq m_0$ implies:

$$d(T_n a, a) < (1 - k)r.$$

So for all such n we will have:

$$(2.1) \quad d(T_n x, a) \leq d(T_n x, T_n a) + d(T_n a, a) < kd(x, a) + (1 - k)r < r$$

and thus if $n \geq m_0, T_n$ maps $K(a, r)$ into itself.

Now for the functions T_i, $i = 1, 2, 3, \ldots, m_0 - 1$, if $d(x, a) > \frac{d(T_i a, a)}{1-k}$ we have:

$$(2.2) \quad \begin{aligned} d(T_i x, a) &\leq d(T_i x, T_i a) + d(T_i a, a) \leq \\ &\leq kd(x, a) + d(T_i a, a) \leq d(x, a) \end{aligned}$$

and if $d(x, a) < \frac{d(T_i a, a)}{1-k}$

$$(2.3) \quad \begin{aligned} d(T_i x, a) &\leq kd(x, a) + d(T_i a, a) < \\ &< k\frac{d(T_i a, a)}{1-k} + d(T_i a, a) = \frac{d(T_i a, a)}{1-k} \end{aligned}$$

From (2.2) and (2.3) we see that if $r_i \geq \frac{d(T_i a, a)}{1-k}$ and $x \in K(a, r_i)$ it follows that $T_i x \in K(a, r_i)$.

Thus if we define $R(x)$ with

$$R(x) = \max\{r, \frac{d(T_i a, a)}{1-k} \quad i = 1, 2, 3, \ldots, m_0 - 1\}$$

from equations (2.1), (2.2) and (2.3) we have that $x \in K(a, R(x))$ implies $T_n x \in K(a, R(x))$ for all $n = 1, 2, 3, \ldots$
It is now clear that we also have: $d(T^n x, a) \leq R(x)$ for $n = 1, 2, 3, \ldots$ and for all $x \in K(a, R(x))$.

Step II. Let $\varepsilon > 0$ be given. Choose $k_1 \in \mathbf{R}$ such that $0 < k < k_1 < 1$. Then for $\varepsilon' = (k_1 - k)\varepsilon > 0$ there exists a positive integer n_0 such that $n \geq n_0$ gives $d(T_n a, a) < \varepsilon'$. So if $d(T^{n-1}x, a) > \frac{\varepsilon'}{k_1 - k} = \varepsilon$ we have

$$d(T^n x, a) = d(T_n T^{n-1} x, a) \leq d(T_n T^{n-1} x, T_n a) + d(T_n a, a) <$$

(2.4)

$$< k d(T^{n-1} x, a) + \varepsilon' \leq k_1 d(T^{n-1} x, a)$$

And if $d(T^{n-1}x, a) < \frac{\varepsilon'}{k_1 - k} = \varepsilon$, it follows that

$$d(T^n x, a) < k d(T^{n-1} x, a) + \varepsilon' < \frac{k_1 \varepsilon'}{k_1 - k} < \varepsilon \tag{2.5}$$

Equations (2.4) and (2.5) imply that there exists an $N \in \mathbf{N}$ such that $n \geq N$ gives

$$d(T^n x, a) < \max\{\varepsilon, k_1^{n-n_0+1} R(x))\} = \varepsilon$$

This completes the proof.

Example 1 of the next paragraph shows that the condition $\sup\{k_n\} = k < 1$ in Theorem 2 cannot be omitted.

3. Main results

We say that the space X has the Heine-Borel property if every closed and bounded subset of X is compact.

Theorem 3. *Let (X, d) be a metric space which has the Heine-Borel property, let $\{T_n\}$ be a sequence of contraction mappings from X into X such that $d(T_n x, T_n y) \leq k_n d(x, y)$, and let $T : X \to X$ be contraction mapping with Lipschitz constant k and fixed point a. If the sequence $\{T_n\}$ converges pointwise to T then the sequence $\{T^n x\}$ converges to a for every $x \in X$.*

Proof. The set $K(a,r) \equiv \{x \in X : d(a,x) \leq r\}$ is a compact subset of X. Thus since $\{T_n\}$ is an equicontinuous sequence of functions converging pointwise to T, it follows that the sequence $\{T_n\}$ converges uniformly on $K(a,r)$ to T. We choose n_0 such that if $n \geq n_0$, then $d(T_n x, Tx) < (1-k)r$. Then if $n \geq n_0$ and $x \in K(a,r)$

$$d(T_n x, a) \leq d(T_n x, Tx) + d(Tx, a) < (1-k)r + kd(x,a) \leq r.$$

This proves that if $n \geq n_0$, then T_n maps $K(a,r)$ into itself.

Now proceeding as in Theorem 2, for any function T_i, $i = 1, \ldots, n_0 - 1$, if $r_i \geq \frac{d(T_i a, a)}{1-k_i}$ and $x \in K(a, r_i)$ we have $T_i x \in K(a, r_i)$. So if we define $R(x)$ by

$$R(x) = \max\{r, \frac{d(T_i a, a)}{1-k_i},\ i = 1, 2, 3, \ldots, n_0 - 1\},$$

then $x \in K(a, R(x))$ implies $T_n x, T^n x \in K(a, R(x))$ for all $n = 1, 2, 3, \ldots$. Now we take $k_1 \in \mathbf{R}$ with $0 < k < k_1 < 1$, and let $\varepsilon > 0$ be given. Then there exists an $N \in \mathrm{N}$, such that $n \geq N$ implies

$$d(T_n x, Tx) < \varepsilon' = \varepsilon(k_1 - k)$$

for all $x \in K(a, R(x))$, so that for these x and for $i \geq N$ if $d(T^{i-1}x, a) \geq \frac{\varepsilon'}{k_1 - k}$ we have:

$$d(T^i x, a) \leq d(T_i T^{i-1} x, T T^{i-1} x) + d(T T^{i-1} x, Ta) <$$

$$< \varepsilon' + kd(T^{i-1}x, a) \leq k_1 d(T^{i-1}x, a)$$

and if $d(T^{i-1}x, a) < \frac{\varepsilon'}{k_1 - k}$ it follows that

$$d(T^i x, a) < \varepsilon' + k\frac{\varepsilon'}{k_1 - k} < \varepsilon$$

Thus in both cases there exists an $n_0 \in \mathrm{N}$, $n_0 > N$, such that $n \geq n_0$ gives

$$d(T^n x, a) < \max\{\varepsilon, k_1^{n-N+1} d(T^{N-1}x, a)\} = \varepsilon,$$

and the theorem is proved.

We conclude with an example which shows that in spaces which has not the Heine-Borel property, a sequence of contraction mappings $\{T_n\}$ may converge pointwise to a contraction mapping T with fixed point a, but the sequence $\{T^n x\}$ does not converge to the point a for $x \in K(a,r)$ for any $r > 0$.

Example 1. Proceeding as in [2], let X be the Hilbert space ℓ^2, $X^*(=\ell^2)$ the first conjugate of X and $E = \{f \in X^* : \|f\| \leq 1\}$.

Since E is a weak* sequentially compact, the sequence $\{e^n = (0_1, 0_2, \ldots, 1_n, 0_{n1}, \ldots)\} \subset X^*$ has a weak* convergent subsequence $\{e_{n_i}\}$. Let e be a the weak* limit of $\{e_{n_i}\}$. Then for each $i = 1,2,3,\ldots$ we define

$$f_i = \frac{e_{n_i} - e}{\|e_{n_i} - e\|}.$$

The sequence $\{f_i\}$ is weak* convergent to the zero linear functional and $\|f_i\| = 1$ for all $i = 1,2,3,\ldots$.
For each $i = 1,2,3,\ldots$ let $a_i \in X$ be such that $\|a_i\| = 1$ and

$$|f_i(a_i)| > \frac{1 - \frac{1}{(i+1)^2}}{1 - \frac{1}{(i+1)^3}}$$

and define $T_i : X \to X$ by

$$T_i x = (1 - \frac{1}{(i+1)^3}) f_i(x) a_{i+1}$$

for all $x \in X$. It is easily seen that $\{T_i\}$ converges pointwise to the zero mapping. Since

$$\|T_i x - T_i y\| = (1 - \frac{1}{(i+1)^3}) |f_i(x) - f_i(y)| \, \|a_{i+1}\| \leq (1 - \frac{1}{(i+1)^3}) \, \|x - y\|$$

for all x and y in X, T_i is contraction mapping for each $i = 1,2,3,\ldots$ and we have

$$T^n x = (1 - \frac{1}{(n+1)^3}) \, (1 - \frac{1}{n^3}) \ldots (1 - \frac{1}{2^3}) \, f_1(x) f_2(a_2) \ldots f_n(a_n) a_{n+1}$$

with

$$\|T^n x\| > \prod_{i=2}^{\infty} (1 - \frac{1}{i^2}) \, |f_1(x)| = \frac{1}{2} |f_1(x)|.$$

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REZIME

KONVERTUJUĆE KONTRAKTIVNA PRESLIKAVANJA

Neka je $\{T_n\}$ niz preslikavanja metričkog prostora (X, d) u samog sebe koji konvergira tačkasto ka kontraktivnom preslikavanju T. Definiše se $T^{n+1} = T_{n+1}T^n$ sa $T_1 = T^1$ i ispituje se konvergencija niza $\{T^n x\}$ za ma koje $x \in X$.

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