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ON EXTRINSIC SPHERES IN A LOCALLY PRODUCT RIEMANNIAN MANIFOLD

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Abstract

Let M be a complete, connected, simply connected and non-invariant extrinsic sphere of dimension ≥ 3 in a locally product Riemannian manifold. If the curvature tensor of the connection induced in the normal bundle satisfies the condition

$$\nabla_k \nabla_h R^x_{jiy} - \nabla_h \nabla_k R^x_{jiy} = 0,$$

then M is isometric to an ordinary sphere.

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1. Introduction

An m-dimensional submanifold, $m \geq 2$, of a Riemannian manifold is called an extrinsic sphere if it is totally umbilical and its mean curvature vector field is non-zero and parallel. Extrinsic spheres have been geometrically characterized by Nomizu and Yano [4]. Since they have the same extrinsic properties as ordinary spheres in a Euclidean space, it is natural to ask when an extrinsic sphere is isometric to an ordinary sphere. In general, an extrinsic sphere is not always isometric with a sphere (see [1] p. 66). Extrinsic spheres in a locally product Riemannian manifold have been studied by Okumura [5] and Nemoto [3]. The present paper also concerns properties of extrinsic spheres in such manifolds. More precisely, we show that a complete, connected, simply connected and non-invariant extrinsic sphere of dimension ≥ 3 is isometric to an ordinary sphere if its curvature tensor of the connection induced in the normal bundle satisfies the condition

$$\nabla_k \nabla_h R^{x}_{jiy} - \nabla_h \nabla_k R^{x}_{jiy} = 0.$$

Our theorem is strictly related to the results of Nemoto in [3].

2. Preliminaries

Let $\overline{M}$ be a locally product Riemannian manifold of dimension n and (F, G) its almost structure (cf. [6], also [2],[7]). Then we have

$$F^{\alpha}_{\gamma} F^{\gamma}_{\beta} = \delta^{\alpha}_{\beta},\ F^{\lambda}_{\alpha} F^{\gamma}_{\beta} G_{\lambda\gamma} = G_{\alpha\beta},\ \nabla_{\gamma} F^{\beta}_{\alpha} = 0, \tag{2.1}$$

and the tensor field of components $F_{\alpha\beta} = F^{\gamma}_{\alpha} G_{\gamma\beta}$ is non-degenerate and symmetric. Here and in the sequel, the Greek indices run over the range { 1,2, ..., n }, and ∇ denotes the covariant differentiation in $\overline{M}$.

Let M (dim $M = m$, $m < n$) be a submanifold of $\overline{M}$.Denote by (x^{α}) and (u^{i}) local coordinates of $\overline{M}$ and M, respectively. The continual indices h, i, j, ... run over the range { 1,2, ..., m }. Let $x^{\alpha} = x^{\alpha}(u^{i})$ be local parametric representation of M in $\overline{M}$. Set $B^{\alpha}_{i} = \frac{\partial x^{\alpha}}{\partial u^{i}}$. M inherits from $\overline{M}$ the Riemannian metric g of local components $g_{ij} = G_{\alpha\beta} B^{\alpha}_{i} B^{\beta}_{j}$. Next we take $(n-m)$ mutually orthogonal unit local vector fields normal to M and denote their local components by N^{α}_{x}. The indices v, x, y, ... run over the range { $m+1$, ..., n }. Denote by g_{xy} the components of the metric tensor induced on the normal bundle $T^{\perp} M$ of M from the metric tensor G of $\overline{M}$, that is $g_{xy} = G_{\alpha\beta} N^{\alpha}_{x} N^{\beta}_{y}$.

Let us express FB_i and FN_x as linear combinations of B_i and N_x as follows

$$F^{\alpha}_{\gamma} B^{\gamma}_{i} = f^{r}_{i} B^{\alpha}_{r} + f^{x}_{i} N^{\alpha}_{x}, \tag{2.2}$$

$$F^{\alpha}_{\gamma} N^{\gamma}_{x} = f^{r}_{x} B^{\alpha}_{r} + f^{y}_{x} N^{\alpha}_{y}. \tag{2.3}$$

Then f_x^y are components of an f-structure in the normal bundle $T^{\perp}M$. It can be easily noted that

$$f_{ij} = f_{ji},\ f_{ix} = f_{xi},\ f_{xy} = f_{yx},$$

where

$$f_{ij} = f_i^r g_{rj},\ f_{ix} = f_i^y g_{xy},\ f_{xi} = f_x^r g_{ri},\ f_{xy} = f_x^v g_{vy}.$$

With the help of (2.2) - (2.3) , we find

$$f_j^r f_{ri} + f_j^x f_{xi} = g_{ij}, \tag{2.4}$$

$$f_i^r f_{rx} + f_i^v f_{vx} = 0, \tag{2.5}$$

$$f_x^v f_{vy} + f_x^r f_{ry} = g_{xy}. \tag{2.6}$$

If $FT_xM \subset T_xM$ for any $x \in M$ (i.e., $f_i^x = 0$), then M is said to be an invariant submanifold of $\overline{M}$.

Denoting by h_{ij}^x the components of the second fundamental tensor of the submanifold M, we have the following equations of Gauss

$$\nabla_j B_i^\alpha = h_{ij}^v N_v^\alpha, \tag{2.7}$$

and of Weingarten

$$\nabla_j N_x^\alpha = -h_j{}^r{}_x B_r^\alpha, \tag{2.8}$$

where $h_j{}^r{}_x = h_{ji}^v g^{ir} g_{vx}$ and ∇ denotes the Van der Waerden-Bortolotti covariant differentiation see, e.g., Yano and Ishihara [8].

The mean curvature vector field of the submanifold M has local components $h^x = (\frac{1}{m}) g^{rs} h_{rs}^x$ and the function h, such that $h^2 = g_{xy}h^x h^y$ is the mean curvature of M.

The submanifold M is said to be totally umbilical if $h_{ij}^x = g_{ij}h^x$ and totally geodesic if $h_{ij}^x = 0$.

If M is a totally umbilical submanifold and its mean curvature vector field is non-zero and parallel ($\nabla_j h^x = 0$), then M is called an extrinsic sphere in $\overline{M}$ [2]. The mean curvature h of an extrinsic sphere is a non-zero constant.

3. Extrinsic spheres

Let M be an extrinsic sphere in a locally product Riemannian manifold $\overline{M}$. Differentiating (2.2) and (2.3) covariantly along the submanifold M and using (2.1) - (2.3), (2.7) and (2.8) , we obtain

$$\nabla_k f_{ij} = v_i g_{jk} + v_j g_{ik}, \tag{3.1}$$

$$\nabla_k f_{ix} = v_x g_{ik} - h_x f_{ik}, \tag{3.2}$$

$$\nabla_k f_{xy} = -h_y f_{kx} - h_x f_{ky}, \tag{3.3}$$

where $v_i = f_{ix} h^x$ and $v_x = f_{vx} h^v$. Note that v_i (resp. v_x) are components of the tangent part (resp. normal part) of the image of the mean curvature vector field by F.

Define P to be the function defined on the whole of M as $P = f_{xy} h^x h^y$.

As consequences of relations (3.2) and (3.3) we obtain the following formulas

$$\nabla_j v_i = -h^2 f_{ij} + P g_{ij}, \tag{3.4}$$

$$\nabla_j v_x = -v_j h_x - h^2 f_{jx}, \tag{3.5}$$

$$\nabla_j P = -2h^2 v_j. \tag{3.6}$$

Thus, the vector field $v = (v_i)$ is a gradient.

In the case of non-constant function P; the following theorem can be stated (see Nemoto [3], Theorem 1).

Theorem 1. *Let M be a complete, connected and simply connected extrinsic sphere in a locally product Riemannian manifold $\overline{M}$. If the 1-form $\omega = dP$ does not vanish identically, M is isometric to an ordinary sphere.*

In the sequel, we shall give another sufficient condition for an extrinsic sphere to be isometric to an ordinary sphere. For this purpose, we prove the following lemmas.

Lemma 1. *If the function P is constant on M, then*

$$h^2 f_{ij} = P g_{ij}. \tag{3.7}$$

Proof. Since $P = const$, by (3.6) we see that the vector field $v = (v_i)$ is zero. Now, by (3.4) we have (3.7), completing the proof. □

Lemma 2. *An extrinsic sphere M in a locally product Riemannian manifold $\overline{M}$ is invariant if and only if $h^2 = |P|$.*

Proof. Let M be an invariant submanifold in $\overline{M}$, that is $f_{ix} = 0$. Then (3.2) yields (3.7) and together with (2.4) gives $h^2 = |P|$.

Conversely, let $h^2 = |P|$ be satisfied on an extrinsic sphere M in $\overline{M}$. Then, by (3.7) we have $f_{ij} = \pm g_{ij}$. This reduces (2.4) to the form $f_i^x f_{jx} = 0$. Consequently $f_i^x = 0$, which completes the proof. □

Lemma 3. *For an extrinsic sphere in a locally product Riemannian manifold, the following relations hold*

$$f_x^r T_{kjir} + f_i^v R_{kjxv} = 0, \tag{3.8}$$

$$\begin{aligned} f_x^r(\nabla_n\nabla_m R_{kjir} - \nabla_m\nabla_n R_{kjir}) + f_i^v(\nabla_n\nabla_m R_{kjxv} - \nabla_m\nabla_n R_{kjxv}) = \\ = h^2(f_{nx}T_{kjim} - f_{mx}T_{kjin} + g_{in}f_x^r T_{kjmr} - g_{im}f_x^r T_{kjnr}), \end{aligned} \tag{3.9}$$

where

$$T_{kjih} = R_{kjih} - h^2(g_{kh}g_{ij} - g_{ki}g_{jh}), \tag{3.10}$$

R_{kjih} and $R_{kjyx} = R^v_{kjy}g_{vx}$ are covariant components of the curvature tensor of M and the curvature tensor of the connexion induced in the normal bundle of M, respectively.

Proof. Differentiating (3.2) covariantly and using (3.1), (3.5) and the Ricci identity

$$\nabla_k\nabla_j f_{ix} - \nabla_j\nabla_k f_{ix} = -R^r_{kji}f_{rx} - R_{kjxv}f_i^v,$$

we obtain

$$f_x^r R_{kjir} + f_i^v R_{kjxv} = h^2(g_{ij}f_{kx} - g_{ik}f_{jx}).$$

This, by making use of (3.10), can be written in the form (3.8).

To verify (3.9), we use the Ricci identities

$$\begin{aligned} \nabla_n\nabla_m R_{jiyx} - \nabla_m\nabla_n R_{jiyx} &= -R_{riyx}R^r_{nmy} - R_{jryx}R^r_{nmi} - \\ &\quad -R_{jivx}R^v_{nmy} - R_{jiyv}R^v_{nmx}, \\ \nabla_n\nabla_m R_{hkji} - \nabla_m\nabla_n R_{hkji} &= -R_{rkji}R^r_{nmh} - R_{hrji}R^r_{nmk} - \\ &\quad -R_{hkri}R^r_{nmj} - R_{hkjr}R^r_{nmi} \end{aligned}$$

and relations (3.8) and (3.10). This completes the proof. □

The following theorem states the main result of the presented paper.

Theorem 2. *Let M be a complete, connected and simply connected extrinsic sphere in a locally product Riemannian manifold $\overline{M}$ and $\dim M \geq 3$. If the curvature tensor of the connexion induced in the normal bundle of M satisfies the condition*

$$\nabla_k \nabla_h R^x_{jiy} - \nabla_h \nabla_k R^x_{jiy} = 0, \tag{3.11}$$

then one of the following cases occurs:

(1) M is isometric to an ordinary sphere,

(2) M is invariant.

Proof. Suppose that M is a complete, connected and simply connected extrinsic sphere M in a locally product Riemannian manifold $\overline{M}$. If the function P is non-constant, M is isometric to an ordinary spheres by virtue of Theorem 1. So, in the sequel, we assume that P is constant. Let M satisfy additionally (3.11). Then, from (3.9) it follows that

$$\begin{aligned} f^r_x(\nabla_n \nabla_m R_{kjir} - \nabla_m \nabla_n R_{kjir}) \;=\;& h^2(f_{nx} T_{kjim} - f_{mx} T_{kjin} + \\ & g_{in} f^r_x T_{kjmr} - g_{im} f^r_x T_{kjnr}). \end{aligned} \tag{3.12}$$

Moreover, applying (3.7) to (2.4) we get

$$h^4 f^x_i f_{xj} = (h^4 - P^2) g_{ij}. \tag{3.13}$$

Next, permuting (3.12) cyclically with respect to indices $(k,\ j,\ i)$, adding the resulting equations and using the relation $T_{kjir} + T_{jikr} + T_{ikjr} = 0$ and the first Bianchi identity, we find

$$f^r_x(g_{in} T_{kjmr} - g_{im} T_{kjnr} + g_{kn} T_{jimr} - g_{km} T_{jinr} + g_{jn} T_{ikmr} - g_{jm} T_{iknr}) = 0.$$

Transvecting this with $h^4 f^x_h g^{im}$ and making use of (3.13), we have

$$(h^4 - P^2)\{(m-3) T_{kjnh} - g_{kn} T_{jh} + g_{jn} T_{kh}\} = 0, \tag{3.14}$$

where $T_{kh} = T_{kjnh} g^{jn}$. Now assume that M is non-invariant. By Lemma 2 $h^4 \neq P^2$. Therefore, from (3.14) we get

$$(m-3) T_{kjnh} = g_{kn} T_{jh} - g_{jn} T_{kh}, \tag{3.15}$$

which by transvection with g^{jn} yields $T_{kh} = 0$ for $m \geq 3$. If $m > 3$, then the relation $T_{kh} = 0$ used in (3.15) yields $T_{kjih} = 0$. As it is known, if $m = 3$, then $T_{kh} = 0$ always implies $T_{kjih} = 0$. Thus, M is a manifold of constant curvature. Since M is complete, connected and simply connected, M is isometric to an ordinary sphere. The proof is complete. □

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REZIME

O SPOLJNIM SFERAMA U LAKALNOJ PROIZVOD RIMANOVOJ MNOGOSTRUKOSTI

neka je M kompletna, prosto povezana i neinvarijantna spoljna sfera dimenzije ≥ 3 u lokalnoj proizvod Rimanovoj mnogostrukosti. Ako tenzor krivine koneksije indukovan u normalnom svežnju ispunjava uslov

$$\nabla_k \nabla_h R^x_{jiy} - \nabla_h \nabla_k R^x_{jiy} = 0,$$

onda je M izometrična normalnoj sferi.

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