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COMPUTATION OF WEAKLY AND NEARLY SINGULAR INTEGRALS OVER TRIANGLES IN $\mathbf{R}^3$

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Abstract

We study the approximation of weakly singular integrals over triangles in general position in $\mathbf{R}^3$, giving explicit formulae where convenient and numerical quadrature in more general cases. Particular models considered concern the collocation and Galerkin methods in the boundary integral approach to the Dirichlet problem for Laplace's equation.

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1. Introduction

In the boundary integral approach to solving partial differential equations over a domain $\Omega \subset \mathbf{R}^3$ one converts to an integral equation over $B = \partial\Omega$ and approximates the domain Ω by a polytope boundary Γ approximates B. The faces of the polytope are polygons, so there is no loss of generality in assuming that Γ consists of triangles. See, e.g., [2, sec. 15.4] for an introduction to piecewise linear methods for approximating surfaces. An efficient implementation including smoothing steps has been developed in [11]. To motivate our general discussion, let us consider the problem of finding a solution u to the first kind integral equation.

$$\int_\Gamma \frac{u(x)}{|x-\xi|} d\Gamma_x = f(\xi), \quad \xi \in \Gamma, \tag{1.1}$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbf{R}^3$. Equation (1.1) stems from the boundary integral approach for the Dirichlet problem

$$\Delta u = 0 \quad \text{in} \ \ \Omega,$$

$$u = f \qquad \text{on} \ \ B,$$

where B is approximated by Γ, see, e.g., the books [3,4,5,7,8].

The numerical solution of (1.1) is usually carried out via a collocation or Galerkin method. Our discussion below pertains to both methods. Let $Q_1, ..., Q_N$ be the vertices of Γ and let $\Delta_1, ..., \Delta_M$ be the faces (triangles) of Γ. We will assume that the domain Ω and hence the approximate boundary Γ are bounded.

Let us seek an approximate solution to (1.1) which is of the form

$$u \approx \sum_{i=1}^{N} A_i \varphi_i,$$

where the φ_i are chosen basis functions and the $A_i \in \mathbf{R}$ are unknown coefficients which must be calculated. Typically, the φ_i are piecewise polynomials. That is,

$$\varphi_i(x) = \sum_{|\alpha| \le n} B_{i,q,\alpha} x^\alpha \quad \text{for} \quad x \in \Delta_q,$$

where we adopt the convention

$$x = (x_1, x_2, x_3) \in \Delta_q,$$

$$\begin{aligned} \mathbf{Z}_+ &:= \text{set of non-negative integers}, \\ \alpha &= (\alpha_1, \alpha_2, \alpha_3) \in \mathbf{Z}_+^3, \\ |\alpha| &:= \alpha_1 + \alpha_2 + \alpha_3, \\ x^\alpha &:= x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}. \end{aligned}$$

The non-negative integer n denotes the degree of the polynominal φ_i.

In the collocation method for (1.1) the unknown coefficients $A_1, ..., A_n$ are determined by solving the system of linear equations

$$\sum_{i=1}^{N} A_i \int_\Gamma \frac{\varphi(x)}{|x - Q_p|} d\Gamma_x = f(Q_p), \quad p = 1, ..., N. \tag{1.2}$$

In (1.2) it is necessary to compute the coefficient matrix:

$$a_{i,p} := \int_\Gamma \frac{\varphi(x)}{|x - Q_p|} d\Gamma_x = \sum_{q=1}^{M} \sum_{|\alpha| \le n} B_{i,q,\alpha} \int_{\Delta_q} \frac{x^\alpha}{|x - Q_p|} d\Gamma_x.$$

Consequently, it becomes necessary to calculate integrals of the from

$$I(p, q, \alpha) := \int_{\Delta_q} \frac{x^\alpha}{|x - Q_p|} d\Gamma_x. \tag{1.3}$$

In the Galerkin method for (1.1) a system of equations of the from

$$\sum_{i=1}^{N} A_i \int_\Gamma \int_\Gamma \frac{\varphi_j(\xi)\varphi_i(x)}{|x - \xi|} d\Gamma_x d\Gamma_\xi = \int_\Gamma \varphi_j(\xi) f(\xi) d\Gamma_\xi \tag{1.4}$$

must be satisfied for $j = 1, ..., N$. Here the φ_i are the same general basis functions as those described in the collocation method.

Now writing

$$I_i(\xi) := \int_\Gamma \frac{\varphi_i(x)}{|x - \xi|} d\Gamma_x,$$

the coefficients of the system (1.4) assume the from

$$a_{i,j} = \int_\Gamma \varphi_j(\xi) I_i(\xi) d\Gamma_\xi.$$

Typically, the coefficients $a_{i,j}$ are computed via a quadrature formula:

$$a_{i,j} \approx \sum_{p=1}^{N} c_p \varphi(Q_p) I_i(Q_p)$$

where the c_p, $p = 1, ..., N$ are weights. Since

$$I_i(\xi) = \int_\Gamma \frac{\varphi_i(x)}{|x-\xi|} d\Gamma_x$$

$$= \sum_{q=1}^{N} \sum_{|\alpha| \le n} B_{i,q,\alpha} \int_{\Delta_q} \frac{x^\alpha}{|x - Q_p|} d\Gamma_x,$$

we are again led to the task of computing integrals of the form (1.3) also in the Galerkin method. Our aim therefore is to develop methods for evaluating or approximating integrals of the form (1.3).

2. Geometric Transformations

Recent related works concerning approximation of surface integrals in the context of boundary element methods are [1,6,9,10]. The book of Hackbusch [5] discusses the panel method in detail and gives some cases in which certain integrals over triangles can be integrated exactly.

Since the triangles $\Delta_q \in \Gamma$ are generally in arbitrary position in space, we begin the treatment of the computation of (1.3) by describing briefly transformations which bring the general configuration of Q_p and Δ_q into a more standard configuration. Let us suppose that Q_p and Δ_q are described in standard 3-dimensional Cartesian co-ordinates (x_1, x_2, x_3), and the vertices of Δ_q are denoted by v^1, v^2, v^3.

An affine transformation $x \mapsto y = (y_1, y_2, y_3)$ brings the general configuration into a standard configuration having the following properties:

(i) Δ_q lies in the plane $y_3 = 0$. That is, $y \in \Delta_q$ implies $y = (y_1, y_2, 0)$.

(ii) The point Q_p lies on the y_3-axis. That is, $Q_p = (0, 0, c)$ for some $c \in \mathbf{R}$.

By introducing the orthonormal basis

$$(2.1) \quad \rho^1 := \frac{v^2 - v^1}{|v^2 - v^1|}, \; \rho^3 := \frac{(v^2 - v^1) \times (v^3 - v^1)}{|(v^2 - v^1) \times (v^3 - v^1)|}, \rho^2 := \rho^3 \times \rho^1,$$

we obtain

$$y = Ax - b \quad \text{or} \quad x = A^t(y + b),$$

where $A := (\rho^1, \rho^2, \rho^3)^t, b := \begin{pmatrix} \rho^1 \cdot v^1 \\ \rho^2 \cdot Q_p \\ \rho^3 \cdot Q_p \end{pmatrix}$.

It is immediately seen that

$$x^\alpha = \sum_{|\gamma| \le |\alpha|} d_\gamma y_1^{\gamma_1} y_2^{\gamma_2} \quad \text{for} \quad x \in \Delta_q,$$

where $\gamma = (\gamma_1, \gamma_2) \in \mathbf{Z}_+^2$ and $d_\gamma \in \mathbf{R}$ are uniquely determined and easily computed from above transformation. Hence we can concentrate upon approximating integrals of the form

$$\hat{I}(p, q, \gamma) := \int_{\Delta_q} \frac{y_1^{\gamma_1} y_2^{\gamma_2}}{| y - Q_p |} d\Gamma_y, \tag{2.2}$$

since (1.3) is merely a linear combination of the integrals in (2.2).

Let us denote by $v^0 := (0, 0, 0)$ the projection of Q_p into the $y_3 = 0$ plane, and the vertices v^i of Δ_q are understood to be expressed in the y−coordinates. The indexing of the vertices v^1, v^2, v^3 will be chosen to correspond to the orientation of Γ, i.e., ρ^1 defined in (2.1) is the outer normal of Γ on Δ_q. The triangle Δ_q oriented in this way will be written as $\Delta_q = [v^1, v^2, v^3]$. The parameter c appearing in the y−co-ordinates of $Q_p = (0, 0, c)$ will play an important role below.

By making use of the orientation, it follows that for any integrand F which is defined on all the domains involved, the following formula holds:

$$\int_{[v^1,v^2,v^3]} F(y) d\Gamma_y = \int_{[v^0,v^1,v^2]} F(y) d\Gamma_y + \int_{[v^0,v^2,v^3]} F(y) d\Gamma_y + \int_{[v^0,v^3,v^1]} F(y) d\Gamma_y.$$

In particular, we have

$$\hat{I}(p, q, \gamma) = \hat{I}_{1,2}(p, q, \gamma) + \hat{I}_{2,3}(p, q, \gamma) + \hat{I}_{3,1}(p, q, \gamma)$$

where

$$\hat{I}_{i,j}(p, q, \gamma) := \int_{[v^0,v^i,v^j]} \frac{y_1^{\gamma_1} y_2^{\gamma_2}}{| y - Q_p |} d\Gamma_y \quad \text{for } i, j \in \{1, 2, 3\}, i \neq j. \tag{2.3}$$

For convenience, let us denote any integral of the form (2.3) by $J(\gamma, c)$, where c is the y_3−co-ordinate of Q_p and the indices of the vertices of the triangles have been supressed. Note that we have reduced our problem to the evaluation of such integrals.

3. Numerical Approximation of $J(\gamma, c)$.

We begin the calculation of $J(\gamma, c)$ by introducing the polar co-ordinates

(3.1) $$y_1 = \rho\cos(\theta), \quad y_2 = \rho\sin(\theta).$$

Then

(3.2) $$J(\gamma, c) = \int_a^b \cos^{\gamma_1}(\theta)\sin^{\gamma_2}(\theta)\int_0^{R(\theta)} \frac{\rho^{|\gamma|+1}}{\sqrt{\rho^2+c^2}} d\rho d\theta,$$

where the limits of the integration $R(\theta), a, b$ are derived corresponding to the following figure a:

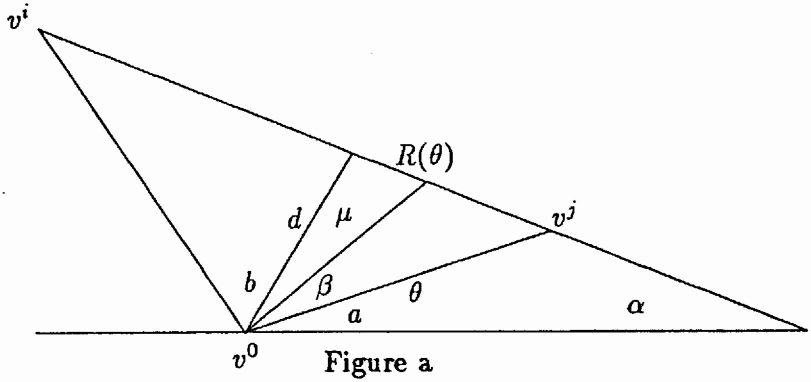

Figure a

Hence we have

$$a = \begin{cases} \frac{\pi}{2} & \text{if } v_1^i = 0, \\ \tan^{-1}(\frac{v_2^i}{v_1^i}) & \text{if } v_1^i \neq 0, \end{cases}$$

$$b = \begin{cases} \frac{\pi}{2} & \text{if } v_1^j = 0, \\ \tan^{-1}(\frac{v_2^j}{v_1^j}) & \text{if } v_1^j \neq 0, \end{cases}$$

(3.3) $$R(\theta) = \frac{d}{\cos(\theta-\beta)} = \frac{d}{\cos(\mu)},$$

$$\beta = \frac{\pi}{2} - \alpha,$$

$$\mu = \theta - \beta,$$

$$\alpha = \begin{cases} \frac{\pi}{2} & \text{if } v_1^j = v_1^i, \\ \tan^{-1}(\frac{v_2^j - v_2^i}{v_1^j - v_1^i}) & \text{if } v_1^j \neq v_1^i. \end{cases}$$

We will consider the integral $J(\gamma, c)$ in the form

$$J(\gamma, c) = \int_a^b \cos^{\gamma_1}(\theta)\sin^{\gamma_2}(\theta) G_{|\gamma|+1}(R(\theta), c)d\theta, \tag{3.4}$$

where

$$G_p(r, c) := \int_0^r \frac{\rho^p}{\sqrt{\rho^2 + c^2}} d\rho. \tag{3.5}$$

Integration by parts yields the recursion formula

$$G_0(r, c) = ln(r + \sqrt{r^2 + c^2}) - ln(|\, c\,|),$$

$$G_1(r, c) = \sqrt{r^2 + c^2} - |\, c\,|, \tag{3.6}$$

$$G_p(r, c) = \frac{1}{p} r^{p-1}\sqrt{r^2 + c^2} - \frac{p-1}{p} c^2 G_{p-2}(r, c) \text{ for } p = 2, 3, ...$$

Note that the function $G_p(r, c)$ can be efficiently calculated via this recursion. The formulae hold for $r > 0$ and $c \neq 0$, and are supplemented by $G_p(r, 0) = \frac{1}{2} r^p$ for $r > 0$ and $p > 0$.

It is clear that for fixed $c \in \mathbf{R}$ and $p = 1, 2, ...$, the function $r \mapsto G_p(r, c)$ is C^∞, and consequently the integrand in (3.4) is C^∞. Hence, the integral $J(\gamma, c)$ can be approximated by highly accurate and efficient numerical quadrature methods. We do not elaborate this point but refer to the standard literature on numerical quadrature.

4. Explicit Integration for the Case $c = 0$

For the case $c = 0$, the integral $J(\gamma, c)$ in (3.4) caan be explicitly calculated as follows. We have $G_p(r, 0) = \frac{1}{2} r^p$ and consequently, see (3.3),

$$\begin{aligned} J(\gamma, c) &= \frac{1}{|\,\gamma\,| + 1} \int_a^b \cos^{\gamma_1}(\theta)\sin^{\gamma_2}(\theta) R^{|\gamma|+1}(\theta) d\theta \\ &= \frac{d^{|\gamma|+1}}{|\,\gamma\,| + 1} \int_a^b \frac{\cos^{\gamma_1}(\theta)\sin^{\gamma_2}(\theta)}{\cos^{|\gamma|+1}(\theta - \beta)} d\theta \\ &= \frac{d^{|\gamma|+1}}{|\,\gamma\,| + 1} \int_{a-\beta}^{b-\beta} \frac{\cos^{\gamma_1}(\mu + \beta)\sin^{\gamma_2}(\mu + \beta)}{\cos^{|\gamma|+1}(\mu)} d\mu. \end{aligned}$$

For convenience, let us set $A := -\sin(\beta)$ and $B := \cos(\beta)$. Then

$$\frac{\cos^{\gamma_1}(\mu+\beta)\sin^{\gamma_2}(\mu+\beta)}{\cos^{|\gamma|+1}(\mu)} = \sum_{p_1=0}^{\gamma_1}\sum_{p_2=0}^{\gamma_2}(-1)^{\gamma_2-p_2}\binom{\gamma_1}{p_1}\binom{\gamma_2}{p_2}A^{\gamma_2+p_1-p_2}$$

$$\cdot B^{\gamma_1+p_2-p_1}\frac{\sin^{p_1+p_2}(\mu)\cos^{|\gamma|-p_1-p_2}(\mu)}{\cos^{|\gamma|+1}(\mu)}.$$

From this we have

$$J(\gamma,0) = \frac{d^{|\gamma|+1}}{|\gamma|+1}\sum_{p_1=0}^{\gamma_1}\sum_{p_2=0}^{\gamma_2}(-1)^{\gamma_2-p_2}\binom{\gamma_1}{p_1} \tag{4.1}$$

$$\cdot\binom{\gamma_2}{p_2}A^{\gamma_2+p_1-p_2}B^{\gamma_1+p_2-p_1}F_{p_1+p_2},$$

where we denoted by F_k the integral

$$F_k := \int_{a-\beta}^{b-\beta}\frac{\sin^k(\mu)}{\cos^{k+1}(\mu)}d\mu.$$

Now integration by parts yields the recursion

$$F_0 = ln(\frac{1+\tan(\mu/2)}{1-\tan(\mu/2)})\,|_{\mu=a-\beta}^{\mu=b-\beta},$$

$$F_1 = \frac{1}{\cos(\mu)}\,|_{\mu=a-\beta}^{\mu=b-\beta},$$

$$F_k = \frac{1}{k}\frac{\sin^{k-1}(\mu)}{\cos^k(\mu)}\,|_{\mu=a-\beta}^{\mu=b-\beta} - \frac{k-1}{k}F_{k-2} \quad \text{for } k = 2,3,\ldots,$$

which makes (4.1) explicit.

5. Asymptotic Estimates for Small $|c|$

In collocation or Galerkin methods, the entries of the system matrix $a_{i,j}$ are typically approximated only to such a precision that the order of the discretization error is not worsened. Hence, it is of interest to investigate the error which is introduced when replacing the integral $J(\gamma,c)$ by $J(\gamma,0)$ for small $|c|$. If this error is small, then the (more efficient) explicit calculations described in section 4 can be used for approximating $J(\gamma,c)$.

We differentiate the recursion (3.6) twice with respect to c and let c tend to zero. Thus the following limits are obtained:

$$\partial_2 G_1(r, 0 \pm 0) = \mp 1,$$

$$\partial_2 G_p(r, 0) = 0 \quad \text{for } p = 2, 3, ...,$$

$$\partial_2^2 G_p(r, 0) = -\frac{r^{p-2}}{p-2} \quad \text{for } p = 3, 4, \ldots,$$

Here ∂_2 indicates the partial derivative with respect to the c−variable, and 0 ± 0 indicates whether the limit $c \to 0$ is taken from the right or the left.

As a consequence, we obtain the following asymptotic estimates for the integral in (3.4).

$$J(\gamma, c) = J(\gamma, 0) + \begin{cases} O(|c|) & \text{for } |\gamma| = 0, \\ o(|c|) & \text{for } |\gamma| = 1, \\ O(|c|^2) & \text{for } |\gamma| = 2, 3, ... \end{cases}$$

Let us recall that the parameter $|c|$ describes the distance of the collocation point Q_p from the plane generated by the triangle Δ_q. On the other hand, the discretization error of the collocation or Galerkin method is usually known and expr ssed as $O(h^r)$ where h indicates the maximal diameter of the triangles Δ_q. Hence, if for example $|\gamma| \geq 2$, then the approximation $J(\gamma, c) \approx J(\gamma, 0)$ is permissible for collocation points Q_p which are near to Δ_q in the sense that the distance $|c|$ does not exceed the order $O(h^{r/2})$.

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REZIME

IZRAČUNAVANJE SLABO I SKORO SLABO SINGULARNIH INTEGRALA NA TROUGLOVIMA U $\mathbf{R}^3$.

Proučava se aproksimacija slabo singularnih integrala na trouglovima u opštem položaju u $\mathbf{R}^3$, koja daje eksplicitne formule gde je to pogodno i numeričke kvadrature u opštem slučaju. Posmatraju se partikularni modeli koji se odnose na kolokacije i Galerkinov metod za rešavanje Dirichleovog problema za Laplasovu jednačinu preko integrala po konturi.

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