Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 22, 1 (1992), 133-140

Review of Research
Faculty of Science
Mathematics Series

SOME NEW RESULTS CONCERNING TWO COUNTERFEIT COINS

Ivica Bošnjak, Ratko Tošić
Institute of Mathematics, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

We consider the problem of ascertaining the minimum number of weighings which suffice to determine the counterfeit (heavier) coins in a set of n coins of the same appearance, given a balance scale and the information that there are exactly two heavier coins present. Some results from [8] are improved by construction of a procedure which is proved to be optimal for all n's belonging to the set

$$\bigcup_{k\geq 2}([\lceil 3^k\sqrt{6}+1\rceil, 4\cdot 3^k]\cup[\lceil 3^k\sqrt{2}+1\rceil, 20\cdot 3^{k-2}])$$

AMS Mathematics Subject Classification (1991): 90B40, 62C20, 90C39
Key words and phrases: optimal algorithm, counterfeit coin.

1. Introduction

Let $S=\{c_1,c_2,...,c_n\}$ be a set of n coins indistinguishable except that exactly m of them are slightly heavier than the rest (in the sence specified bellow). Given a balance scale, we want to find an optimal Weighing procedure, i.e. a procedure which minimizes the maximum number of steps (weighings) which are required to identify all heavier coins.

We suppose that all heavier coins are of equal weight, and so are the light coins. If w is the weight of the light (good) coin, then the weight of a heavy (defective) coin is the less than $\frac{m+1}{m}w$, so that the larger of two numerically unequal subsets of S is always the heavier. It means that no information is gained by balancing two numerically unequal sets. We also suppose that the scale reveals which, if either, of two subsets of S is heavier but not by how much.

For two numerically equal disjoint subsets A,B of S step (A, B) will mean the balancing of A against B. The possible outcomes are:
(a) The sets balance, symbolized by $A = B$.
(b) The sets do not balance, symbolized by $A \neq B$. We use the notation, if necessary, $A > B$, $A < B$, where $>$ between two sets means "is heavier than".

We denote by $P_n^m(l)$ any algorithm which enables us to identify all heavier coins if there are m of them in a set S of n coins, l being the maximum number of weighings to be required. We write $\mu_m(n) = l$ if $P_n^m(l)$ is optimal.

It follows by information theoretical reasonings that

$$\mu_m(n) \geq \left\lceil \log_3 \binom{n}{m} \right\rceil \tag{1}$$

where $\lceil x \rceil$ denotes the least integer $\geq x$. It is well known that

$$\mu_1(n) = \lceil \log_3(n) \rceil \tag{2}$$

The case $m = 1$ is a well known puzzle [1], [2], [3]. There are several proofs showing that the lower bound (1) is sharp.

There have been some investigations concerning the cases $m \geq 2$, [3], [6], [8], [9], and some related problems [4], [5], [7], [10].

In the case $m = 2$, we denote two counterfeit coins by x and y.

2. The Results

For $m = 2$, the following statement holds.

Theorem 1. *(Tošić [8])*

$$\left\lceil \log_3 \binom{n}{2} \right\rceil \leq \mu_2(n) \leq \left\lceil \log_3 \binom{n}{2} \right\rceil + 1 \tag{3}$$

The corresponding algorithm is constructed inductivelly.

The proof of Theorem 1. combines the following statements:

$$n \leq 2 \cdot 3^k \Rightarrow \mu_2(n) \leq 2k + 1 \quad (k = 0, 1, ...) \tag{4}$$

$$n \leq 3^{k+1} \Rightarrow \mu_2(n) \leq 2k + 2 \quad (k = 0, 1, ...) \tag{5}$$

In [8] also an infinite set of n's is determined for which the lower bound $\lceil \log_3 \binom{n}{2} \rceil$ is reached in the constructed algorithm. So, for those n's the constructed algorithm is optimal.

We also use $\mu_1^r(n_1, ..., n_r)$ to denote the minimum number of weighings which enable us to identify all heavier coins if there are exactly one of them in each of the sets $S_1, ..., S_r$ where $|S_1| = n_1, ..., |S_r| = n_r$. We denote the coresponding optimal procedure by $P^1_{n_1,...,n_r}(l)$, where $l = \mu_1^r(n_1, ..., n_r)$.

In this paper we improve the results from [8]. First we prove the following lemma.

Lemma 1.

$$\mu_1^r(mn_1, ..., mn_r) \leq \mu_1^r(n_1, ..., n_r) + \mu_1^r(m, ..., m) \tag{6}$$

Proof. We partition each set S_i, $|S_i| = mn_i$, $(i = 1, 2, ..., r)$ into n_i disjoint subsets of the same cardinality m. Such a subset is said to be counterfeit if it contains a counterfeit coin. First we apply an optimal algorithm $P^1 n_1, ..., n_r(l)$, where $l = \mu_1^r(n_1, ..., n_r)$, to identify the unique counterfeit subset of each set S_i. Then we apply an optimal algorithm $P^1_{m,...,m}(s)$, where $s = \mu_1^r(m, ..., m)$, to determine all counterfeit coins (exactly one from each counterfeit subset). Hence follows (6).

Corollary 1.

$$\mu_1^r(m_1 \cdot 3^k, ..., m_r \cdot 3^k) \leq rk + \mu_1^r(m_1, ..., m_r) \tag{7}$$

Proof. Follows from Lemma 1. and the very well known fact that $\mu_1^r(3^k, ..., 3^k) = rk$

We shall often use the special case of Corollary 1., when $r = 2$:

$$\mu_1^r(m \cdot 3^k, n \cdot 3^k) \leq 2k + \mu_1^2(m, n) \tag{8}$$

We shall also use the following equalities, which can be easily verified:

$$\mu_1^2(2,3) = 2 \tag{9}$$

$$\mu_1^2(2,4) = 2 \tag{10}$$

$$\mu_1^2(4,6) = 3 \tag{11}$$

The following two theoremes enable us to improve the results from [8].

Theorem 2. *If* $3^{k+1} < n \leq 4 \cdot 3^k$, $k \geq 0$, *then* $\mu_2(n) \leq 2k + 2$.

Proof. The proof is by induction. For $k = 0$, the statement is true. In that case, the first step is $(\{c_1\}, \{c_2\})$, the second step is $(\{c_2\}, \{c_3\})$, and it is obviously sufficient for detection of counterfeits.

Let $k > 0$. We construct an algorithm $P_n^2(2k+2)$, for $3^{k+1} < n \leq 4 \cdot 3^k$, in the following way.

The first step is (A, B), where $A = \{c_1, ..., c_r\}$, $B = \{c_{r+1}, ..., c_{2r}\}$ and r is the least even integer $\geq \frac{n}{3}$.

(1.) If $A > B$, the second step is (C_1, C_2), where $C = S \setminus (A \cup B)$, $C_1 \subseteq C$, $|C_1| = \left\lceil \frac{|C|}{2} \right\rceil$, $C_2 = C \setminus C_1$ if $|C|$ is even and $C_2 = C \setminus C_1 \cup \{c_{r+1}\}$ if $|C|$ is odd.

(1.1.) If $C_1 > C_2$, then $x \in C_1, y \in A$. Since $|C_1| \leq 2 \cdot 3^{k-1}$ and $|A| \leq 4 \cdot 3^{k-1}$, according to (8) and (10), we can find x and y using at most $2k$ additional steps.

(1.2.) The case $C_1 < C_2$ is quite similar to (1.1.).

(1.3.) If $C_1 = C_2$, then $x, y \in A$ and we can apply $P_{|A|}^2(2k)$ which exists by induction hypothesis, since $|A| \leq 4 \cdot 3^{k-1}$.

(2.) The case $A < B$ is quite analogous to (1.).

(3.) If $A = B$, then either $x \in A, y \in B$ or $x, y \in C$. Now the second step is (A_1, A_2) where $|A_1| = |A_2|$, $A_1 \cup A_2 = A$, $A_1 \cap A_2 = \emptyset$.

(3.1.) If $A_1 > A_2$, then $x \in A_1, y \in B$, $|A_1| \leq 2 \cdot 3^{k-1}$, $|B| \leq 4 \cdot 3^{k-1}$, and according to (8) and (10), we can find x and y using at most $2k$ additional steps.

(3.2.) The case $A_1 < A_2$ is quite similar to (3.1.).

(3.3.) If $A_1 = A_2$, then $x, y \in C$, then the statement follows by induction hypothesis, since $|C| \leq 4 \cdot 3^{k-1}$.

Theorem 3. *If* $2 \cdot 3^{k+1} < n \leq 20 \cdot 3^{k-1}$, $k \geq 1$, *then* $\mu_2(n) \leq 2k + 3$.

Proof. We adopt the following notations:

$$A = \{c_1, \ldots, c_{6 \cdot 3^{k-1}}\}, \quad B_1 = \{c_{6 \cdot 3^{k-1}+1}, \ldots, c_{9 \cdot 3^{k-1}}\},$$

$$B_2 = \{c_{9 \cdot 3^{k-1}+1}, \ldots, c_{12 \cdot 3^{k-1}}\}, \quad B = B_1 \cup B_2, \quad C = S \backslash (A \cup B), \quad C_1 \cup C_2 = C,$$

$$C_1 \cap C_2 = \emptyset, \text{ such that } 0 \leq |C_1| - |C_2| \leq 1; \quad C' = C_2 \cup \{c_{9 \cdot 3^{k-1}}\},$$

$$A_3 = \{c_1, \ldots, c_{4 \cdot 3^{k-1}}\}, \quad C_3 = \{c_{12 \cdot 3^{k-1}+1}, \ldots, c_{16 \cdot 3^{k-1}}\}$$

We construct an algorithm $P_n^2(2k+3)$ in the following way.

The first step is (A, B).

(1.) If $A > B$, then the second step is (C_1, C_2) if $|C|$ is even, and (C_1, C') if C is odd.

(1.1.) If $C_1 > C_2$ $(C_1 > C')$, then $x \in C_1, y \in A$, and according to (8) and (11), we can find x and y using at most $2k + 1$ additional steps.

(1.2.) The case $C_1 < C_2$ $(C_1 < C')$ is quite analogous to (1.1.).

(1.3.) If $C_1 = C_2$ $(C_1 = C')$, then $x, y \in A$, and according to (4), we can determine x and y using at most $2k + 1$ additional steps.

(2.) The case $A < B$ is quite analogous to (1.).

(3.) If $A = B$, then either $x \in A, y \in B$ or $x, y \in C$. In that case, the second step is (A_3, C_3).

(3.1.) If $A_3 > C_3$, then $x \in A_3, y \in B$, and according to (8) and (11), we can identify x and y using at most $2k + 1$ additional steps.

(3.2.) If $A_3 < C_3$, then either $x, y \in C_3$ or $x \in C_3, y \in C \setminus C_3$. Now, we partition $C \setminus C_3$ into the subsets D_1 and D_2 such that $0 \leq |D_1| - |D_2| \leq 1$. The third step is (D_1, D_2) if $|D_1| = |D_2|$ and (D_1, D_2'), where $D_2' = D_2 \cup \{c_1\}$ if $|D_1| = |D_2| + 1$.

(3.2.1.) If $D_1 \neq D_2$ ($D_1 \neq D_2'$), then according to (8) and (10), we can find x and y using at most $2k$ additional steps.

(3.2.2.) If $D_1 = D_2$ ($D_1 = D_2'$), then $x, y \in C_3$, and according to Theorem 2., we can find x and y using at most $2k$ additional steps.

(3.3.) If $A_3 = C_3$, then either $x, y \in C \setminus C_3$ or $x \in A \setminus A_3, y \in B$. Now, the third step is (B_1, B_2).

(3.3.1.) If $B_1 \neq B_2$, then $x \in A \setminus A_3, y \in B_1$, and according to (8) and (9), we can find x and y using at most $2k$ additional steps.

(3.3.2.) If $B_1 = B_2$, then $x, y \in C \setminus C_3$, and according to Theorem 2. we can determine x and y using at most $2k$ additional steps.

Remark 1. *In the proofs of Theorems 2. and 3., we often use the algorithms* $P^1_{m \cdot 3^{k-1}, n \cdot 3^{k-1}}(l)$. *We suppose, when necessary, that we may add some coins which are proved to be good, in order to obtain the sets* S_1 *and* S_2 *of the cardinality* $m \cdot 3^{k-1}$ *and* $n \cdot 3^{k-1}$ *respectively. It can be easily checked that in each particular case we have at our disposal a sufficient amount of good coins.*

Theorem 4. *(a) If* $n \leq 4 \cdot 3^k$ *then* $\mu_2(n) \leq 2k + 2$.
(b) If $n \leq 20 \cdot 3^{k-1}$ *then* $\mu_2(n) \leq 2k + 3$.

Proof. Follows from Theorems 1,2 and 3.

Theorem 5. *The optimal algorithm* $P_n^2(l)$, *i.e. such that* $l = \mu_2(n)$, *exists at least for all integers belonging to the set*

$$\bigcup_{k \geq 2} ([\lceil 3^k \sqrt{6} + 1 \rceil, 4 \cdot 3^k] \cup [\lceil 3^k \sqrt{2} + 1 \rceil, 20 \cdot 3^{k-2}]),$$

where $[p, q]$ *denotes the set of all integers* n *such that* $p \leq n \leq q$.

Proof. Follows from Theorems 1,2 and 3 and from the inequalities:

$$\binom{\lceil 3^k \sqrt{2} + 1 \rceil}{2} > 3^{2k}, \quad \binom{\lceil 3^k \sqrt{6} + 1 \rceil}{2} > 3^{2k+1}$$

Theorem 5. is also a significant improvement of Theorem 2. from [8].

Conjecture If $3^{k-1} < \binom{n}{2} \leq 3^k$, then $\mu_2(n) = k$.

References

[1] Bellman, R.: Dynamic Programming, Princeton, Princeton Univ. Press, 1957.

[2] Bellman, R., Gluss, B.: On Various Versious of the Defective Coin Problem, Inf. Control 4, 119-131 (1961).

[3] Cairns, S.S.: Balance Scale Sorting, Amer. Math. Mon. 70, 136-143 (1963).

[4] Manvel, B.: Counterfeit Coin Problem, Math. Mag. 50, 90-92 (1977).

[5] Mead, D.G.: The Average Number of Weighings to Locate a Counterfeit Coin, IEEE Trans. Inf. Theory 25, 616-617 (1979).

[6] Pyber, L.: How to Find Many Counterfeit Coins, Graphs and Combinatorics 2, 173-177 (1986).

[7] Smith, C.A.B.: The Counterfeit Coin Problem, Math. Gaz. 31, 31-39 (1947).

[8] Tošić, R.: Two Counterfeit Coins, Discrete Math. 46, 295-298 (1983).

[9] Tošić, R.: Five Counterfeit Coins, J. Stat. Plan. Inf. 22, 197-202 (1989).

[10] Winkelmann, K.: An Improved Strategy for a Counterfeit Coin Problem, IEEE Trans. Inf. Theory 28, 120-122 (1982).

REZIME

NEKI NOVI REZULTATI O DVA NEISPRAVNA NOVČIĆA

U radu je razmatran problem odredjivanja minimalnog broja merenja koji je potreban za pronalaženje neispravnih (težih) novčića u skupu od n novčića pri čemu je poznato da su tačno dva novčića neispravna. Neki rezultati iz [8]

su poboljšani nalaženjem algoritma koji je optimalan za beskonačno mnogo prirodnih brojeva n.

Received by the editors October 28, 1991