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# A NOTE ON WEAK PARTIAL CONGRUENCE ALGEBRAS

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**Abstract**

In the paper is introduced for a given algebra $\mathcal{A}$, the notion of a weak partial congruence algebra $K_w(\mathcal{A}) = (C_w(A), \wedge, \vee, 0, ^{-1}, \Delta, \sigma, A^2$. $C_w$(A) is the set of weak congruences of the algebra $\mathcal{A}$ i.e. of all the symmetric and transitive subalgebras of $\mathcal{A} \times \mathcal{A}$ (see [3]). It is shown that $K_w(\mathcal{A})$ gives more information on $\mathcal{A}$ than just lattice $C_w(\mathcal{A})$. Also considered is the corresponding abstract weak partial congruence algebra and proved that a theorem of representation does not hold for it.

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## 1.

Let $\mathcal{A} = (A, F)$ be an algebra and $C \subseteq A$ set a its constants (iff $C = \emptyset$, then we consider the empty set as a subalgebra of $\mathcal{A}$).
A weak congruence relation $\rho$ on $\mathcal{A}$ ([3]) is a symmetric, transitive binary relation on $\mathcal{A}$, satisfying the usual substitution property (if $f \in F_n \subseteq F, n \geq 1$, and $x_i \rho y_i, x_i, y_i \in A, i = 1, ..., n$, then $f(x_1, ..., x_n) \rho f(y_1, ..., y_n)$, and a weak reflexivity: if $c \in C$, then $c \rho c$.
The notation here will be as follows:

$C_w(\mathcal{A})$ is the set of all the weak congruence on an algebra $\mathcal{A}$;

$S(\mathcal{A})$ is the set of all the subalgebras of $\mathcal{A}$;

$C(\mathcal{A})$ is the set of all the (ordinary) congruences on $\mathcal{A}$;

$C(\mathcal{B})$ is the set of all the congruences on $\mathcal{B}$ is a subalgebra of $\mathcal{A}$;

$C_w(\mathcal{A})$ coincides with the lattice of all the congruences on all the subalgebra of $\mathcal{A}$ (under set inclusion), and $(C_w(A), \wedge, \vee)$ is the algebraic lattice (see [3]). Moreover ([3]), $C(\mathcal{A})$ is a sublattice of $C_w(\mathcal{A})$ (as a filter generated by $\Delta = \{(x,x) \mid x \in A\}$), and $S(\mathcal{A})$ is a retract of $C_w(\mathcal{A})$ (ideal generated by $\Delta$). Subalgebras are represented in $C_w(A)$ by diagonal relations ($\mathcal{B} \in \mathcal{S}(\mathcal{A})$ which correspond to $d_{B^2} = \{(x,x) \mid x \in B\}$.)
Let then be given in algebra $\mathcal{A}$. By its weak partial congruence algebra we mean the partial algebra:

$$K_w(\mathcal{A}) = (C_w(A), \wedge, \vee, o,^{-1}, \Delta, \sigma, A^2),$$

where

$$(C_w(A), \wedge, \vee) = C_w(\mathcal{A}),$$

o-is a composition of relation (binary partial operation on $C_w(A)$,$^{-1}$- is a unary inverse operation for relations, $\sigma$ - is a diagonal relation corresponding to the least subalgebra on $S(\mathcal{A})$.
The next example shows that $K_w(\mathcal{A})$ "knows" more about algebra $\mathcal{A}$ than $C_w(\mathcal{A})$ (much more than $S(\mathcal{A})$ and $C(\mathcal{A})$).
**Example.** Let us consider the following algebras $\mathcal{A}_1, \mathcal{A}_2$, both with some domain $B = \{0, 1, ..., pq-1\}$ which is also the set of constants with $p, q, p \neq q$ primes:

$$\mathcal{A} = (B, *_1, f_1, B),$$

$$\mathcal{A} = (B, *_2, f_2, B),$$

where for any $i, j < p \cdot q$, $i *_1 j = 0$;
$f_1(i) = i + 1$, for $i < p \cdot q - 1$, and $f_1(p \cdot q - 1) = 1$ : $*_2$ is an addition module $pq$, and $f_2$ is an identity mapping of $B$. It is easi to verify that

$$C_w(\mathcal{A}_1) \cong C_w(\mathcal{A}),$$

while

$$K_w(\mathcal{A}_1) \cong K_w(\mathcal{A}_2).$$

(Let us note that $(B, {}^*_1)$ belongs to the class of algebras whose lattice of congruences coincides with the lattice of equivalence relations).
It is an easy exercise to see that for any two-element algebra $\mathcal{A}_1, \mathcal{A}_2$ of the same type it holds that

$$C_w(\mathcal{A}_1) \cong C_w(\mathcal{A}_2) \text{iff } K_w(\mathcal{A}_1) \cong K_w(\mathcal{A}_2).$$

For the above, the next two problems are natural.

**Problem A.** Answer the following question? do there exist two algebras $\mathcal{A}_1, \mathcal{A}_2$ of the same type of the same cardinalaity less than six such that

$$C_w(\mathcal{A}_1) \cong C_w(\mathcal{A}_2)$$

but

$$K_w(\mathcal{A}_1) \cong K_w(\mathcal{A}_2)$$

**Problem B.** Describe the class of the same type of algebra such that for any two of its elements $\mathcal{A}_1, \mathcal{A}_2$ it holds that

$$K_w(\mathcal{A}_1) \cong K_w(\mathcal{A}_2) \text{ implies } \mathcal{A}_\infty \cong \mathcal{A}_\in.$$

The following theorem holds:

**Theorem 1.** *Let $\mathcal{A}$ be an algebra and $K_w(\mathcal{A}) = (C_w(\mathcal{A}), \wedge, \vee, o, {}^{-1}, \Delta, \sigma, A^2)$ its weak partial congruence algebra. Then, for any $\rho, \theta \in C_w(\mathcal{A})$ it holds that*

*(1)* $\Delta o (\Delta \vee \rho) = \Delta \vee \rho$,

*(2)* $\Delta o (\Delta \wedge \rho) = \Delta \wedge \rho$,

*(3)* $\Delta \wedge (\rho \vee \theta) = (\Delta \wedge \rho) \vee (\Delta \wedge \theta)$,

*(i.e. $\Delta$ is a co-distributive element of $C_w(\mathcal{A})$),*

*(4)* $\rho o \theta \in C_w(\mathcal{A})$ *iff* $\rho o \theta = \theta o \rho$

*(5) if* $\rho o \theta \in C_w(\mathcal{A})$ *then* $\rho \vee \theta = \rho o \theta$ *iff* $\Delta \wedge \rho = \Delta \wedge \theta$.

*Proof.* Items (1), (2) and (3) are obvious.
(4) The same as for the (ordinary) congruence.
(5) Let $\Delta \wedge \rho = \Delta \wedge \theta$. Then, as in [2], $\rho \cup \theta \subseteq \rho o \theta \subseteq \rho \vee \theta$.
Really, $(x,y) \in \rho \cup \theta$ iff $((x,y) \in \rho$ or $(x,y) \in \theta)$, iff $((x,y) \in \rho$ and $(y,y) \in \rho)$ or $((x,x) \in \theta$ and $(x,y) \in \theta)$, iff $((x,y) \in \rho)$ and $(y,y) \in \theta)$ or $((x,x) \in \rho)$ and $(x,y) \in \theta)$, iff $(x,y) \in \rho o \theta$.
We use the fact that $(x,y) \in \rho$ implies, by symmetry and transitivity, that $(x,x) \in \rho$ and $(y,y) \in \theta$. The same holds for $\theta$.
Thus, $\rho \cup \theta \subseteq \rho o \theta$.
If $(x,z) \in \rho$ and $(z,y) \in \theta$, then $(x,z),(z,y) \in \rho \cup \theta$, and clearly $(x,z),(z,y) \in \tau$, for every $\tau \in C_w(\mathcal{A})$, such that $\rho \cup \theta \subseteq \tau$. Hence, $\rho o \theta \subseteq \rho \cup \theta$. Since $\rho o \theta \in C_w(\mathcal{A})$, it follows that $\rho o \theta = \rho \vee \theta$.
Let now $\rho o \theta = \rho \vee \theta$. Then, $\rho, \theta \subseteq \rho o \theta$. We also have $(x,x) \in \rho o \theta$, iff there is $z \in A$, such that $(x,z) \in \rho$ and $(z,x) \in \theta$, iff $(x,x) \in \rho$ and $(x,x) \in \theta$. Since $\rho, \theta \subseteq \rho o \theta$, it follows that $\Delta \wedge \rho = \Delta \wedge \theta$.

## 2.

Previous considerations enable us to introduce the notion of an abstract weaka partial congruence algebra.
$L_w = (L, \wedge, \vee, \bullet, ^{-1}, a, o, 1)$ where the following conditions are satisfied:

(1) $(L, \wedge, \vee, 0, 1)$ is an algebraic lattice (with zero - 0, and unity 1);

(2) $\bullet$ - is partial operation on $L$ with $x \cdot x = x$ for each $x \in L$ and if $x \bullet y$ is defined then $y \bullet x$ is also defined and

$$x \bullet y = y \bullet x :$$

(3) $^{-1}$ - is a unary operation on $L$ such that for any $x, x^{-1} = x$,

(4) $a$ - is a constant such that for any $x, y \in L$ it holds that
(i) $a \bullet (x \vee y)$ is defined and
$a \bullet (x \vee a) = x \vee a :$
(ii) $a \bullet (x \wedge a)$ is defined and
$a \bullet (x \wedge a) = x \wedge a;$
(iii) if $x \bullet y$ is defined and

$x \vee y = x \bullet y$ iff $a \wedge x = a \wedge y$;
(iv) $a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y)$.

**Theorem 2.** *The representation theorem does not hold for an abstract weak partial congruence algebra, i.e. that is, there exists $L_w$ such that for any algebra*

$$L_w \neq K_w(\mathcal{A}).$$

*Proof.* Let us consider the following example for $L_w : (L, \wedge, \vee, 0, 1)$ is the chaian $0 = a_0 < a_1 < ... < a_k = 1$, $k \geq 4$ and constant a is $a_3$. Then, for any algebra $\mathcal{A}, L_w \cong K_w(\mathcal{A})$.

Problem C. Find the conditions which should be saatisfied by an abstract weak partial congruence algebra in order that the representation theorem should hold.

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**REZIME**

PRIMEDBA O SLABIM PARCIJALNIM KONGRUENCIJSKIM ALGEBRAMA

U radu se uvodi slaba parcijalna kongruencijska algebra $K_w(\mathcal{A})$ date alagebre $\mathcal{A}$. Pokazuje se da $K_w(\mathcal{A})$ daje više informacija o algebri $\mathcal{A}$ nego mreža slabih kongruencija. Takodje, posmatra se odgovarajuća apstraktna slaba parcijalna kongruencijska algebra i pokazuje se da za nju važi teorema reprezentacije.

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