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A NOTE ON THE REGULARIZATION OF DISTRIBUTIONS

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Abstract

In this paper the regularization of functions and improper integrals is analyzed. Some definitions based on Hadamard's finite part method and Gel'fand-Šilov's regularizations are compared and another modification of Hadamard's finite part is used. In particular, it turns out that this regularization and type II Gel'fand-Šilov regularization (see below) are consistent on a rather wide set of (in general) non-integrable functions.

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1. Two types of Gel'fand-Šilov regularization

In this section we shall compare two regularizations in the manner of Gel'-fand-Šilov. Troughout the paper we shall use the notations from [4]. Let c be a real number and f a locally integrable function on $\mathbf{R}\backslash\{c\}$. Then any distribution T with the property that for each $\phi \in \mathcal{D}$, whose support does not contain the point $c \in \mathbf{R}$

$$(1) \qquad \langle T, \phi \rangle = \int_{-\infty}^{\infty} f(x)\phi(x)dx$$

is called a *regularization* of function f.

It is well known that there exist functions that cannot be regularized; as an example, one can take $f(x) = \exp(1/x^2)$, for x different from zero. A sufficient condition for regularization gives the following.

Theorem 1. *([4],p23) Let f be a locally integrable function on $\mathbf{R}\setminus\{c\}$ such that for some $m \in \mathbf{N}$ the function $(x-c)^m f(x), x \in \mathbf{R}$, becomes locally integrable on $\mathbf{R}$. Then there exist a regularization of the function f, denoted by $\bar{f}$ such that*

$$(2) \quad \langle \bar{f}, \phi \rangle = \int_{-\infty}^{\infty} f(x)(\phi x) - (\sum_{j=0}^{m-1} \frac{\phi^{(j)}(c)}{j!}(x-c)^j)\Theta(\chi - |x-c|))dx,$$

for some $\chi > 0, \chi \neq c$ and each $\phi \in \mathcal{D}$, where Θ denotes the Heaviside function.

We shall call this regularization a *type I regularization.* The last integral is also called the regularization of the improper integral in (1). It is convenient to denote the set of locally integrable functions on $\mathbf{R}$ $\{c\}$ that satisfy the following conditions:

For some $m \in \mathbf{N}$ the function $(x-c)^m f(x), x \in \mathbf{R}$, becomes locally integrable on $\mathbf{R}$ and it holds that

$$\lim_{x\to c}(x-c)^{m+1}f(x) = 0$$

by $\mathcal{R}_m(c)$.

Then we put

$$\mathcal{R}(c) = \bigcup_{m=1}^{\infty} \mathcal{R}_m(c)$$

From now on we shall observe only the case $c = 0$ and we write simply $\mathcal{R}_m$ and $\mathcal{R}$ for $\mathcal{R}_m(0)$ and $\mathcal{R}(0)$ respectively. We shall observe only functions f

with the property $f(x) = 0$ for $x < 0$. This is not a restriction, since any regularizable function f can be written as a sum of two such functions:

$$f(x) = f_1(x) + f_2(-x),\ \text{supp } f_i \subset [0,\infty),\ \ i = 1,2.$$

Then (2) becomes for $f = f_1$

$$(3) \qquad \langle \bar{f}, \phi \rangle = \int_0^\infty f(x)(\phi(x) - (\sum_{j=0}^{m-1} \frac{\phi^{(j)}(0)}{j!} x^j)\Theta(\chi - x))dx$$

Let now $f \in \mathcal{R}_m \backslash \mathcal{R}_{m-1}$ have the property:

$$(4) \qquad x^{m-1} f(x) \in L^1((\chi,\infty))$$

Then, instead of (3), it is handier to use the following regularization

$$(5) \qquad \langle f_s, \phi \rangle = \int_0^\infty f(x)(\phi(x) - \sum_{j=0}^{m-1} \frac{\phi^{(j)}(0)}{j!} x^j)dx$$

for $\phi \in \mathcal{D}$. The regularization f_s (in (5)) will be called a *type II regularization.*

We remark that neither of these two regularizations preserves the addition. However, one can prove that the type II regularization is consistens with the derivative, as it was done in [4].

2 . A modified definition of the regularization

In this we shall analyze another definition of the regularization of functions which reduces to the one given by Estrada and Kanwal ([2]), provided the power functions are used in the one dimensional case. This regularization given by (6) below we shall call *Hadamard's finite part regularization.*

Lemma 1. *If $f \in \mathcal{R}_m$, then the function $f^{(-1)}(x)x^{m-1}$ is locally integrable on $\mathcal{R}$.*

Proof. The function $f^{(-1)}(x)x^{m-1}$ is continuous on $\mathbf{R}$. Using partial integration, we have for $0 < \epsilon < \chi$

$$\int_\epsilon^\chi f(x)x^m dx = f^{(-1)}(\chi)\chi^m - f^{(-1)}(\epsilon)\epsilon^m -$$

$$-m\int_{\epsilon}^{\chi} f^{(-1)}(x)x^{m-1}dx.$$

Since $f(x)\in\mathcal{R}_m$, we have

$$\lim_{\epsilon\to 0} f^{(-1)}(\epsilon)\epsilon^m = \lim_{\epsilon\to 0}\frac{f^{(-1)}(\epsilon)}{\epsilon^{-m}} =$$

$$= \lim_{\epsilon\to 0}\frac{f(\epsilon)}{-m\epsilon^{-m-1}} = -m^{-1}\lim_{\epsilon\to 0} f(\epsilon)\epsilon^{m+1} = 0,$$

so $f^{(-1)}x^{m-1}\in L^1_{loc}(\mathbf{R})$. Q.E.D.

Let $\mathcal{R}'_m$ be a set of all functions from $\mathcal{R}_m$ such that for every $i, 1\le i\le m-1, f^{(-i)}\in\mathcal{R}_{m-i}$. Let $\mathcal{R}'$ be the union of $\mathcal{R}'_m, m\in N$. In view of Lemma 1, we can define Hadamard's finite part of any function from $\mathcal{R}'$.

If $f\in\mathcal{R}_1$, then for some $\epsilon>0$ and $\phi\in\mathcal{D}$ we have:

$$\int_{\epsilon}^{\infty} f(x)\phi(x)dx = -f^{(-1)}(\epsilon)\phi(\epsilon)-$$

$$-\int_{\epsilon}^{\infty} f^{-1}(x)\phi'(x)dx = -J(\epsilon)+F(\epsilon)$$

where $f^{(-1)}(x) = -\int_x^a f(t)dt, a\in[\chi,\infty)$.

We can see (from Lemma 1) that there exists $\lim_{\epsilon\to 0}F(\epsilon)$, so we have by definition

$$F.P.\int_0^{\infty} f(x)\phi(x)dx = -\int_0^{\infty} f^{(-1)}(x)\phi'(x)dx = \lim_{\epsilon\to 0}F(\epsilon).$$

If $\mathcal{R}'_m$, $m>1$, we define by induction

$$F.P.\int_0^{\infty} f(x)\phi(x)dx = -F.P.\int_0^{\infty} f^{(-1)}(x)\phi'(x)dx.$$

Continuing that process, we get for $\mathcal{R}'_m$

$$F.P.\int_0^{\infty} f(x)\phi(x)dx = (-1)^m\int_0^{\infty} f^{(-m)}(x)\phi^{(m)}(x)dx, \tag{6}$$

hence

$$F.P.\int_0^{\infty} f(x)\phi(x)dx =$$

$$= (-1)^m \int_0^\chi f^{(-m)}(x)\phi^{(m)}(x)dx + \int_\chi^\infty f(x)\phi(x)dx +$$

$$+ \sum_{s=1}^{m}(-1)^{s+1} f^{(-s)}(\chi)\phi^{(s-1)}(\chi),$$

for all $\phi \in \mathcal{D}$.

By definition, we take in the sequel

$$\langle \text{Pf}(f), \phi \rangle = F.P. \int_0^\infty f(x)\phi(x)dx$$

(Pf standard for "pseudofunction").

We say that two regularizations of a function are consistent if the result of these regularizations is the same distribution.

If f is in $\mathcal{R}'_m$ and has a locally integrable derivative on $\mathbf{R}\backslash\{0\}$, then the regularization is consistent with derivation: on $\mathbf{R}\backslash\{0\}$, then

$$\langle \text{Pf}(f'), \phi \rangle = F.P. \int_0^\infty f'(x)\phi(x)dx$$

$$\langle \text{Pf}(f)', \phi \rangle = -F.P. \int_0^\infty f(x)\phi'(x)dx,$$

hence $\text{Pf}(f') = (\text{ Pf}(f))'$.

This regularization and the *type II regularization* are the only ones from this paper which have this property.

The next theorem shows the relation between Hadamard's finite part and the type I regularization.

Theorem 2. *Let f be in $\mathcal{R}'_m, f_1$ and f_2 be its Hadamard's finite part and type I regularizations respectively. Then we have*

$$\langle f_2, \phi \rangle - \langle f_1, \phi \rangle = \sum_{i=0}^{m-1} \sum_{k=0}^{i} \frac{\chi^k}{k!} \langle \delta^{(i)}, \phi \rangle, \chi > 0.$$

Proof. The type I regularization is:

$$\langle f_2, \phi \rangle = \int_0^\infty f(x)(\phi(x) - (\phi(0) + \ldots + \frac{\phi^{(m-1)}(0)}{(m-1)!} x^{m-1})\Theta(\chi - x))dx$$

Using partial integration we have

$$\langle f_2,\phi \rangle = f^{(-1)}(\chi)(\phi(\chi) - (\phi(0) + \ldots + \frac{\phi^{(m-1)}(0)}{(m-1)!}\chi^{m-1}))-$$

$$-f^{(-2)}(\chi)(\phi'(\chi) - (\phi'(0) + \ldots + \frac{\phi^{(m-1)}(0)}{(m-1)!}\chi^{m-2})) + \ldots$$

$$\ldots + (-1)^{m+1} f^{(-m)}(\chi)(\phi^{(m-1)}(\chi) - \phi^{(m-1)}(0))+$$

$$+(-1)^m \int_0^\chi f^{(-m)}(x)\phi^{(m)}(x)dx + \int_\chi^\infty f(x)\phi(x)dx =$$

$$= (-1)^{(m)} \int_0^\chi f^{(-m)}(x)\phi^{(m)}(x)dx + \int_\chi^\infty f(x)\phi(x)dx+$$

$$+\sum_{s=1}^{m}(-1)^{s+1} f^{(-s)}(\chi)\phi^{(s-1)}(\chi) + \sum_{s=0}^{m-1}\phi^{(s)}(0)\sum_{k=0}^{s}\frac{\chi^k}{k!},$$

so using (6) we have the result of the Theorem. Q.E.D.

A set of S_0-functions is the set of functions $f \in \mathcal{R}'$ which can be written in the form $f(x) = f_1(x) + \ldots + f_k(x)$, where f_i are integrable on $[\epsilon,\infty]$ for $\epsilon > 0$, and they can be regularized by *type II regularization*

$$\langle \bar{f}_i,\phi \rangle = \int_0^\infty f_i(x)(\phi(x) - \sum_{s=0}^{m-1}\frac{\phi^{(s)}(0)}{s!}x^s)dx.$$

We define $\bar{f} := \bar{f}_1 + \ldots + \bar{f}_k$.

Theorem 3. *The type II Gel'fand-Šilov and Hadamard's finite part regularization of S_0-functions are consistent.*

Proof. Let f be an S_0-function, and $f(x) = f_1(x) + \ldots + f_k(x)$

$$\int_0^\infty f_i(x)(\phi(x) - \sum_{s=0}^{m-1}\frac{\phi^{(s)}(0)}{s!}x^s)dx =$$

$$= f_i^{(-1)}(x)(\phi(x) - \sum_{s=0}^{m-1}\frac{\phi^{(s)}(0)}{s!}x^s)|_0^\infty -$$

$$-f_i^{(-2)}(x)(\phi'(x)-\sum_{s=0}^{m-1}\frac{\phi^{(s+1)}(0)}{s!}x^s)|_0^\infty+\ldots+$$

$$+(-1)^{m+1}f_i^{(m-1)}(x)(\phi^{(m-1)}(x)-\phi^{(m-1)}(0))|_0^\infty+$$

$$+(-1)^m\int_0^\infty f_i^{(-m)}(x)\phi^{(m)}(x)dx=$$

$$=(-1)^m\int_0^\infty f_i^{(-m)}(x)\phi^{(m)}(x)dx=F.P.\int_0^\infty f_i(x)\phi(x)dx.Q.E.D.$$

We end this section with a hypothesis:

The set of S_0-functions is the largest subset of the set of $\mathcal{R}$-functions for which the type II and Hadamard's finite part regularization are consistent.

3 . Lavoine's version of Hadamard's finite part regularizations

In [6] J. Lavoine analyzed a set of functions which we shall call " the class $\mathcal{A}$". It consist of locally integrable functions g on $(0,\infty)$ that can be written in the form

$$g(x)=\sum_{k=1}^{K}(a_k+\sum_{j=1}^{J_k}\alpha_{k_j}\ln^j x)x^{-\nu_k}+\sum_{k=1}^{K}(b_k+ \tag{7}$$

$$+\sum_{j=1}^{J_k}\beta_{k_j}\ln^j x)x^{-k}+h(x),$$

where $k\in N,\nu_k>0,\nu_k\notin \mathrm{N}$ for $k=1,\ldots,K$ and h is a measurable and bounded function on $(0,\chi)$; they depend on the function g.

Observe that for all $\phi\in\mathcal{D}$ and $f\in\mathcal{A}$ their product $g=f\phi$ is also in $\mathcal{A}$: we just have to write $\phi=\sum_{i=0}^{L}\frac{\phi^{(i)}(0)}{i!}x^i+h(x),L=\max\{k,[\nu_k]\}$.

Let us calculate the improper integral $\int_\epsilon^\infty f(x)\phi(x)dx=\int_\epsilon^\infty g(x)dx$, by Hadamard's finite part method used in ([6]).

$$\int_\epsilon^\infty g(x)dx=\int_\epsilon^\infty(\sum_{k=1}^{K}(a_k+\sum_{j=1}^{J}\alpha_{k_j}\ln^j x)x^{-\nu_k}+$$

$$+\sum_{k=1}^{K}(b_k+\sum_{j=1}^{J}\beta_{k_j}\ln^j x)x^{-k}+h(x))dx+\int_\epsilon^\infty g dx=$$

$$=\sum_{j=1}^{9} I_j+\int_\epsilon^\infty g dx,$$

where

$$I_1=-\sum_{k=1}^{K}\frac{a_k}{\nu_k-1}\chi^{-\nu_k+1},$$

$$I_2=\sum_{k=1}^{K}\frac{a_k}{\nu_k-1}\epsilon^{-\nu_k+1},$$

$$I_3=\sum_{k=1}^{K}\sum_{j=1}^{J}\alpha_{k_j}\sum_{i=1}^{j+1}\frac{(-1)^{i+1}j!\chi^{-1-\nu_k}\ln^{j-i+1}\chi}{(1-\nu_k)^i(j-i+1)!},$$

$$I_4=\sum_{k=1}^{K}\sum_{j=1}^{J}\alpha_{k_j}\sum_{i=1}^{j+1}\frac{(-1)^{i+1}j!\epsilon^{-1-\nu_k}\ln^{j-i+1}\epsilon}{(1-\nu_k)^i(j-i+1)!},$$

$$I_5=-\sum_{k=1}^{K}\frac{b_k}{k-1}\chi^{-k+1}+b_1\ln\chi,$$

$$I_6=-\sum_{k=2}^{K}\frac{b_k}{k-1}\epsilon^{-k+1}+b_1\ln\epsilon,$$

$$I_7=\sum_{k=1,k\neq 2}^{K}\sum_{j=1}^{J}\beta_{k_j}\sum_{i=1}^{j+1}\frac{(-1)^{i+1}j!\chi^{2-k}\ln^{j-i+1}\chi}{(2-k)^i(j-i+1!)}-\sum_{j=1}^{J}\beta_{2_j}\frac{\ln^{j+1}\chi}{j+1},$$

$$I_8=\sum_{k=1,k\neq 2}^{K}\sum_{j=1}^{J}\beta_{k_j}\sum_{i=1}^{j+1}\frac{(-1)^{i+1}j!\epsilon^{2-k}\ln^{j-i+1}\epsilon}{(2-k)^i(j-i+1)!}-\sum_{j=1}^{J}\beta_{2_j}\frac{\ln^{j+1}\epsilon}{j+1}.$$

We define the finite part of integral $I=\int_0^\infty g(x)dx$ by

$$L.F.P.. := I_1+I_3+I_5+I_7+I_9+\int_\chi^\infty g(x)dx$$

where $L.F.P.$ stands for *Lavoine's finite part*. This regularization is a sort of Hadamard's finite part regularization.

Obviously, Lavoine's regularization is defined on a wider class of functions that Edward's [1] of Hörmander's [5] version of Hadamard's finite part regularizations. On the intersection of these classes these regularizations are equivalent. However Lavoine's regularization is also equivalent to Fisher's one defined in [3] and the type I Gel'fand-Šilov's regularization. In fact all the sums I_2, I_4, I_6, I_8 from Lavoine's regularizations are sums of neutrices in the sense of Fisher, while it is proved in [3] that his regularization is consistent with the type I Gel'fand-Šilov regularization.

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REZIME

PRIMEDBA O REGULARIZACIJI DISTRIBUCIJA

U radu se upoređuju različite regularizacije uopštenih funkcija i nesvojstvenih integrala i to koristeći Hadamard-ovu regularizaciju, regularizaciju Gel'fand-Šilova i Lavoanovu regularizaciju. Pokazuje se saglasnost ovih regularizacija na jednoj širokoj klasi neintegrabilnih funkcija.

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