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# DIHEDRAL QUADRUPLE SYSTEMS

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**Abstract**

A class of quadruple systems called dihedral quadruple systems (DQSs), which lie between Mendelsohn and Steiner quadruple systems, is defined and considered. A ternary quasigroup which is invariant under conjugation by every permutation from $D_4$, the dihedral group, is called dihedral. It is proved that DQSs are equivalent to generalized idempotent dihedral ternary quasigroups. Some properties of such ternary quasigroups and DQSs are described and their spectrum determined. It is proved that a DQS of order $v$ exists if and only if $v \equiv 0\ (mod\ 2)$, $v \geq 4$.

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## 1. Introduction

A $t-(v,k,\lambda)$ design is a set $V$ of cardinality $v$ and a collection $B$ of $k$-subsets of $V$ called blocks with the property that every $t$-subset of $V$ is contained in exactly $\lambda$ blocks (where repeated blocks are allowed in $B$). A $2-(v,3,1)$ design is called a Steiner triple system and a $3-(v,4,1)$ design is a Steiner quadruple system (SQS). An ordered analogue of SQSs are

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Mendelsohn quadruple systems (MQSs). A MQS is a pair $(S,T)$ where $S$ is a finite set and $T$ is a collection of directed quadruples $\langle abcd\rangle$, $a,b,c,d$ distinct elements of $S$, such that every ordered triple of distinct elements from $S$ belongs to exactly one directed quadruple from $T$. A directed quadruple $\langle abcd\rangle$ contains ordered triples $(abc),(bcd),(cda),(dab)$.

Several generalizations and modifications of Steiner triple systems have been studied. Our aim is to define and consider a new class of quadruple systems representing another generalization of Steiner systems which lies between SQSs and MQSs.

The sequence $x_m, x_{m+1}, \ldots, x_n$ is denoted by $\{x_i\}_{i=m}^n$ or by $x_m^n$. If $m > n$, then $x_m^n$ will be considered empty.

An $n$-ary groupoid ($n$-groupoid) $(Q,f)$ is called an $n$ -quasigroup if the equation $f(a_1^{i-1}, x, a_{i+1}^n) = b$ has a unique solution $x$ for every $a_1^n, b \in Q$ and every $i \in \{1,\ldots,n\}$.

By $S_n$ we denote the symmetric group of degree $n$ and by $D_n$ its dihedral subgroup.

If $(Q,f)$ is an $n$-quasigroup and $\sigma \in S_{n+1}$, then the $n$-quasigroup $(Q,f^\sigma)$ defined by

$$f^\sigma(\{x_{\sigma(i)}\}_{i=1}^n)) = x_{\sigma(n+1)} \iff f(x_1^n) = x_{n+1}$$

is called a $\sigma$-conjugate (or simply conjugate) of $f$.

If $(Q,f)$ is an $n$-quasigroup, the set of all permutations $\sigma \in S_{n+1}$ such that $f = f^\sigma$ is a subgroup of $S_{n+1}$ which will be denoted by $\Pi(f)$.

An $n$-quasigroup $(Q,f)$ is called totally symmetric (TS) if $f = f^\sigma$ for all $\sigma \in S_{n+1}$.

An $n$-quasigroup $(Q,f)$ is called cyclic ([3],[4]) if $f = f^\sigma$ for all $\sigma \in H$, where $H$ is a subgroup of $S_{n+1}$ generated by the cycle $(12\ldots n{+}1)$.

## 2. Dihedral quadruple systems

First we shall define a class of quadruple systems and later show that they lie between SQSs and MQSs.

**Definition 1.** *Let $S$ be a finite set of $v$ elements. A dihedral quadruple $\langle abcd\rangle$, where $a,b,c,d$ are distinct elements of $S$, is the following set of 8* *ordered triples*

$$\langle abcd\rangle = \{(abc), (bcd), (cda), (dab), (cba), (dcb), (adc), (bad)\}.$$

*A dihedral quadruple system (DQS) of order $v$ is a pair $(S,T)$, where $T$ is a family of dihedral quadruples of elements of $S$ called blocks, such that every ordered triple of distinct elements from $S$ belongs to exactly one block of $T$.*

It is obvious that $\langle abcd\rangle = \langle bcda\rangle = \langle cdab\rangle = \langle dabc\rangle = \langle dcba\rangle = \langle cbad\rangle = \langle badc\rangle = \langle adcb\rangle$.

Note that the dihedral quadruple $\langle abcd\rangle$ is cyclicaly ordered in two directions ($a < b < c < d < a$ and $a > b > c > d > a$), unlike Mendelsohn systems where blocks are also cyclicaly ordered but in only one direction ($a < b < c < d < a$).

That DQSs are really a class lying between SQSs and DQSs will be proved later, when we determine an algebraic characterization of DQSs.

## 3. Dihedral ternary quasigroups

In [5] a class of $n$-quasigroups called dihedral $n$-quasigroups was introduced.

**Definition 2.** *([5]). An $n$-groupoid $(Q,f)$ is called dihedral iff for every permutation $\sigma \in D_{n+1}$ and every $x_1^{n+1} \in Q$*

$$f(x_1^n) = x_{n+1} \iff f(\{x_{\sigma(i)}\}_{i=1}^n) = x_{\sigma(n+1)}.$$

Every such $n$-groupoid is necessarily an $n$-quasigroup called dihedral $n$-quasigroup. Since every TS $n$-quasigroup is dihedral and every dihedral $n$-quasigroup is cyclic, if we denote the class of all TS $n$-quasigroups by $\mathcal{A}(TS)$, the class of all dihedral $n$-quasigroups by $\mathcal{A}(D)$ and the class of all cyclic $n$-quasigroups by $\mathcal{A}(C)$, it follows

$$\mathcal{A}(TS) \subset \mathcal{A}(D) \subset \mathcal{A}(C).$$

The class $\mathcal{A}(D)$ lies between $\mathcal{A}(TS)$ and $\mathcal{A}(C)$, that $\mathcal{A}(D)$ is different from both $\mathcal{A}(TS)$ and $\mathcal{A}(C)$ follows from [2], where it was proved that for every subgroup $G \subset S_{n+1}$ and every $m > n$, $p \geq 2$ there exists an $n$-quasigroup $(Q, f)$ of order $mp$ such that $\Pi(f) = G$.

For $n = 2$, since $D_3 = S_3$, it follows that dihedral binary quasigroups are totally symmetric quasigroups. So, dihedral $n$-quasigroups can be considered as a generalization of totally symmetric binary quasigroups.

Since the group $D_4$ is generated by permutations (1234) and (24), we get that a 3-groupoid $(Q, f)$ is a dihedral 3-quasigroup iff for all $x, y, z, u \in Q$

$$f(x, y, z) = u \iff f(y, z, u) = x$$

and

$$f(x, y, z) = u \iff f(x, u, z) = y.$$

The first of the two preceding equivalences is equivalent to

$$f(y, z, f(x, y, z)) = x,$$

and the second is eqivalent to

$$f(x, f(x, y, z), z) = y.$$

Hence the following theorem is valid.

**Theorem 1.** *A 3-groupoid $(Q, f)$ is a dihedral 3-quasigroup iff for all $x, y, z \in Q$*

$$\begin{cases} f(y, z, f(x, y, z)) = x, \\ f(x, f(x, y, z), z) = y. \end{cases}$$

A 3-groupoid $(Q, f)$ satisfying the identities

$$f(x, y, y) = f(y, x, y) = f(y, y, x) = x \tag{1}$$

is called a generalized idempotent (GI) 3-groupoid.

A dihedral 3-groupoid which is GI will be called a GID-3-groupoid. From Theorem 1 and (1), we obtain the following theorem.

**Theorem 2.** *A 3-groupoid $(Q, f)$ is a GID-3-quasigroup iff the following identities hold*

$$\begin{cases} f(y, z, f(x, y, z)) = x, \\ f(x, f(x, y, z), z) = y, \\ f(x, y, y) = f(y, x, y) = x. \end{cases}$$

Since GID-3-quasigroups are defined by identities, the class of all GID-3-quasigroups is a variety, which implies that the class of all GID-3-quasigroups is closed under the formation of direct products, subalgebras and quotient algebras.

## 4. Coordinatization of DQS

Now we shall establish an equivalence between DQSs and GID-3-quasigroups.

**Theorem 3.** *Every DQS of order $v$ defines and is defined by an GID-3-quasigroup of order $v$.*

P r o o f. Let $(S,T)$ be a DQS of order $v$. We shall define a ternary operation $f$ on $S$ in the following way. If $a, b, c$ are arbitrary distinct elements from $S$, then the ordered triple $(abc)$ belongs to exactly one dihedral quadruple $\langle abcd \rangle$ from $T$. Then we define

$$f(a,b,c) = d.$$

For every $x, y \in S$, let

$$f(x,y,y) = f(y,x,y) = f(y,y,x) = x.$$

Hence a 3-groupoid $(S,f)$ is defined and it is easy to see that $(S,f)$ is a GID-3-quasigroup of order $v$.

Now, let $(S,f)$ be a GID-3-quasigroup. Let $a, b, c$ be distinct elements from $S$ and $f(a,b,c) = d$. Suppose that $d \in \{a,b,c\}$, say $d = a$, then since $(S,f)$ is dihedral $f(a,a,b) = c$, and since $f$ is also GI it follows that $b = c$ which is a contradiction. Hence $d$ is different from $a, b, c$.

For every ordered triple $(abc)$ of distinct elements from $S$ we define a dihedral quadruple $\langle abcf(abc) \rangle$ and denote by $T$ the set of all such quadruples. Since $(S,f)$ is dihedral it follows that for every $\langle abcd \rangle \in T$

$$\langle abcd \rangle = \langle bcda \rangle = \langle cdab \rangle = \langle dabc \rangle = \langle dcba \rangle = \langle cbad \rangle = \langle badc \rangle = \langle adcb \rangle.$$

This means that every ordered triple of distinct elements from $S$ belongs to exactly one dihedral quadruple from $S$, thus $(S,T)$ is a DQS.

Since SQSs are equivalent to GI totally symmetric 3-quasigroups and MQSs are equivalent to GI cyclic 3-quasigroups, from Theorem 3 and the fact that

$$\mathcal{A}(TS) \subset \mathcal{A}(D) \subset \mathcal{A}(C),$$

we obtain that the class of DQSs lies between SQSs and MQSs. If we denote the class of all SQSs of order $v$ by $\mathcal{S}(v)$, the class of all DQSs of order $v$ by $\mathcal{D}(v)$ and the class of all MQSs of order $v$ by $\mathcal{M}(v)$, we get the following

$$\mathcal{S}(v) \subset \mathcal{D}(v) \subset \mathcal{M}(v).$$

On the other hand, since GID-3-quasigroups are a generalization of idempotent totally symmetric binary quasigroups, which are equivalent to Steiner triple systems, we get that DQSs represent another generalization of Steiner triple systems.

## 5. On the spectrum of DQS

Directly from the definition of DQSs follows a necessary condition for their existence.

**Theorem 4.** *If $(S,T)$ is a DQS of order $v$, then*

$$|T| = \frac{v(v-1)(v-2)}{8}.$$

DQSs are related to $t-(v,k,\lambda)$ designs as follows. If $(S,T)$ is a DQS of order $v$ and we replace every dihedral quadruple $\langle abcd \rangle$ from $T$ by the set $\{a,b,c,d\}$, then we get a system $T'$ of four element subsets of $S$ (allowing repeated elements in $T'$). Every 3-subset of $S$ belongs to exactly three 4-subsets from $T'$, hence $(S,T')$ is a $3-(v,4,3)$ design. Using the necessary and sufficient conditions for the existence of $3-(v,4,3)$ designs ([1]), we get the following theorem.

**Theorem 5.** *If $(S,T)$ is a DQS of order $v$, then*

$$v \equiv 0 \pmod 2.$$

Now we shall prove that the necessary condition for the existence of DQS given in Theorem 5 is also sufficient.

**Theorem 6.** *There exists a DQS of order 6.*

P r o o f. If $S = \{a, b, c, d, e, f\}$ and

$$T = \{\langle a\,b\,c\,d\rangle, \langle a\,b\,f\,c\rangle, \langle a\,c\,b\,e\rangle, \langle a\,d\,b\,f\rangle, \langle a\,b\,e\,f\rangle, \langle a\,c\,d\,f\rangle, \langle a\,e\,c\,f\rangle, \langle a\,d\,f\,e\rangle, \langle b\,d\,c\,e\rangle, \langle b\,c\,e\,f\rangle, \langle b\,e\,d\,f\rangle, \langle c\,d\,e\,f\rangle, \langle a\,b\,d\,e\rangle, \langle a\,d\,e\,c\rangle, \langle b\,c\,f\,d\rangle\},$$

then $(S, T)$ is a DQS of order 6.

**Theorem 7.** *A DQS of order $v$ exists if and only if $v \equiv 0\ (mod\ 2)$, $v \geq 4$.*

P r o o f. That the given condition for the existence of DQS is necessary is proved in Theorem 5.

In [1] it was proved that for every $v \equiv 0\ (mod\ 2)$, $v \geq 4$, there exist a $3 - (v, \{4, 6\}, 1)$ design (a 3-wise balanced design of index $\lambda = 1$ with block sizes 4 and 6 ). If in this design every block $\{a, b, c, d\}$ of size 4 is replaced by three dihedral quadruples $\langle abcd\rangle$, $\langle bacd\rangle$, $\langle acbd\rangle$, and every block $\{a, b, c, d, e, f\}$ of size 6 is replaced by the dihedral quadruples from the family $T$ constructed in the proof of Theorem 6, then a DQS of order $v$ is obtained.

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**REZIME**

DIEDARSKI SISTEMI ČETVORKI

Definisan je i razmatran diedarski sistem četvorki (DSČ), koji se nalazi između Mendelsonovih i Štajnerovih sistema četvorki. Ternarna kvazigrupa invarijantna u odnosu na konjugovanje svakom permutacijom iz diedarske grupe $D_4$, naziva se diedarska. Dokazano je da su DSČ ekvivalentni sa generalisanim idempotentnim diedarskim ternarnim kvazigrupama. Određene su neke osobine ovih ternarnih kvazigrupa i njihov spektar. Dokazano je da DSČ postoje ako i samo ako je $v \equiv 0 \ (mod\, 2)$, $v \geq 4$.

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