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ON CEP AND SEMIMODULARITY IN THE LATTICE OF WEAK CONGRUENCES

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Abstract

The congruence extension property (CEP) is characterized in the paper by a lattice identity: an algebra $\mathcal{A}$ has the CEP if and only if the diagonal relation and an arbitrary congruence form a modular pair in the lattice of weak congruences of $\mathcal{A}$. It is know that in the lattice of a finite length, the semimodularity can be characterized by modular pairs. It is shown in the paper that the CEP is one of the sufficient conditions under wich the lattice of weak congruences of an algebra is semimodular. Another sufficient condition is the weak congruence intersection property, the wCIP. Necessary conditions for the semimodularity of that lattice are also given. Two particular cases are considered: if $\mathcal{A}$ is a lattice, and if $\mathcal{A}$ is a unary algebra. In both cases, necessery and sufficient conditions for the semimodularity of a weak congruence lattice are formulated.

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1. Preliminaries

1. An element a of a lattice L is codistributive if for all $x, y \in L$

$$a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y),$$

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or, equivalently, if and only if the mapping $m_a : x \mapsto x \wedge a$ is a homomorphism of L into the ideal $(a]$.An element a of L is a said to be modular, if for all $x, y \in L$

$$a \leq y \text{ implies } a \vee (x \wedge y) = (a \vee x) \wedge y.$$

Two elements b and c from L form a modular pair, if for every $x \in L$

$$x \leq c \text{ implies } x \vee (b \wedge c) = (x \vee b) \wedge c.$$

As it is know, a lattice L is semimodular (upper semimodular), if for all $x, y \in L$

$$x \succ x \wedge y \text{ implies } x \vee y \succ y,$$

where $x \succ y$ means that x covers y. A lattice L has a finite length if all its maximal chains are finite.

2. Algebraic results and definitions are from [6] and [7]. The lattice Cw$\mathcal{A}$ of weak congruences of an algebra $\mathcal{A} = (A, F)$ is the algebraic lattice of all the symmetric and transitive subalgebras of $\mathcal{A}^2$, under the set inclusion. The diagonal relation $\Delta = \{(x, x) \mid x \in A\}$ is a codistributive element in that lattice, the filter $[\Delta)$ is a lattice of congruences Con$\mathcal{A}$, and the ideal $(\Delta]$, consisting of all the diagonal relations is isomorphic with Sub$\mathcal{A}$, under the mapping $\rho \rightarrow \{x \mid x\rho x\}$. Recall that $\mathcal{A}$ has the congruence extension property (CEP), if every congruence on a subalgebra of $\mathcal{A}$ is a restriction of a congruence on $\mathcal{A}$. $\mathcal{A}$ is said to have the weak congruence intersection property (wCIP) if for $\Theta \in Con\mathcal{A}$ and $\rho \in Con\mathcal{B}$, $\mathcal{B} \in Sub\mathcal{A}$

$$(\rho \cap \Theta)_A = \rho_A \cap \Theta$$

where ρ_A is the least congruence on $\mathcal{A}$ extending ρ (see [5] and [8]). Since $\rho_A = \rho \vee \Delta$ in Cw$\mathcal{A}$, $\mathcal{A}$ has wCIP if and only if Δ is a modular element in Cw$\mathcal{A}$.

2. Lattice characterization of CEP

In the following we shall consider a codistributive element a of a lattice L, and suppose that classes of the congurence induced by $m_a : x \rightarrow x \wedge a$ have maximum elements. For $x \in L, \bar{x}$ is the maximum element in the corresponding class, i.e. $m_a(x) = m_a(\bar{x})$.

Proposition 1. *If a is a codistributive element of a lattice L, then the following statements are equivalent:*

$$(i) \qquad \text{for all } x, y \in L$$
$$x \leq y \text{ implies } x \vee (a \wedge y) = (x \vee a) \wedge y;$$
$$(ii) \qquad \text{for all } x, y \in L$$
$$x \wedge a = y \wedge a \text{ and } x \vee a = y \vee a \text{ imply } x = y.$$

Proof. (ii) $\Rightarrow$ (i). Let $x, y \in L$ and $x \leq y$. a) Let $x \wedge a = y \wedge a$. Then, $x \vee (a \wedge y) = x \vee (a \wedge x) = x$, and since $((x \vee a) \wedge y) \vee a \leq x \vee a$, it follows that $((x \vee a) \wedge y) \vee a = x \vee a$. Since $(x \vee a) \wedge y \wedge a = x \wedge a$, it follows by (ii) that $(x \vee a) \wedge y = x$ i.e. $(x \vee a) \wedge y = x \vee (a \wedge y)$. b) Let $x \wedge a < y \wedge a$. Then by a), $x \vee (a \wedge y) \vee (a \wedge y) = (x \vee (a \wedge y) \vee a) \wedge y$, since $(x \vee (a \wedge y)) \wedge a = (x \wedge a) \vee (y \wedge a) = y \wedge a$ and $x \vee (a \wedge y) \leq y$ (a is a codistributive element!). Thus, $x \vee (a \wedge y) = x \vee (a \wedge y) \vee (a \wedge y) = (x \vee (a \wedge y) \vee a) \wedge y = (x \vee a) \wedge y$.

(i) $\Rightarrow$ (ii). Suppose that (ii) does not hold, i.e. that there are $x, y \in L$ such that $x \wedge a = y \wedge a, x \vee a = y \vee a$, and $x \neq y$. Then,

$$x \vee (a \wedge y) = x \vee (a \wedge x) = x \neq y = (y \vee a) \wedge y = (x \vee a) \wedge y,$$

i.e. (i) is not satisfied. □

Condition (i) from Proposition 1 means that for every element y of the filter $[a)$, (a, y) is a modular pair in L. This condition is satisfied even under some weaker assumptions.

Proposition 2. *If a is a codistributive element of a lattice L, then the following are equivalent:*

$$(i) \qquad \text{for all } x, y \in L$$
$$x \leq y \text{ implies } x \vee (a \wedge y) = (x \vee a) \wedge y$$
$$(i') \qquad \text{for all } x, y \in L$$
$$x \leq \bar{y} \text{ implies } x \vee (a \wedge \bar{y}) = (x \vee a) \wedge \bar{y}$$

($\bar{y}$ is a maximum element in the class of the congruence induced by m_a : $x \to x \wedge a$, to which y belongs).

Proof. (i') → (i). For all $x, y \in L, x \leq y$ implies $x \leq \bar{y}$ and by (*i'*) $x \vee (a \wedge \bar{y}) = (x \vee a) \wedge \bar{y}$. Since $a \wedge y = a \wedge \bar{y}$, it follows that $x \vee (a \wedge \bar{y}) = (x \vee a) \wedge y$. Now, $y \leq \bar{y}$ implies $(x \vee a) \wedge y \leq (x \vee a) \wedge \bar{y}$, and $x \vee (a \wedge y) \geq (x \vee a) \wedge y$. The opposite inequality always holds, and thus

$$x \vee (a \wedge y) = (x \vee a) \wedge y.$$

(i) ⇒ (i'). Obvious. □

It was proved in [7] that an algebra $\mathcal{A}$ has the CEP if and only if for $\rho, \Theta \in Con\mathcal{B}, \mathcal{B} \in Sub\mathcal{A}$,

$$\text{from } \rho \vee \Delta = \Theta \vee \Delta \ (\text{in} Cw\mathcal{A}), \text{ it follows that } \rho = \Theta.$$

In [8], the CEP was characterized by the following modular property: $\mathcal{A}$ has the CEP, if and only if for $\rho \in Con\mathcal{C}$, $\mathcal{B}, \mathcal{C} \in Sub\mathcal{A}$,

$$\rho \leq B^2 \text{ implies } \rho \vee (\Delta \wedge B^2) = (\rho \vee \Delta) \wedge B^2, \text{ in} Cw\mathcal{A}.$$

Since in the lattice of weak congruences B^2(for $\mathcal{B} \in Sub\mathcal{A}$) is a maximum element in the class of the congruence induced by $m_\Delta : \rho \to \rho \wedge \Delta$, to which $\Delta_B = \{(x, x) | x \in B\}$ belongs (i.e. in Con $\mathcal{B}$), we can sum up the following lattice characterization of the CEP.

Theorem 1. *The following conditions are equivalent for an algebra $\mathcal{A}$:*

(*i*) $\mathcal{A}$ *has the CEP*

(*ii*) *for* $\rho, \Theta \in Con\mathcal{B}, \mathcal{B} \in Sub\mathcal{A}$,
$\rho \vee \Delta = \Theta \vee \Delta$ *implies* $\rho = \Theta$

(*iii*) *for* $\rho \in Cw\mathcal{A}, \mathcal{B} \in Sub\mathcal{A}$,
$\rho \leq B^2$ *implies* $\rho \vee (\Delta \wedge B^2) = (\rho \vee \Delta) \wedge B^2$

(*iv*) *for* $\rho, \Theta \in Cw\mathcal{A}$,
$\rho \leq \Theta$ *implies* $\rho \vee (\Delta \wedge \Theta) = (\rho \vee \Delta) \wedge \Theta$.

Proof. By Propositions 1 and 2 and by the above consideration. □

3. Semimodularity in the lattice of weak congruences

The conditions (iii) and (iv) from Theorem 1, can be formulated in terms of modular pairs, which are closely related to the semimodularity of the lattice, provided that its length is finite.

In the following we shall consider the conditions under which the lattice of weak congruences of an algebra is semimodular.

Proposition 3. *If L is a semimodular lattice of a finite length and a is a codistributive element of L, then for all $x, y \in L$*

$$x \leq y \text{ implies } x \vee (a \wedge y) = (x \vee a) \wedge y.$$

Proof. By Theorem 9, IV ,2, in [1], the semimodularity of L implies the equivalency of the following conditions for $a \in L$:

$$(i) \qquad \text{for all } x, y \in L$$
$$x \leq y \text{ implies } x \vee (a \wedge y) = (x \vee a) \wedge y, \text{ and}$$
$$(ii) \qquad \text{for all } x, y \in L$$
$$y \leq a \text{ implies } (y \vee x) \wedge a = y \vee (x \wedge a).$$

Now, since a is a codistributive element of L, $y \leq a$ implies

$$(y \vee x) \wedge a = (y \wedge a) \vee (x \wedge a) = y \vee (x \wedge a),$$

and (ii) holds. Hence, (i) is satisfied as well. □

Corollary 1. *If the lattice of weak congruences of an algebra $\mathcal{A}$ is semimodular and has a finite length, then $\mathcal{A}$ satisfies the CEP.*

Proof. Immediately by Theorem 1 and Proposition 3. □

Theorem 2. *If the lattice of weak congruences of an algebra $\mathcal{A}$ is semimodular, then the following are satisfied:*

(*i*) *$Con\mathcal{B}$ is a semimodular lattice for every $\mathcal{B} \in Sub\mathcal{A}$;*

(ii) *SubA is a semimodular lattice;*

(iii) *for $\rho, \Theta \in ConC$, $C \in SubB$, $B \in SubA$,*
$\rho \succ \Theta$ implies $\rho \vee \Delta_B \succ \Theta \vee \Delta_B$.

Proof. The lattices $ConB$ (for$B \in SubA$) and $SubA$ are convex sublattices of CwA. Thus, they are semimodular, proving (i) and (ii). To prove (iii), we use the fact that $(\Theta \vee \Delta_B) \wedge \rho = \Theta$ (since $(\Theta \vee \Delta_B) \wedge \rho \geq \Theta$, and $(\Theta \vee \Delta_B) \wedge \rho > \Theta$ would contradict the assumption $\rho \succ \Theta$), and $\rho \vee \Delta_B \vee \Theta = \rho \vee \Delta_B$. The proof of (iii) now follows by the semimodularity of L. □

Now, we shall show that two sufficient conditions for the semimodularity of the lattice of weak congruences are the CEP and the wCIP. To prove this, we need the following proposition.

Proposition 4. [8] *If A satisfies the CEP and the wCIP, then for every $B \in SubA$*

$$ConB \cong (B^2 \vee \Delta]_{conA}.$$

Theorem 3. *If $ConA$ and $SubA$ are semimodular lattices, and A has the CEP and wCIP, then CwA is semimodular.*

Proof. Let $\rho \in ConC$, $\Theta \in ConB$, $B,C \in SubA$, and assume that $\rho \succ \rho \wedge \Theta$. We shall consider three cases: a) $C = B$, b) $C < B$, and c) B and C are not comparable.

a) Because of the CEP and wCIP, by Proposition 4, we have that

$$ConC \cong (C^2 \vee \Delta]_{conA}, \quad (*)$$

and hence, since $ConA$ is semimodular,

$$\rho \vee \Theta \succ \Theta.$$

b) From $C,< B$ it follows that $\rho \wedge \Theta \in ConC$. Now, the CEP and the wCIP imply that

$$\rho \vee \Delta_B \succ (\rho \wedge \Theta) \vee \Delta_B.$$

Moreover, by the wCIP, since $\Delta_B \leq \Theta$,

$$\Delta_B \vee (\rho \wedge \Theta) = (\Delta_B \vee \rho) \wedge \Theta.$$

Finally, the semimodularity of $Con\mathcal{B}$, (the proof of which is similar to the one of (*)) implies

$$(\rho \vee \Delta_B) \vee \Theta = \rho \vee \Theta \succ \Theta.$$

c) $\mathcal{C}$ and $\mathcal{B}$ are incomparable, and $\rho \succ \rho \wedge \Theta$ implies $\rho = (\rho \wedge \Theta) \vee \Delta$. Hence,

$$\rho \vee \Theta = (\rho \wedge \Theta) \vee \Delta_c \vee \Theta = \Theta \vee \Delta_c = \Theta \vee \Delta_{B \vee C}.$$

$Sub\mathcal{A}$ is semimodular, Δ is a codistributive element of $Cw\mathcal{A}$, and hence

$$(\{Con\mathcal{B} | \mathcal{B} \in Sub\mathcal{A}\}, \leq) \cong Sub\mathcal{A}.$$

Thereby, $\mathcal{C} \succ \mathcal{B} \wedge \mathcal{C}$ implies $\mathcal{B} \vee \mathcal{C} \succ \mathcal{B}$, and hence

$$\rho \vee \Theta = \Theta \vee \Delta_{B \vee C} \succ \Theta$$

(since by the CEP in $ConB$, there is no $\Theta_1 > \Theta$ such that $\Theta_1 < \rho \vee \Theta$). □

Theorem 4. *Let $\mathcal{A}$ be an algebra whose lattice of weak congruences has a finite length. Now, if for $\rho, \Theta \in Con\mathcal{C}$, $\mathcal{C} \in Sub\mathcal{A}$,*

$$\rho \succ \Theta \textit{ implies } \rho \vee \Delta \succ \Theta \vee \Delta, \quad (*)$$

then $\mathcal{A}$ has the CEP.

Proof. If $\mathcal{A}$ does not satisfy the CEP, and $Cw\mathcal{A}$ has a finite length, then there are $\rho, \Theta \in Con\mathcal{C}$, such that $\rho \succ \Theta$ and $\rho \vee \Delta = \Theta \vee \Delta$, contrary to the assumption (*). □

Now, it is possible to formulate necessary and sufficient conditions under which the lattice of weak congrvences of an algebra is semimodular.

Theorem 5. *Let $\mathcal{A}$ be an algebra whose lattice of weak congruences $Cw\mathcal{A}$ has a finite length. Then, $Cw\mathcal{A}$ is a semimodular lattice if and only if the following three conditions are satisfied:*

(*i*) *$Con\mathcal{B}$ is a semimodular lattice for every $\mathcal{B} \in Sub\mathcal{A}$;*

(*ii*) *$Sub\mathcal{A}$ is a semimodular lattice;*

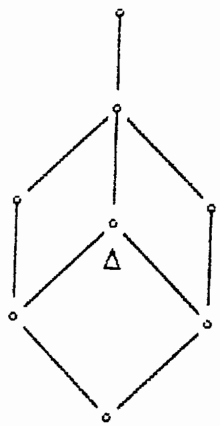
(*iii*) *for all $\rho, \Theta \in Con\mathcal{C}$, $\mathcal{C} \in Sub\mathcal{B}$, $\mathcal{B} \in Sub\mathcal{A}$,*

$$\rho \succ \Theta \textit{ implies } \rho \vee \Delta_B \succ \Theta \vee \Delta_B.$$

Proof. ($\Leftarrow$)
Let $\rho, \Theta \in Cw\mathcal{A}$, $\rho \in Con\mathcal{C}$, $\Theta \in Con\mathcal{B}$, $\mathcal{B},\mathcal{C} \in Sub\mathcal{A}$, and let $\rho \succ \rho \wedge \Theta$. We shall consider three cases: a) $\mathcal{B} = \mathcal{C}$, b) $\mathcal{C} < \mathcal{B}$, and c) $\mathcal{B}$ and $\mathcal{C}$ are not comparable. In the case a), $\rho \vee \Theta \succ \Theta$ holds because of (i). b) $\mathcal{C} < \mathcal{B}$ implies $\rho \wedge \Theta \in Con\mathcal{C}$, and by (iii), $\rho \vee \Delta_B \succ (\rho \wedge \Theta) \vee \Delta_B$. Obviously, $(\rho \wedge \Theta) \vee \Delta_B \leq (\rho \vee \Delta_B) \wedge \Theta$, and thus $(\rho \wedge \Theta) \vee \Delta_B = (\rho \vee \Delta_B) \wedge \Theta$. Now, $(\rho \vee \Delta_B) \wedge \Theta = \rho \vee \Theta \succ \Theta$, since $Con\mathcal{B}$ is a semimodular lattice. The proof of case c) is similar to the one of Theorem 3. provided that Theorem 4. is used.
($\Rightarrow$). Immediately by Theorem 2. □

Example 1.

	a	b	c	d
a	b	b	d	c
b	b	a	c	d
c	b	b	d	c
d	a	b	c	c



CwG
*Fig.*1

The groupoid G ([9]) has a semimodular lattice of weak congruences, given in Fig.1. *ConG* and *SubG* are distributive lattices, and G satisfies the CEP and the wCIP.

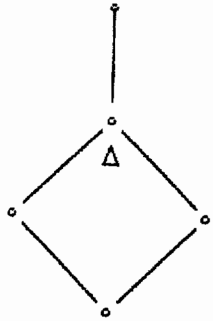
4. Some applications

4.1 Semimodularity of CwL, where L is a lattice

For a lattice L as an algebra,the following statements are known:
a) SubL is a semimodular lattice if and only if L is a chain ([4]).
b) For every L, ConL is a distributive lattice.
c) If L is a distributive lattice, then L has the CEP ([3]).

Example 2.

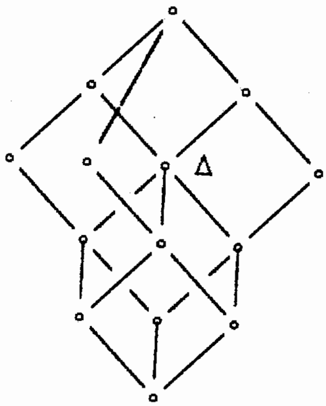
Let L_2 be a two-element chain. CwL_2 is a semimodular lattice (Fig.2)



CwL_2
Fig.2

Example 3.

The lattice of weak congruences CwL_3 of a three-element chain L_3 is given in Fig.3.



CwL_3
Fig.3

This lattice is not semimodular.

Theorem 6. *The lattice L has a semimodular lattice of weak congruences if and only if L is a two-element chain.*

Proof. If CwL is a semimodular lattice,then SubL is semimodular as well. Hence, by a), L is a chain. On the other hand, if L is a chain with more than two elements, then CwL has a convex sublattice CwL_3, which is not semimodular (Example 2.) Thus, CwL is not semimodular. Since CwL_2 is semimodular, the proof is complete. □

4.2 Semimodularity of $Cw\mathcal{A}$, where $\mathcal{A}$ is a unary algebra

The following are known for a unary algebra $\mathcal{A}$:
a) Sub$\mathcal{A}$ is a distributive lattice
b) $\mathcal{A}$ has the CEP ([3])

Lemma 1. *If $\mathcal{A}$ is a unary algebra,then for* $\rho \in Cw\mathcal{A}$

$$\rho \vee \Delta = \rho \cup \Delta$$

Proof. Straightforward, since $\rho \cup \Delta \in Con\mathcal{A}$. □

Lemma 2. *Every unary algebra $\mathcal{A}$ satisfies the wCIP.*

Proof. Let $\Theta \in Con\mathcal{A}$ i.e.$\Delta \leq \Theta$ Then, by Lemma 1.

$$\Delta \vee (\rho \wedge \Theta) = \Delta \cup (\rho \wedge \Theta) = (\Delta \cup \rho) \wedge (\Delta \cup \Theta) = (\Delta \cup \rho) \wedge \Theta = (\Delta \vee \rho) \wedge \Theta.$$

□

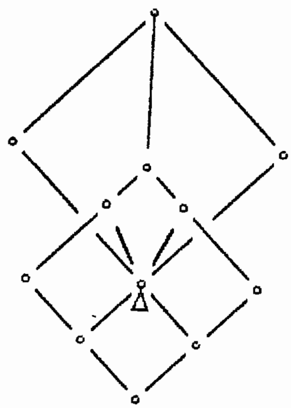
Theorem 7. *The lattice of weak congruences of a unary algebra is semimodular if and only if its lattice of congruences is semimodular.*

Remark. Necessary and sufficient conditions under which, for a unary algebra $\mathcal{A}$, Con$\mathcal{A}$ is semimodular, were given in [1].
Proof ($\Rightarrow$) Obvious.
($\Leftarrow$) By Theorem 2. , by a), b), c), and by Lemma 2. □

Example 4.

$\mathcal{A} = (A, f)$, $A = \{a, b, c, d,\}$

A	a	b	c	d
f	c	d	a	b



CwA
Fig.4

For this unary algebra $\mathcal{A}$ Con$\mathcal{A}$ is not semimodular, and neither is Cw$\mathcal{A}$.

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REZIME

O CEP-U I SEMIMODULARNOSTI U MREŽI SLABIH KONGRUENCIJA

U radu se daju neki potrebni i dovoljni uslovi da proizvoljna algebra ima svojstvo proširenja kongruencija (CEP). Uslovi se svode na neke modularne zakone u mreži slabih kongruencija algebre. U nastavku se daju potrebni i dovoljni uslovi da mreža slabih kongruencija proizvoljne algebre bude polumodularna. Posebno se, kao primer, ti uslovi razmatraju u slučaju kada je

data algebra mreža, odnosno kada je ona unarna.

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