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SOME RESULTS ON AXIOMATIZABILITY

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Abstract

The aim of this paper is to generalize the main idea of [1] where we prove that the class of semigroup-relation algebras is not axiomatizable. In the present paper we give some conditions for non-axiomatizability of some classes of cylindric algebras and generally, for any class of universal algebra.

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1. Introduction

Algebraic logic in the modern sense began 1935, with Tarski's paper on the foundations of the calculus of systems. In this paper Tarski introduced the algebra of propositional formulas. He defined a relation $\equiv$ on the set of all propositional formulas and asserted that $\equiv$ forms what we now call a congruence relation on the algebra of formulas, and that the corresponding quotient algebra is a Boolean algebra. Subsequently, a number of different logics were algebraized in this or a similar way. Cylindric and relation algebras are two different algebraizations of the first-order logic.

Cylindric algebras are Boolean algebras enriched with some constants and some unary operations such that these new constants and unary operations satisfy some additional equations.

Definition 1.1. *Let α be any ordinal number. By a* cylindric algebra of dimension α *we mean an algebraic structure*

$$\mathcal{A} = (A, +, \cdot, -, 0, 1, c_\kappa, d_{\kappa\lambda})_{\kappa,\lambda<\alpha}$$

such that 0,1 and $d_{\kappa\lambda}$ are distinguished elements of A (for all $\kappa, \lambda < \alpha$), $-$ and c_κ are unary operations on A (for all $\kappa < \alpha$), $+$ and $\cdot$ are binary operations on A and such that the following postulates are satisfied for any $x, y \in A$ and any $\kappa, \lambda, \mu < \alpha$:

1. *The structure $(A, +, \cdot, -, 0, 1)$ is a Boolean algebra (*BA*);*
2. $c_\kappa 0 = 0$;
3. $x + c_\kappa x = c_\kappa x$;
4. $c_\kappa(x \cdot c_\kappa y) = c_\kappa x \cdot c_\kappa y$;
5. $c_\kappa c_\lambda x = c_\lambda c_\kappa x$;
6. $d_{\kappa\kappa} = 1$;
7. *if $\kappa \neq \lambda, \mu$ then $d_{\lambda\mu} = c_\kappa(d_{\lambda\kappa} \cdot d_{\kappa\mu})$;*
8. *if $\kappa \neq \lambda$, then $c_\kappa(d_{\kappa\lambda} \cdot x) \cdot c_\kappa(d_{\kappa\lambda} \cdot -x) = 0$.*

The class of all the cylidric algebras is denoted by CA, *and the class of all the cylindric algebras of dimension α by* CA_α.

Relation algebras were historically the first algebraic version of a portion of first-order logic. They appeared by the abstraction of some concrete algebras of binary relations. In [6] one can find the realization of building the whole of mathematics inside relation algebras - the realization of the idea of Tarski and Chin.

Definition 1.2. *Let $\mathcal{A} = (A, +, \cdot, -, 0, 1, \circ, 1', ^{-1})$ be an algebra of type (2, 2, 1, 0, 0, 2, 0, 1). It is called a* relation algebra (RA) *if the following axioms are satisfied:*

1. *$(A, +, \cdot, -, 0, 1)$ is a Boolean algebra;*
2. *$(A, \circ, 1')$ is a monoid;*

3. *Operation* $^{-1}$ *is an involution of the semigroup* $(A, \circ)$, *i.e.*

$$(\forall x)(\forall y)(x \circ y)^{-1} = y^{-1} \circ x^{-1};$$

$$(\forall x)(x^{-1})^{-1} = x;$$

4. *Operations* $^{-1}$ *and* $\circ$ *are distributive over* $+$ *i.e.*

$$(\forall x)(\forall y)(x + y)^{-1} = x^{-1} + y^{-1},$$

$$(\forall x)(\forall y)(\forall z) x \circ (y + z) = (x \circ y) + (x \circ z);$$

5. $(\forall x)(\forall y)(x^{-1} \circ (\overline{\overline{x} \circ y})) \cdot y = 0.$

There is a very close connection between RA and CA . Here we shall introduce the standard method of associating an RA with a CA_α, according to Henkin and Tarski. First, we shall introduce some notions and notations. Let $\mathcal{A} \in CA_\alpha$, and $x \in A$. Then

$$\Delta x = \{\kappa < \alpha : c_\kappa x \neq x\}.$$

If $\Gamma \subseteq \alpha$, then we define

$$Cl_\Gamma \mathcal{A} = \{x \in A : \Delta x \cap \Gamma = \emptyset\}.$$

Let $\beta < \alpha$. Then,

$$\mathcal{R}\delta_\beta \mathcal{A} = (A, +, \cdot, -, 0, 1, c_\lambda, d_{\kappa\lambda})_{\kappa, \lambda < \beta}.$$

So, algebra $\mathcal{R}\delta_\beta \mathcal{A}$ is the β-reduct of the algebra $\mathcal{A}$. Algebra $\eta r_\beta \mathcal{A}$ is defined as a special subreduct of $\mathcal{A}$:

$$\eta r_\beta \mathcal{A} = \mathcal{B} < \mathcal{R}\delta_\beta \mathcal{A},$$

such that

$$B = Cl_{\alpha \setminus \beta} \mathcal{A}.$$

In other words, the carrier of the algebra $\eta r_\beta \mathcal{A}$ is the set of such elements $x \in A$ which has the property

$$\text{if } \lambda \in \alpha \setminus \beta \text{ then } c_\lambda x = x.$$

The carrier of $\eta r_\beta \mathcal{A}$ will be denoted by $Nr_\beta A$.
Let $\mathcal{A} \in \mathrm{CA}_\alpha, x \in A$, and $\kappa, \lambda \in \alpha$. We define the unary operations s^κ_λ in following way:

$$s^\kappa_\lambda x = \begin{cases} x & \text{if } \kappa = \lambda \\ c_\kappa(d_{\kappa\lambda} \cdot x) & \text{if } \kappa \neq \lambda \end{cases}$$

If $\kappa, \lambda, \mu \in \alpha$ and $x \in A$, then

$$_\mu s(\kappa, \lambda)x = s^\mu_\kappa s^\kappa_\lambda s^\lambda_\mu x.$$

Definition 1.3. *Let $\mathcal{A} \in \mathrm{CA}_\alpha, \alpha \geq 3$. We define the algebra*

$$\mathcal{R}a\mathcal{A} = (Nr_2\mathcal{A}, +, \cdot, -, 0, 1, \circ, d_{01}, ^{-1})$$

such that for all $x, y \in Nr_2\mathcal{A}$ we have

$$x \circ y = c_2(s^1_2 x \cdot s^0_2 x)$$

$$x^{-1} = s_2(0, 1).$$

So, the carrier of algebra $\mathcal{R}a\mathcal{A}$ is the set of all the elements $x \in A$ which has the property:

$$\text{for all } \lambda < \alpha, \text{ if } \lambda \neq 0 \text{ and } \lambda \neq 1, \text{ then } c_\lambda x = x.$$

The following theorem can be proved:

Theorem 1.1. *(see [3]) If $\mathcal{A} \in \mathrm{CA}_\alpha, \alpha \geq 4$, then $\mathcal{R}a\mathcal{A} \in \mathrm{RA}$.*

Something more holds. Namely, any relation algebra can be obtained from a cylindric algebra of dimension 3, via the correspondence $\mathcal{R}a$. More precisely, let M be the following class of cylindric algebras of dimension 3:

$$M = \{\mathcal{A} \in \mathrm{S}Nr_3\mathrm{CA}_4 : \mathcal{A} = \sigma g^{\mathcal{A}}\{x \in A : \Delta x \subseteq 2\}\},$$

where for $X \subseteq A, \sigma g^{\mathcal{A}} X$ denote the subalgebra of $\mathcal{A}$ generated by X.

Remark 1.1. *In the rest of the present paper M will always denote this class of cylindric algebras.*

The following can be proved:

Theorem 1.2. *(see [3])*

1. $\mathcal{R}a^*M = \mathrm{RA}$ *(i.e. $\mathcal{R}a$ maps M onto* RA*).*
2. *$\mathcal{R}a$ induces a one-to-one mapping of isomorphism types of members of M onto isomorphism types of* RA*'s.*

2. Ultraproducts and the mapping $\mathcal{R}a$

Since the mapping $\mathcal{R}a : M \to \mathrm{RA}$ is onto, this means that for any relation algebra $\mathcal{A}$ there exists (up to isomorphism a unique) a cylindric algebra $\mathcal{B} \in M \subseteq \mathrm{CA}_3$ such that $\mathcal{R}a(\mathcal{B}) = \mathcal{A}$. Hence, we can define for any $K \subseteq \mathrm{RA}$ the class

$$\mathcal{R}a^{-1}(K) = \{\mathcal{B} \in M : \mathcal{R}a(\mathcal{B}) \in K\}.$$

We shall call this class the *corresponding class* of cylindric algebras (relative to K). Also, because of T1.1. (i.e. $\mathcal{R}a\mathcal{A} \in \mathrm{RA}$, for any $\mathcal{A} \in \mathrm{CA}_\alpha, \alpha \geq 4$), we can define for any $K \subseteq \mathrm{RA}$ and $\alpha \geq 4$, the *corresponding class* $\mathcal{R}a_\alpha^{-1}(K)$ *of cylindric algebras of dimension* α:

$$\mathcal{R}a_\alpha^{-1}(K) = \{\mathcal{B} \in \mathrm{CA}_\alpha : \mathcal{R}a(\mathcal{B}) \in K\}.$$

In this paragraph we shall give some conditions, under which the property "to be not elementary" is preserved by the mappings $\mathcal{R}a^{-1}$ and $\mathcal{R}a_\alpha^{-1}$.(In the sequel, K will always be an abstract class, i.e. K will be closed under isomorphisms).

Proposition 2.1. *Let $K \subseteq \mathrm{RA}$ be a class of relation algebras which is not closed under the ultraproducts. If for all the ultraproducts $\prod_{i\in I} \mathcal{A}_i/D, (\mathcal{A}_i \in \mathcal{R}a^{-1}(K))$ it holds that*

$$\mathcal{R}a(\prod_{i\in I} \mathcal{A}_i/D) \cong \prod_{i\in I} \mathcal{R}a\mathcal{A}_i/D, \tag{1}$$

then the corresponding class $\mathcal{R}a^{-1}(K)$ is not an elementary class.

Proof. Since the class K is not closed under ultraproducts, there exist $\mathcal{B}_i \in K$ $(i \in I)$ and an ultrafilter D over I, such that

$$\prod_I \mathcal{B}_i/D \notin K. \tag{2}$$

Since we have that $\mathcal{R}a^*M = \mathrm{RA}$, then for all $i \in I$ there is $\mathcal{A}_i \in M$ such that $\mathcal{R}a(\mathcal{A}_i) = \mathcal{B}_i$. We shall prove that $\prod_I \mathcal{A}_i/D \notin \mathcal{R}a^{-1}(K)$.

Suppose the opposite, i.e. $\prod_I \mathcal{A}_i/D \in \mathcal{R}a^{-1}(K)$. This is equivalent to $\mathcal{R}a(\prod \mathcal{A}_i) \in K$. But , from assumption (1) of this proposition, we have

$$\mathcal{R}a(\prod \mathcal{A}_i/D) \cong \prod \mathcal{R}a\mathcal{A}_i/D = \prod \mathcal{B}_i/D \in K,$$

which is in contradiction with (2). Hence, $\prod \mathcal{A}_i/D \notin \mathcal{R}a^{-1}(K)$ and $\mathcal{R}a^{-1}(K)$ is not an elementary class.

The condition (1) speaks about the permutability of operator $\mathcal{R}a$ and the operator of the ultraproduct Up. We already know (see [3]) that the operator $\mathcal{R}a$ commutire with the operator of direct products P :

$$\mathcal{R}a^*\mathrm{P}(K) = \mathrm{P}\mathcal{R}a^*(K), \text{ for } K \subseteq \mathrm{CA}_\alpha, \ \ \alpha \geq 3.$$

Similarly, (see [3]) it holds that

$$\mathcal{R}a^*\mathrm{S}(K) = \mathrm{S}\mathcal{R}a^*(K), \text{ for } K \subseteq \mathrm{S}Nr_3\mathrm{CA}_4 \subseteq M.$$

We wonder if

$$\mathcal{R}a^*\mathrm{Up}(K) = \mathrm{Up}\mathcal{R}a^*(K), \text{ for } K \subseteq M?$$

We are going to prove the following:

Theorem 2.1. *Let $\mathcal{A}_i \in \mathrm{CA}_\alpha, i \in I, \alpha \geq 3$. Then, for any ultrafilter D over I it holds that $\prod_I \mathcal{R}a\mathcal{A}_i/D$ is isomorphic to a subalgebra of $\mathcal{R}a\prod_I \mathcal{A}_i/D$.*

Proof. The carrier of the algebra $\mathcal{R}a\mathcal{A}_i$ is the set $Nr_2\mathcal{A}_i$. Let us define the mapping $\Psi : Nr_2\mathcal{A}_i/D \to Nr_2(\prod \mathcal{A}_i/D)$ in the following way:

$$\Psi(x/D) = \{y \in \prod \mathcal{A}_i : \{i \in I : x_i = y_i\} \in D\}.$$

Notice, that $\Psi(x/D)$ is the class containing the element $x \in \prod Nr_2\mathcal{A}_i$ in the algebra $\prod \mathcal{A}_i/D$. The proof of the theorem will be divided into four steps.

1. First we are going to prove that $\Psi(x/D) \in Nr_2(\prod \mathcal{A}_i/D)$. Because of the definition of operator Nr_2 it is true if and only if

$$c_\lambda(\Psi(x/D)) = \Psi(x/D), \text{ for all } \lambda \geq 2 \ \ (\lambda < \alpha).$$

According to the definition of the operations on an ultraproduct, we know that the operation c_λ "works" on some class of elements in the following way: we have to choose a representative x of given class, apply operation c_λ on this representative, and finally the result is the class of elements which contains the element $c_\lambda x$.

Because of the definition of the mapping Ψ,we have that $x \in \Psi(x/D)$. So, in $Nr_2(\prod \mathcal{A}_i/D)$ we have:

$$c_\lambda(\Psi(x/D)) = c_\lambda x/D = x/D, \text{ for } \lambda \geq 2,$$

because for all $x \in Nr_2\mathcal{A}_i$ we have that $c_\lambda x = x$ for all $\lambda \geq 2$.

2. We shall prove now that Ψ is well-defined i.e. if $x/D = y/D$ in $\prod Nr_2\mathcal{A}_i/D$, then $\Psi(x/D) = \Psi(y/D)$. Now we have that

$$\Psi(x/D) = \{z \in \prod \mathcal{A}_i : \{i : x_i = z_i\} \in D\},$$

$$\Psi(y/D) = \{u \in \prod \mathcal{A}_i : \{i : y_i = u_i\} \in D\}.$$

Let us prove that $\Psi(x/D) \subseteq \Psi(y/D)$. Let $z \in \Psi(x/D)$, then

$$\{i \in I : x_i = z_i\} = D_1 \in D.$$

Since $x/D = y/D$ in $\prod Nr_2\mathcal{A}_i/D$, then

$$\{i \in I : x_i = y_i\} = D_2 \in D.$$

Because of the property of ultrafilters, we have that $D_1 \bigcap D_2 \in D$ i.e.

$$\{i \in I : x_i = y_i = z_i\} = D_1 \bigcap D_2 \in D.$$

But $\{i \in I : y_i = z_i\} = D_1 \bigcap D_2$, and again because of the properties of the ultrafilters, we have that

$$\{i \in I : y_i = z_i\} \in D$$

and consequently $z \in \Psi(y/D)$.

3. Now, we shall prove that Ψ is a one-to-one mapping. Let $x/D, y/D \in \prod Nr_2\mathcal{A}_i/D$ and $\Psi(x/D) = \Psi(y/D)$. This means that

$$x \in \{u \in \prod \mathcal{A}_i : \{i : x_i = u_i\} \in D\} = \{z \in \prod \mathcal{A}_i : \{i : y_i = z_i\} \in D\} \Rightarrow$$

$$x \in \{z \in \prod \mathcal{A}_i : \{i : y_i = z_i\} \in D\} \Rightarrow$$

$$\{i : y_i = x_i\} \in D \Rightarrow x/D = y/D \text{ in } \prod \mathcal{N}r_2\mathcal{A}_i/D.$$

4.Ψ is a homomorphism, because if $x \in \prod \mathcal{N}r_2\mathcal{A}_i$ then Ψ maps the class containing x in $\prod \mathcal{N}r_2\mathcal{A}_i/D$ to the class containing x in $\mathcal{N}r_2 \prod \mathcal{A}_i/D$, and the operations on an ultraproduct are defined via repesentatives of the corresponding classes. □

Of course, if D is a principal ultrafilter on I, then

$$\prod_I \mathcal{R}a\mathcal{A}_i/D \cong \mathcal{R}a \prod_I \mathcal{A}_i/D.$$

Suppose that D is non-principal. For this case we can prove the following:

Corollary 2.1. *Let $\mathcal{A}_i \in \mathrm{CA}_\alpha, (i \in I), \alpha \geq 3, D$ a non-principal ultrafilter over I. Then, $\prod_I \mathcal{R}a\mathcal{A}_i/D \cong \mathcal{R}a \prod_I \mathcal{A}_i/D$ if for all $x \in \mathcal{N}r_2\Pi\mathcal{A}_i$ there is an element $y \in \Pi\mathcal{A}_i$, such that*
(i) $c_\lambda y = y$, for all $\alpha > \lambda \geq 2$,
(ii) $\{i \in I : x_i = y_i\} \in D$.

Proof. Conditions (i) and (ii) imply that for all $x/D \in \mathcal{N}r_2\Pi\mathcal{A}_i/D$ there exists an element y/D from $\Pi\mathcal{N}r_2\mathcal{A}_i/D$ such that $\Psi(y/D) = x/D$, and this means that Ψ is "onto". □

Conjecture 2.1. There are algebras $\mathcal{A}_i \in \mathrm{CA}_\alpha(i \in I), \alpha \geq 3$, such that for some (non- principal) ultrafilter D over I it holds

$$\prod_I \mathcal{R}a\mathcal{A}_i/D \ncong \mathcal{R}a \prod_I \mathcal{A}_i/D.$$

In Proposition 2.1. we have a sufficient condition to transfer the property of being not elementary from class $K \subseteq \mathrm{RA}$ to the corresponding class $\mathcal{R}a^{-1}K$. Using T2.1. we can give an other sufficient condition.

Proposition 2.2. *Let $K \subseteq \mathrm{RA}$ be a class closed under the taking subalgebras, which is not closed under ultraproducts. Then, the corresponding class $\mathcal{R}a^{-1}K$ is not elementary.*

Proof. Since the class K is not closed under ultraproducts, there are $\mathcal{B}_i \in K, i \in I$, such that for some ultrafilter D over I it holds that

$$\prod_I \mathcal{B}_i/D \notin K. \tag{3}$$

Let $\mathcal{A}_i \in M, i \in I$, such that

$$(4) \qquad \mathcal{R}a(\mathcal{A}_i) = \mathcal{B}_i, i \in I.$$

(Of course, $\mathcal{A}_i \in \mathcal{R}a^{-1}(K)$.) Then, because of T2.1. we have

$$\Pi \mathcal{R}a\mathcal{A}_i/D \cong \mathcal{A} < \mathcal{R}a\Pi\mathcal{A}_i/D.$$

Suppose $\Pi\mathcal{A}_i/D \in \mathcal{R}a^{-1}(K)$. This means that $\mathcal{R}a(\Pi\mathcal{A}_i/D) \in K$ and because of $\mathrm{S}(K) = K$ we obtain $\Pi\mathcal{R}a\mathcal{A}_i/D \in K$, a contradiction to (3) and (4). So, $\mathcal{R}a^{-1}K$ is not an elementary class. □

3. The case of semigroup-relation algebras

In [2] and [4] we introduced and studied a new class of relation algebras, namely the class of semigroup relation algebras (S_Φ). The " construction Φ" used there enables us, among other things, to show, how we can " transfer" the result about the unsolvability of the word problem from the class of semigroups to the class of relation algebras, and give a new proof of the undecidability of the equational theory of relation algebras (Tarski, 1953). It is well known that every Boolean algebra can be "enriched" to be a relation algebra. But, for semigroups the situation is different. Namely, for any cardinal number $\lambda \geq 3$ there exists a semigroup $\mathcal{S}$, cardinality λ, such that $\mathcal{S}$ is not a semigroup reduct for any relation algebra (see [2]). But, it is easy to see that we can embed any semigroup into the semigroup reduct of some relation algebra. For example, we can do the following (see [4]):
Let $\mathcal{S}$ be a semigroup. Every semigroup is embeddable into a semigroup with an identity element, so we can suppose that $\mathcal{S}$ has an identity. The semigroup $\mathcal{S}$ can be represented as a semigroup of transformation $\mathcal{T}(\mathcal{S})$ in the following way: every element $s \in S$ is represented as the right translation $\rho_s = \{(x, x \cdot s) : s \in S\}$, and the operation $\cdot$ of $\mathcal{S}$ is represented as the composition of functions. Of course,

$$T(S) = \{\rho_s : s \in S\} \subseteq \mathcal{P}(S^2).$$

The relation algebra $\Phi(\mathcal{S})$ is defined as the subalgebra of the full relation algebra $\mathcal{R}(S) = (\mathcal{P}(S^2), \cup, \cap, -, \emptyset, S^2, \circ, \Delta_S, ^{-1})$ generated by the elements of $\mathcal{T}(\mathcal{S})$. Of course, $\mathcal{S}$ is a subreduct of $\Phi(\mathcal{S})$. For a relation algebra $\mathcal{A}$ we say that it is a *semigroup relation algebra* if there is a semigroup $\mathcal{S}$ such that $\mathcal{A} = \Phi(\mathcal{S})$. In [4] we gave a characterization of a proper RA to be a semigroup relation algebra. However, this characterization is not in the first order language. In [1] it is proved that the class S_Φ of all the relation algebras which are isomorphic to some semigroup relation algebra is not first order axiomatizable. Can we say something about the corresponding classes of cylindric algebras $\mathrm{CA}_\Phi = \mathcal{R}a^{-1}(S_\Phi)$ and $\mathrm{CA}_\Phi^\alpha = \mathcal{R}a_\alpha^{-1}(S_\Phi)$?

Note first that the class S_Φ is not closed under the subalgebras (see [4]), so, in general case, we cannot use the criteria for non elementarity given in Proposition 2. Starting from Proposition 1. and from the proof of non-elementarity of the class S_Φ, we can prove three sufficient conditions for the non-elementarity of the class CA_Φ.

Corollary 3.1. *Class* $\mathrm{CA}_\Phi = \mathcal{R}a^{-1}(S_\Phi)$ *is not elementary if one of the following conditions holds:*
(i) Let $\mathcal{B}$ *be an infinite Boolean group,* $\mathcal{A} \in M \subseteq \mathrm{CA}_3$ *such that* $\mathcal{R}a(\mathcal{A}) = \Phi(\mathcal{B})$. *Then there is a non-principal ultrafilter* D *over* ω, *such that* $\mathcal{R}a(\prod_\omega \mathcal{A}/D)$ *is not a semigroup relation algebra.*
(ii) For every $\mathcal{A} \in \mathrm{CA}_\Phi$ *and any ultrafilter* D *over* ω *it holds that*

$$\mathcal{R}a(\prod_\omega \mathcal{A}/D) \cong \prod_\omega \mathcal{R}a\mathcal{A}/D.$$

(iii) For every infinite Boolean group $\mathcal{B}$ *and any non-principal ultrafilter* D *over* ω *it holds that if* $\mathcal{R}a(\mathcal{A}) = \Phi(\mathcal{B})$, *then*

$$\mathcal{R}a(\prod_\omega \mathcal{A}/D) \cong \prod_\omega \mathcal{R}a\mathcal{A}/D.$$

Proof. (i) Follows from Lemma 3. and Proposition 1. in [1] and the definition of the class CA_Φ.
(ii) Follows from Proposition 2.1.
(iii) Follows from Proposition 2.1. (in the present paper) and Lemma 3., Proposition 1. from [1] . □

4. Generalization to classes different from S_Φ

The aim of this paragraph is to generalize the main idea of [1] , and to obtain some similar criteria for being non-elementary in the case of any class of universal algebra. The notions connected with labelling are almost the same as the case of relation algebras. Let L be a first order universal algebraic language (i.e. with no relation symbols). By π^L we shall denote the set of all the terms in this language. In the sequel we shall fix the language L, so instead of π^L we can simply write π.

Definition 4.1. *Let π be the set of all the terms in the language L. We call the family $\{\pi_n : n \in \omega\} \subseteq \mathcal{P}(\pi)$ a labelling of the set π, if*

1. $n \leq m \Rightarrow \pi_n \subseteq \pi_m$,
2. $\bigcup\{\pi_n : n \in \omega\} = \pi$.

Example 4.1. Let π_n be the set of those terms belonging to π which have at most n variables. Then, $\{\pi_n : n \in \omega\}$ is a labelling of π.
Example 4.2. Let π_n be the set of terms from π which have at most n function symbols. Then, $\{\pi_n : n \in \omega\}$ is a labelling of π.

Let $\mathcal{A}$ be a universal algebra (of the language L), and $X \subseteq A$. Then, as in the case of relation algebras, by $\pi_n(X)$ we denote the set of all the elements from A, which are values of terms from π_n, over the set X.

Lemma 4.1. *Let $\{\pi_n : n \in \omega\}$ be a labelling of the set of all the terms π of the language L, $\mathcal{A}$ a universal algebra of the same language, and $G \subseteq A$. Then, G generates $\mathcal{A}$ iff $A = \bigcup_{n\in\omega} \pi_n(G)$.*

Proof. It follows from the definition of the generating set of some algebra and from D4.1. □

In literature we can find several "variations" of the following definition.

Definition 4.2. *Let $\mathcal{A}$ be a universal algebra of the language L. For the set $B \subseteq A$ we say that it is* definable in $\mathcal{A}$ *if there is a formula $\phi(x)$ in L, such that for all $b \in A$*

$$b \in B \text{ iff } \mathcal{A} \models_{x=b} \phi(x).$$

In this case we say that B is definable with $\phi(x)$ in $\mathcal{A}$.

Example 4.3. Let $\mathcal{Z} = (Z, +)$ be the additive group of integers. Then the set of even numbers E is definable in $\mathcal{Z}$ because

$$a \in E \text{ iff } Z \models_{x=a} (\exists y)(y + y = x).$$

Example 4.4. Let $\mathcal{N} = (\omega\backslash\{0\}, \cdot, 1)$, and $\cdot$ is the usual multiplication. The set of prime numbers P is definable in this monoid because

$$a \in P \text{ iff } \mathcal{N} \models_{x\equiv a} x \neq 1 \wedge (\forall y)((\exists k)(k \cdot y = x) \Rightarrow (y = 1 \vee y = x)).$$

Let $\mathcal{A}$ be an algebra, $\phi(x)$ a formula on the language of $\mathcal{A}$. Denote by $\phi\{\mathcal{A}\}$ the set of all elements $a \in A$ such that $\mathcal{A} \models_{x=a} \phi(x)$. So, if $\mathcal{A}$ is a universal algebra of the language L, then $B \subseteq A$ is definable in $\mathcal{A}$ iff there is a formula $\phi(x)$ in L, such that $B = \phi\{\mathcal{A}\}$.

Theorem 4.1. *Let K be a class of universal algebras of the language L, π the set of all the terms on the same language, and $\{\pi_n : n \in \omega\}$ a labelling of π. Let $\phi(x)$ be a formula on L such that*
(i) every algebra $\mathcal{A} \in K$ is generated by $\phi\{\mathcal{A}\}$;
(ii) there exist $\mathcal{A} \in K$ such that $(\forall n \in \omega)(\exists a_n \in A)(a_n \notin \pi_n(\phi\{\mathcal{A}\}))$.
Then class K is not axiomatizable.

Proof. Let $\mathcal{A}$ be an algebra from condition (ii) of the present theorem and D some non-principal ultrafilter over ω. We shall prove that the ultrapower $\mathcal{B} = \prod_\omega \mathcal{A}/D$ is not in K. Because of (ii) we have that

$$a = (a_n : n \in \omega)/D \in B.$$

Suppose $\mathcal{B} \in K$. Then, because of (i) and Lemma 4.1. we conclude that there is n such that

$$a \in \pi_n(\phi\{\mathcal{B}\}).$$

This means that $a = t[f^1, f^2, ..., f^k]$ for some term $t \in \pi_n$ and $f^1, f^2, ..., f^k \in \phi\{\mathcal{B}\}$. Because of the definition of ultraproducts,

$$D_1 = \{i \in \omega : a_i = t[f_i^1, f_i^2, ..., f_i^k]\} \in D.$$

Since $\mathcal{B} \models_{x=f^j} \phi(x)$, for $j = 1, 2, \ldots, k$, and ϕ is a first order formula, then because of the theorem of Loš we have

$$\{i \in \omega : \mathcal{A} \models \phi[f_i^j]\} \in D.$$

In other words,

$$A_1 = \{i \in \omega : f_i^1 \in \phi\{\mathcal{A}\}\} \in D,$$

$$A_2 = \{i \in \omega : f_i^2 \in \phi\{\mathcal{A}\}\} \in D,$$

...

$$A_k = \{i \in \omega : f_i^k \in \phi\{\mathcal{A}\}\} \in D.$$

Because of the properties of ultrafilters,

$$\cap\{A_s : s = 1, 2, ..., k\} = D_2 \in D,$$

and since $D_1 \in D$, then $D_1 \cap D_2 \in D$. D is a non-principal ultrafilter, so every $X \in D$ is infinite. This means that there are infinitely many indices $j \in \omega$, such that

$$f_j^1, f_j^2, ..., f_j^k \in \phi\{\mathcal{A}\}, a_j = t[f_j^1, f_j^2, ..., f_j^k].$$

So, $a_j \in \pi_n(\phi\{\mathcal{A}\})$ for infinitely many indices j . However, by assumption (ii) from the present theorem we have

$$a_s \notin \pi_s(\phi\{\mathcal{A}\}), \text{ for } s > n.$$

Since $\pi_n \subseteq \pi_s$, this implies

$$a_s \notin \pi_n(\phi\{\mathcal{A}\}), \text{ for all } s > n,$$

which is a contradiction. So, $\mathcal{B} \notin K$ and K is not an elementary class. □

Note, that assumption (i) in the previous theorem is not "too strong", because for any algebra $\mathcal{A}$ there is a formula $\phi(x)$ such that $\phi\{\mathcal{A}\}$ generates $\mathcal{A}$. Namely, for $\phi(x)$ we can always take the formula $x = x$. In the case of semigroup-relation algebras, for the formula $\phi(x)$ we can take

$$((x^{-1} \circ x) \cdot 1' = x^{-1} \circ x) \wedge ((x \circ x^{-1}) \cdot 1' = 1'),$$

so Proposition 1. in [1] is in some sense a corollary of T4.1. in the present paper.

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REZIME

NEKI REZULTATI O AKSIOMATIZABILNOSTI

Cilj ovog rada je da se uopšti glavna ideja iz rada [1], gde je dokazano da klasa semigrupnih relacionih algebri nije aksiomatizabilna. U ovom radu su dati uslovi pod kojima su neke klase cilindričnih algebri takodje ne-aksiomatizabilne. Takodje su dati kriterijumi pod kojima je neka proizvoljna klasa univerzalnih algebri ne-aksiomatizabilna.

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