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NEWTON-RAPHSON'S METHOD AND CONVEXITY

Miguel A. Hernández Veron
Dpto de Matemática Aplicada de la Universidad de Zaragoza
Colegio Universitario de La Rioja
C/ Obispo Bustamante 3
26001 Logroño (La Rioja), Spain

Abstract

The influence that the convexity of a real function f has in the Newton-Raphson's method is studied, in order to get the solution of $f(x) = 0$, and a method for accelerating this iterative process is obtained. A new iterative process of third order is obtained, as well.

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1. Introduction

The sequence $\{x_n\}$ obtained by the Newton-Raphson's method for the solution of the equation (1) $f(x) = 0$, consists in applying the iterative process

$$x_n = F(x_{n-1}) \text{ with } F(x) = x - \frac{f(x)}{f'(x)}. \tag{2}$$

We will consider $f \in C^m([a,b])$, $m \geq 2$, $[a,b] \subset \mathbf{R}$, a convex and strictly increasing function in $[a,b]$, with $f(a) < 0 < f(b)$, then (1) has one and only one root s in $[a,b]$, [4]. Besides $\{x_n\}$ converges to s when $x_0 \in [a,b]$ and this convergence is quadratic, [1]. Moreover, if we take $x_0 \in [a,b]$ such that $f(x_0) > 0$, then $\{x_n\}$ is a decreasing sequence.

In general, the Newton-Raphson's method converges for any function f provided that $f' \neq 0$ and f'' does not change sign on $[a,b]$, [4]. Therefore, changing $f(x)$ for $f(-x)$, $-f(x)$ or $-f(-x)$ we obtain our conditions.

The aim in this paper is to study the influence of the convexity in the Newton-Raphson's method. In order to study the convexity of f, we are going to use the log-degree of convexity [3], which will provide us with a measure of the convexity of f at each point. The log-degree of convexity of f at u_0 is defined to be the positive real number given by

$$U[f](u_0) = f''(u_0)[f'(u_0)]^{-2}. \tag{3}$$

If u_0 is a minimum of f we set $U[f](u_0) = +\infty$.

We will study the influence of the log-degree of convexity of f on the sequence $\{x_n\}$. Then we will prove that if the log-degree of convexity of f decreases, in some conditions, the sequence $\{x_n\}$ converges faster to s. So, we obtain a procedure for accelerating the Newton-Raphson's method. By applying this procedure we obtain an acceleration in the form $y_n = G(x_{n-1})$. Besides, this type of acceleration will provide us with a new iterative process of the third order, and also we will prove sufficient conditions for it's convergence.

2. Influence of the convexity on Newton-Raphson's method

Let g be a function, in the same conditions that f in $[a,b]$, such that $g(s) = 0$. We consider the sequence

$$y_n = G(y_{n-1}) \text{ being } G(x) = x - \frac{g(x)}{g'(x)} \text{ and } y_0 = x_0.$$

Then, by means of the log-degree of convexity, we are going to compare the sequences $\{x_n\}$ and $\{y_n\}$. We will denote $L_h(x) = h(x)U[h](x)$ for each h function.

Theorem 1. *If $L_f(x) > L_g(x)$ for $f(x) > 0$, then the sequence $\{y_n\}$ converges faster to s than $\{x_n\}$.*

Proof. In our conditions $\{x_n\}$ and $\{y_0\}$ are decreasing sequences to s. Therefore, we are going to prove by using induction that $y_n < x_n$ for all $n \in \mathrm{N}$ such that $x_n \neq s$. Besides $y_n = s$ if $x_{n-1} \neq s$ and $x_n = s$.

Taking into account that $F(s) = G(s) = s$ and $x_0 \neq s$, we have that

$$x_1 - y_1 = (F - G)(x_0) - (F - G)(s)$$

and therefore there exists $\xi_0 \in (s, x_0)$ such that

$$x_1 - y_1 = (F - G)'(\xi_0)(x_0 - s).$$

On the other hand as

$$(F - G)'(x) = L_f(x) - L_g(x)$$

and $f(\xi_0) > 0$ we obtain that $x_1 - y_1 > 0$.

Now we assume that $x_k > y_k$ for $k = 1, 2, \ldots, n-1$, if $x_n \neq s$ then $x_{n-1} \neq s$ and

$$x_n - y_n = F(x_{n-1}) - G(y_{n-1}) \geq (F - G)(x_{n-1})$$

since G is an increasing function in $[s, b]$ and $\{y_n\} \subset [s, b]$. So, in a way similar to the case $k = 1$, we obtain that $x_n - y_n > 0$.

By applying the above arguments it is easy to prove that if $x_{n-1} \neq s$ and $x_n = s$ then $y_n = s$. □

Below we are going to translate this result by means of the log-degree of convexity.

Corollary 1. *If $U[f](x) > U[g](x)$ and $g'(s) \leq f'(s)$ then the sequence $\{y_n\}$ converges faster to s than $\{x_n\}$.*

Proof. Since $U[f](x) > U[g](x)$, for all $x \in (s, b)$, taking into account (3), it is easy to prove that

$$0 < \int_s^x (U[f](t) - U[g](t))dt = \frac{1}{g'(x)} - \frac{1}{f'(x)} - \left(\frac{1}{g'(s)} - \frac{1}{f'(s)}\right)$$

and therefore $f'(x) > g'(x)$ in (s, b). So, $f(x) > g(x)$ for $f(x) > 0$, i.e. in (s, b), and then $L_f(x) > L_g(x)$ for $f(x) > 0$. Therefore, by applying Theorem 1 we obtain the thesis. □

Note that if $U[f](x) > U[g](x)$ for $f(x) > 0$ the previous result is valid too.

3. An acceleration of Newton-Raphson's method

As a result of Corollary 1 we obtain that if the log-degree of convexity of f decreases, the sequence $\{x_n\}$ converges faster to s. Besides, if we consider $y_n = G(x_{n-1})$ then $\{y_n\}$ is an acceleration of $\{x_n\}$.

Now, we are going to define a function that verifies the conditions of Corollary 1, and thus we will obtain an acceleration of Newton-Raphson's method.

Since straight lines have the minimum log-degree of convexity if we consider "$f'(s)(x-s)$", i.e. the tangent line of f at s, this function verifies the conditions of Corollary 1. Then, we take

$$g(x) = f(x) - [f''(s)/2!](x-s)^2$$

that is the Taylor aproximation to the tangent line of f at s. But, as s is an unknown point, taking into account that we will define $y_n = G(x_{n-1})$ and that $\{x_n\}$ is a decreasing sequence to s, we can consider for $k = 1, 2$:

$$f''(s)(x_{n-1}-s)^k \sim f''(x_{n-1})(x_{n-1}-x_n)^k = f''(x_{n-1})[f'(x_{n-1})]^{-k} f(x_{n-1})^k.$$

Since

$$\lim_n \frac{f''(s)(x_{n-1}-s)^k}{f''(x_{n-1})(x_{n-1}-x_n)^k [\frac{f'(b)}{f'(x_{n-1})}]^k} = 1,$$

we have

$$g(x_{n-1}) \sim f(x_{n-1}) - \frac{f''(x_{n-1})}{2!}[f'(x_{n-1})]^{-2} f(x_{n-1})^2$$

$$g'(x_{n-1}) \sim f'(x_{n-1}) - f''(x_{n-1})[f'(x_{n-1})]^{-1} f(x_{n-1}).$$

Therefore we obtain the sequence

$$y_n = x_{n-1} - \frac{f(x_{n-1})}{2f'(x_{n-1})} \frac{2 - L_f(x_{n-1})}{1 - L_f(x_{n-1})}. \tag{4}$$

Corollary 2. *Sequence (4) is an acceleration of Newton-Raphson's method.*

Proof. It suffices to prove that $\lim_n[|y_n - s|/|x_n - s|] = 0$, where $\{x_n\}$ is the sequence given by (2). □

Note that as $\lim_n x_n = s$ and $L_f(s) = 0$, we can take x_0 such that $L_f(x_n) < 1$ for all n.

4. A new iterative process of third order

Acceleration (4) will provide us with a new iterative process, for the solution of (1), given by the expression

$$(5) \qquad z_n = z_{n-1} - \frac{f(z_{n-1})}{f'(z_{n-1})} H(L_f(z_{n-1})) \text{ with } H(z) = \frac{1}{2} - \frac{1}{2(1-z)}.$$

Now, we obtain the result of global convergence for this iterative process by means of the value of $L_{f'}$, so we will consider f a function in the previous conditions with $m \geq 3$, besides $z_0 \in [a,b]$ with $f(z_0) > 0$.

Theorem 2. *If $a, b \in \mathbf{R}$ and $M = \max\{U[f](x)/x \in [a,b]\}$ such that $M < 1/f(b)$ and $L_{f'}(x) \leq 0$ in $[a,b]$ then the sequence $\{z_n\}$, given by (5), decreases to s.*

Proof. As $x_0 > s$ and $L_f(x) < f(b)U[f](x) < 1$ in $[s,b]$, then (5) is a decreasing sequence.

To finish, we are going to prove that $z_n \geq s$ for all $n \in \mathbf{N}$. It is clear that

$$(6) \qquad z_n = \frac{1}{2}[F(z_{n-1}) + G(z_{n-1})]$$

(where $F(x) = x - \frac{f(x)}{f'(x)}$ and $G(x) = x - \frac{g(x)}{g'(x)}$ with $g = \frac{f}{f'}$). Besides, $F'(x) = L_f(x)$ and $G'(x) = L_g(x)$. Then, taking into account that

$$L_g(x) = -\frac{f(x)L'_f(x)}{f'(x)(1 - L_f(x))^2}$$

and

$$L_{f'}(x) = \frac{f''(x)}{f'(x)}[1 - L_f(x)(2 - L_{f'}(x))]$$

it follows that

$$L_f(x) + L_g(x) = L_f(x)[1 - L_f(x)]^{-2}[L_f(x) - L_{f'}(x)]$$

and therefore $F'(x) + G'(x) \geq 0$ in $[s, b]$. On the other hand,

$$z_1 - s = [(F + G)/2](z_0) - [(F + G)/2](s),$$

so there exists $\xi_0 \in (s, z_0)$ such that

$$z_1 - s = [(L_f + L_g)/2](\xi_0)(z_0 - s)$$

and then $z_1 - s \geq 0$. By induction, it is easy to prove that $z_n - s \geq 0$ for all $n \in \mathbf{N}$.

As $z_n \geq s$ for all $n \in \mathbf{N}$ and $\{z_n\}$ is a decreasing sequence then there exists $lim_n z_n = u \geq s$, and since $L_f(u) < 1$, one can conclude that $u = s$ taking limits in (5). □

Now we are going to study the case where $L_{f'}(x) \geq 0$.

Lemma. *Let f be as before with $m \geq 3$. If*

$$3/f'(b) \underset{(i)}{\geq} U[f'](x) \underset{(ii)}{\geq} 2/f'(a)$$

in $[a, b]$, $U[f](b) < 1/K$ with $K = \max\{|f(a)|, f(b)\}$, then $|L_f(x)| < 1$ in $[a, b]$. Moreover, if $L_f(z_0) < 1/4$ then $|L_g(x)| < 1$ in $[a, z_0]$, with $g = f/f'$.

Proof. It is clear that $U[f]$ is an increasing function in $[a, b]$ if and only if $U[f'](x) \geq 2/f'(x)$ in $[a, b]$, then for (ii) we obtain that $U[f]$ is an increasing function in $[a, b]$. Since $U[f](x) > 0$ in $[a, b]$ and $U[f](b) < 1/K$, as $L_f = fU[f]$, it follows that $|L_f(x)| < 1$.

Now, taking into account that $|L_f(x)| < 1$, (i), (ii) and

$$L'_f(x) = \frac{f''(x)}{f'(x)}[1 - L_f(x)(2 - L_{f'}(x))]$$

it follows that L_f is an increasing function in $[a, b]$.

On the other hand, $|L_g(x)| < 1$ if and only if

$$(*) \qquad |f(x)(-L'_f(x))| < f'(x)(1 - L_f(x))^2$$

and this is always true in $[a, s]$. If $x \in (s, b]$, $(*)$ is equivalent to

$$L_f(x)[3(1 - L_f(x)) + L_f(x)L_{f'}(x)] < 1.$$

Hence, as L_f is in an increasing function in $[a, b]$ and $3(1 - L_f(x)) < 3$ turns out to be suficient, taking into account that $L_f(z_0) < 1/4$ and $L_{f'}(x) \leq 3$ for (i). □

Theorem 3. *Let f be as before. If*

$$(iii) \qquad z_0 - a > 7f(z_0)/6f'(z_0),$$

then the iterative process given by (5) is convergent to s.

Proof. As

$$z_n = \frac{1}{2}[F(z_{n-1}) + G(z_{n-1})]$$

by (6), from (iii) and the Lemma we have that $b > z_0 > z_1 > a$, $|L_f(x)| < 1$ and $|L_g(x)| < 1$ in $[a, z_0]$.

Hence,

$$F(z_0) - F(s) = L_f(\beta_0)(z_0 - s)$$

and

$$G(z_0) - G(s) = L_g(\mu_0)(z_0 - s)$$

with $\beta_0, \mu_0 \in (s, z_0)$, and therefore $|z_1 - s| \leq C|z_0 - s|$, with $C < 1$. Reiterating, we obtain that $z_n \in [a, b]$ and $\{|z_n - s|\}$ is a strictly decreasing sequence to zero. □

It is known [2], that if we have an iterative process in the form given by (5), with $H(0) = 1$, $H'(0) = 1/2$ and $|H''(0)| < +\infty$, then this iterative process has cubic convergence. Therefore, if $m \geq 4$, it is easy to prove that the iterative process given by (5) is of the third order.

Note that as $L_f(s) = 0$, it is easy to take a,b and z_0 as in the previous results.

5. Practical remarks

In applying the new iterative process given by (5), it is necessary to compute f,f' and f'' for each step, besides the specific calculations. But, the required calculations do not have the negative effect on practice since, for the solution of equations, each of the usual iterative processes of the third order [2], needs a similar number of calculations.

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REZIME

NEWTON - RAPHSON -OV METOD I KONVEKSNOST

Proučava se uticaj konveksnosti realne funkcije f na Newton - Raphson - ov metod za nalaženje rešenja jednačine $f(x) = 0$. Dobijen je metod za ubrzanje ovog iterativnog postupka, koji ima red konvergencije tri.

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