Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 22, 2 (1992), 11-21

Review of Research
Faculty of Science
Mathematics Series

CONVERGENCE OF THE UNSYMMETRIC SUCCESSIVE OVERRELAXATION METHOD FOR SDD MATRICES

Ljiljana Cvetković, Nataša Krejić
Institute of Mathematics, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

In this paper a class of SDD matrices is considered for two reasons. First, for this subclass of H - matrices convergence area, which depends only on the matrix elements, is derived, so it is easy to calculate. This area can be wider than the known one for the H - matrices. Second, convergence results for the SDD matrices can be generalized to the whole class of H – matrices using the fact that for every H – matrix A there exists a regular diagonal matrix W such that AW is an SDD matrix.

AMS Mathematics Subject Classification (1991): 65F10
Key words and phrases: iterative methods, relaxation methods, SDD matrices.

1. Introduction

Unsymmetric Successive Overrelaxation Method is a two parameter generalization of the Successive Overrelaxation Method. Sufficient convergence conditions for this method are known for some special classes of matrices like H – matrices (Herceg, Krejić, [5]), p– cyclic matrices (Saridakis, [12], Li, Varga, [6]), matrices with red–black ordering (Krishna, [7], Krishna, Matrins, [8]) and so on.

It is well known that the class od strictly diagonaly dominant (SDD) matrices has great practical importance. Also, every H – matrix can be transform into a SDD matrix. Convergence theorem in the paper Herceg, Krejić, [5] deals with the H – matrices using the theory of H – and M – matrices. In this paper we derive the convergence area for the SDD matrices based on bounds for spectral radius of the iteration matrix given in the paper Cvetković, [2].

As every SDD matrix is an H - matrix, it is interesting to compare these two areas. It can be seen that they have non – empty intersection and in general, neither of them is subset of the another one. It is also impotrant that there is a possibility to generalize the conditions acheived in this paper to the whole class of H – matrices.

2. USSOR Method

Let us consider the system of linear equations

$$Ax = b,$$

where A is a strictly diagonaly dominant matrix from C^{nn}.

Standard splitting of the matrix A is

$$A = D(E - L - U).$$

Here, D is diagonal matrix and L, U are strictly lower and strictly upper triangular matrices, respectively and E denotes the $n \times n$ identity matrix.

USSOR metod is a two half iteration method. The first half iteration is the same as the SOR method with parameter σ, while the second half iteration is the backward SOR method with parameter ω, i.e.,

(1)
$$\begin{aligned} x^{n+\frac{1}{2}} &= \mathcal{L}_\sigma x^n + (E - \sigma L)^{-1}\sigma b \\ x^{n+1} &= \mathcal{U}_\omega x^{n+\frac{1}{2}} + (E - \omega U)^{-1}\omega b, \end{aligned}$$

where

(2)
$$\mathcal{L}_\sigma = (E - \sigma L)^{-1}((1 - \sigma)E + \sigma U)$$

$$\mathcal{U}_\omega = (E - \omega U)^{-1}((1-\omega)E + \omega L).$$

So, the iteration matrix of the USSOR method is

$$\mathcal{S}_{\sigma\omega} = \mathcal{U}_\omega \mathcal{L}_\sigma. \tag{3}$$

If $\omega = \sigma$ USSOR method becomes Symmetric Successive Overrelaxation method with iteration matrix $\mathcal{S}_{\sigma\sigma}$. Convergence of this method is investigated in many papers, see [13], [1], [9], [10], [8], etc.

3. Convergence results

First, let us introduce some notation:

$$P_i(A) = \sum_{i=1}^{n} |a_{ij}|,$$

$$l_i = P_i(L), \quad u_i = P_i(U),$$

$$\Omega(A) = \{C = [c_{ij}] \in C^{n,n} : |c_{ij}| = |a_{ij}|, 1 \le i,j \le n\},$$

$$C_\pi^{nn} = \{A \in C^{nn} : a_{ii} \neq 0, 1 \le i \le n\}.$$

Following convergence theorem is given in the paper Herceg, Krejić, [5].

Theorem 1. *Let $A = [a_{ij}] \in C_\pi^{n,n}$, $n \ge 2$. Then the following statements are equivalent:*
a) A is nonsingular H-matrix;
b) For all $C \in \Omega(A)$ and all $(\sigma, \omega) \in \mathcal{O}$ USSOR method is convergent; where the set $\mathcal{O}$ is defined as

$$\sigma \in \left(-\frac{1-\rho}{2\rho}, \frac{\rho+1}{2\rho}\right), \ \ \textit{with}\ \rho = \rho(|L| + |U|),$$

$$\omega \in \left(max\left\{\frac{|1-\sigma|+|\sigma|\rho-1}{|1-\sigma|+\rho(|\sigma|+1)}, \frac{|1-\sigma|+|\sigma|\rho-1}{|1-\sigma|(1-\rho)}\right\}, min\left\{\frac{1+|1-\sigma|+\rho|\sigma|}{\rho(1+|\sigma|)+|1-\sigma|}, \frac{1+|1-\sigma|-\rho|\sigma|}{|1-\sigma|(1+\rho)}\right\}\right).$$

As every SDD matrix is an H - matrix, this theorem can be used to derive the convergence area. The set $\mathcal{O}$ depends on $\rho = \rho(|L| + |U|)$, which is the same for the whole set $\Omega(A)$, but it's value is usually unknown. In this paper we derive convergence conditions for SDD matrices in terms of matrix elements, more precisely in terms of $l_i, u_i,\ i = 1, \dots, n$, using bound for spectral radius of the iteration matrix given in the paper Cvetković, [2].

Theorem 2. *[2] If A is an SDD matrix and $1-|\omega|u_i>0,\ \ 1-|\sigma|l_j> 0,\ \ 1\le i,j,\le n$, then for the iteration matrix of the USSOR method it stands*

$$\rho(S_{\sigma\omega})\le \max_{1\le i,j\le n}\frac{|1-\omega|+|\omega|l_i}{1-|\omega|u_i}\cdot\frac{|1-\sigma|+|\sigma|u_j}{1-|\sigma|l_j}.$$

Let us define some functions.

$$f(x,l_i,u_i,l_j,u_j)=\frac{x(1-l_i-u_i)}{1+l_j+u_j-x(1+u_j-l_i-l_iu_j+l_ju_i)}$$

$$g(x,l_i,u_i,l_j,u_j)=\frac{2-x(1-l_i+u_i)}{1+l_j+u_j-x(1+u_j-l_i-l_iu_j+l_ju_i)}$$

$$h(x,l_i,u_i,l_j,u_j)=\frac{2-x(1+l_i+u_i)}{x(1+l_i+u_j+l_iu_j-l_ju_i)-(1-l_j+u_j)}$$

$$s(x,l_i,u_i,l_j,u_j)=\frac{x(1-l_j+u_j)}{x(1+l_i+u_j+l_iu_j-l_ju_i)-(1+l_i-u_i)}$$

$$f1(x)=\min_{1\le i,j\le n} f(x,l_i,u_i,l_j,u_j)$$

$$f2(x)=\min_{1\le i,j\le n} f(x,u_j,l_j,u_i,l_i)$$

$$g1(x)=\min_{1\le i,j\le n} g(x,l_i,u_i,l_j,u_j)$$

$$g2(x)=\min_{1\le i,j\le n} g(x,u_j,l_j,u_i,l_i)$$

$$h1(x)=\min_{1\le i,j\le n} h(x,l_i,u_i,l_j,u_j)$$

$$h2(x)=\min_{1\le i,j\le n} h(x,u_j,l_j,u_i,l_i)$$

$$s1(x)=\min_{1\le i,j\le n} s(x,l_i,u_i,l_j,u_j)$$

$$t=\min_{1\le i\le n}\frac{2}{1+l_i+u_i}$$

Theorem 3. *If A is an SDD matrix then the USSOR method is convergent for all (σ,ω) which satisfy one of the following conditions:*

(a) $0<\omega\le 1,\ \ -f1(\omega)<\sigma<g1(\omega)$;

(b) $0<\sigma\le 1,\ \ -f2(\sigma)<\omega<g2(\sigma)$;

(c) $1 < \sigma < g1(1), \; 1 < \omega < s1(\sigma);$

(d) $1 < \omega < t, \; -h1(\omega) < \sigma < 0;$

(e) $1 < \sigma < t, \; -h1(\sigma) < \omega < 0.$

Proof. Let us suppose that the parameters (σ,ω) satisfy conditions (a). It means that $0 < \omega \le 1$ and

$$-\min_{i,j} \frac{\omega(1-l_i-u_i)}{1+l_j+u_j-\omega(1+u_j-l_i-l_iu_j+l_ju_i)} < \sigma <$$

$$\min_{i,j} \frac{2-\omega(1-l_i+u_i)}{1+l_j+u_j-\omega(1+u_j-l_i-l_iu_j+l_ju_i)}.$$

It is easy to see that $1+l_j+u_j-\omega(1+u_j-l_i-l_iu_j+l_ju_i) > 0$ for $0 < \omega \le 1$. Now, there are two possibilies: $\sigma < 0$ or $\sigma > 0$. If $\sigma < 0$, we have

$$-\omega(1-l_i-u_i) - \sigma(1+l_j+u_j-\omega(1+u_j-l_i-l_iu_j+l_ju_i)) < 0,$$

for all indices $1 \le i,j, \le n$, what is equivalent with

$$\frac{1-\sigma-\sigma u_j}{1+\sigma l_j} \cdot \frac{1-\omega+\omega l_i}{1-\omega u_i} < 1, \;\; 1 \le i,j, \le n.$$

As $0 < \omega \le 1$ and $\sigma < 0$, we have

$$\frac{|1-\omega|+|\omega|l_i}{1-|\omega|u_i} \cdot \frac{|1-\sigma|+|\sigma|u_j}{1-|\sigma|l_j} < 1, \;\; 1 \le i,j \le n$$

and

$$\max_{i,j} \frac{|1-\omega|+|\omega|l_i}{1-|\omega|u_i} \cdot \frac{|1-\sigma|+|\sigma|u_j}{1-|\sigma|l_j} < 1.$$

Using Theorem 2, we can conclude that in this case $\rho(S_{\sigma\omega}) < 1$, i.e. USSOR method is convergent. Simillarly, if $\sigma > 0$, we have

$$\sigma(1+l_j+u_j-\omega(1-l_i+u_j-l_iu_j+u_il_j))-2-\omega(1-l_i+u_i) < 0, \;\; 1 \le i,j, \le n,$$

or

$$\frac{1-\sigma+\sigma u_j}{1-\sigma l_j} \cdot \frac{1-\omega+\omega l_i}{1-\omega u_i} < 1, \;\; 1 \le i,j, \le n.$$

As $g1(\omega) < 1$ for $0 < \omega \leq 1$, we have

$$\frac{|1-\omega| + |\omega| l_i}{1 - |\omega| u_i} \cdot \frac{|1-\sigma| + |\sigma| u_j}{1 - |\sigma| l_j} < 1, \quad 1 \leq i, j \leq n,$$

what means

$$\max_{i,j} \frac{|1-\omega| + |\omega| l_i}{1 - |\omega| u_i} \cdot \frac{|1-\sigma| + |\sigma| u_j}{1 - |\sigma| l_j} < 1,$$

and USSOR method is convergent by Theorem 2.

In other cases (b) - (e) the proof is analoques to this case.

Figure 1 illustrates the application of this theorem and relation beetwen area $\mathcal{O}$ from Theorem 1 and the convergence area from this theorem on a simple example.

As a special case we can consider SDD matrices with the property that there exists $k \in \{1, \ldots, n\}$ such that

$$(4) \qquad ||L||_\infty = P_k(L) = l, \quad ||U||_\infty = P_k(U) = u.$$

Then, the spectral radius of the iteration matrix can be bounded like

$$\rho(S_{\sigma\omega}) \leq \frac{|1-\omega| + |\omega| l}{1 - |\omega| u} \frac{|1-\sigma| + |\sigma| u}{1 - |\sigma| l}.$$

Analysing this bound, we obtain the following convergence conditions.

Theorem 4. *Let A be an SDD matrix with the property (4). Then the USSOR method is convergent for all (σ, ω) which satisfy one of the following conditions:*

(a) $0 < \omega \leq 1, \; -\frac{\omega(1-l-u)}{1+l+u-\omega(1-l+u)} < \sigma < \frac{2-\omega(1-l+u)}{1+l+u-\omega(1-l+u)}$

(b) $0 < \sigma \leq 1, \; -\frac{\sigma(1-l-u)}{1+l+u-\sigma(1+l-u)} < \omega < \frac{2-\sigma(1+l-u)}{1+l+u-\sigma(1+l-u)}$

(c) $1 < \sigma < \frac{1+l-u}{2l}, \; 1 < \omega < \frac{\sigma(1-l+u)}{\sigma(1+l+u)-(1+l-u)}$

(d) $1 < \omega < \frac{2}{1+l+u}, \; -\frac{2-\omega(1+l+u)}{\omega(1+l+u)-(1-l+u)} < \sigma < 0$

(e) $1 < \sigma < \frac{2}{1+l+u}, \; -\frac{2-\sigma(1+l+u)}{\sigma(1+l+u)-(1+l-u)} < \omega < 0.$

Application of this theorem is illustrated in Figure 2.

We have mentioned above that for every H – matrix A, there is a regular diagonal matrix W such that AW is an SDD matrix. For some subclasses of H – matrices matrix W is known, see Cvetković, Herceg, [3] and Herceg, Cvetković, [4]. Also,

$$AW = DW(E - W^{-1}LW - W^{-1}UW),$$

so

$$\mathcal{L}_\sigma(AW) = (E - \sigma W^{-1}LW)^{-1}[(1-\sigma)E + \sigma W^{-1}UW] = W^{-1}\mathcal{L}_\sigma(A)W,$$

and

$$\mathcal{U}_\omega(AW) = (E - \omega W^{-1}UW)^{-1}[(1-\omega)E + \omega W^{-1}LW] = W^{-1}\mathcal{U}_\omega(A)W,$$

what gives

$$\mathcal{S}_{\sigma\omega}(AW) = \mathcal{U}_\omega(AW)\mathcal{L}_\sigma(AW) = W^{-1}\mathcal{S}_{\sigma\omega}(A)W,$$

and this relation means that

$$\rho(\mathcal{S}_{\sigma\omega}(A)) = \rho(\mathcal{S}_{\sigma\omega}(AW)).$$

Using this fact we can generalize Theorem 3 to the whole class of H – matrices.

Remark. For an H – matrix A convergence area of the USSOR method is the area derived in Theorem 3 with

$$l_i = P_i(W^{-1}LW),\ u_i = P_i(W^{-1}UW),\ \ 1 \le i \le n.$$

4. Numerical Examples

Let us consider matrix

$$A = \begin{bmatrix} 0 & 1/9 \\ 1/4 & 0 \end{bmatrix}.$$

For this matrix $\rho(|L| + |U|) = \frac{1}{6}$. Figure 1 represents areas obtained by Theorem 1 and Theorem 3. Shaded area is obtained by Theorem 3.

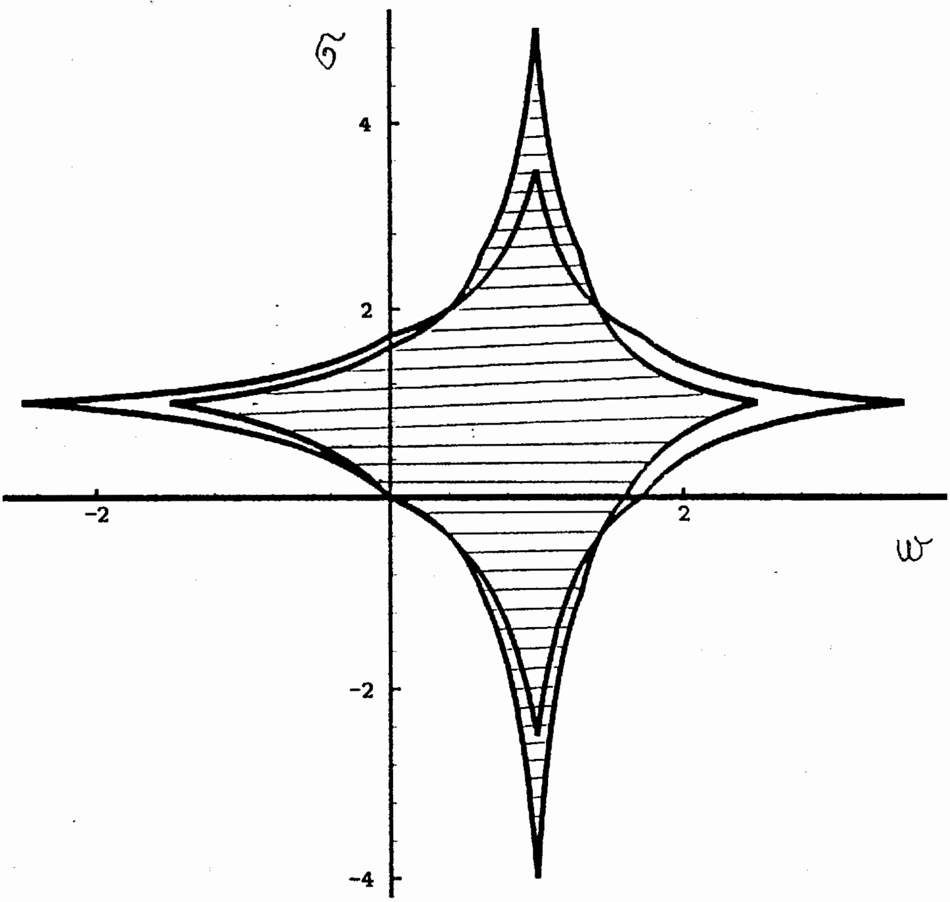

Figure 1.

For illustration of Theorem 4 we consider $nm \times nm$ block three diagonal matrix

$$B = \begin{bmatrix} R & -E & 0 & \dots & 0 \\ -E & R & -E & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & R & -E \\ 0 & 0 & \dots & -E & R \end{bmatrix},$$

where R is $m \times m$ matrix

$$R = \begin{bmatrix} 6 & -1 & 0 & \dots & 0 \\ -1 & 6 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 6 & -1 \\ 0 & 0 & \dots & -1 & 6 \end{bmatrix},$$

and E denotes $m \times m$ identity matrix. For $m = 6$ and $n = 100$ spectral radius of Jacobi matrix is $\rho(|L| + |U|) = 0.6218472286921422$, and $l = 2/6$, $u = 2/6$. Areas obtained by the application of Theorem 1 and Theorem 4 are shown in figure 2. Shaded area is obtained by Theorem 4.

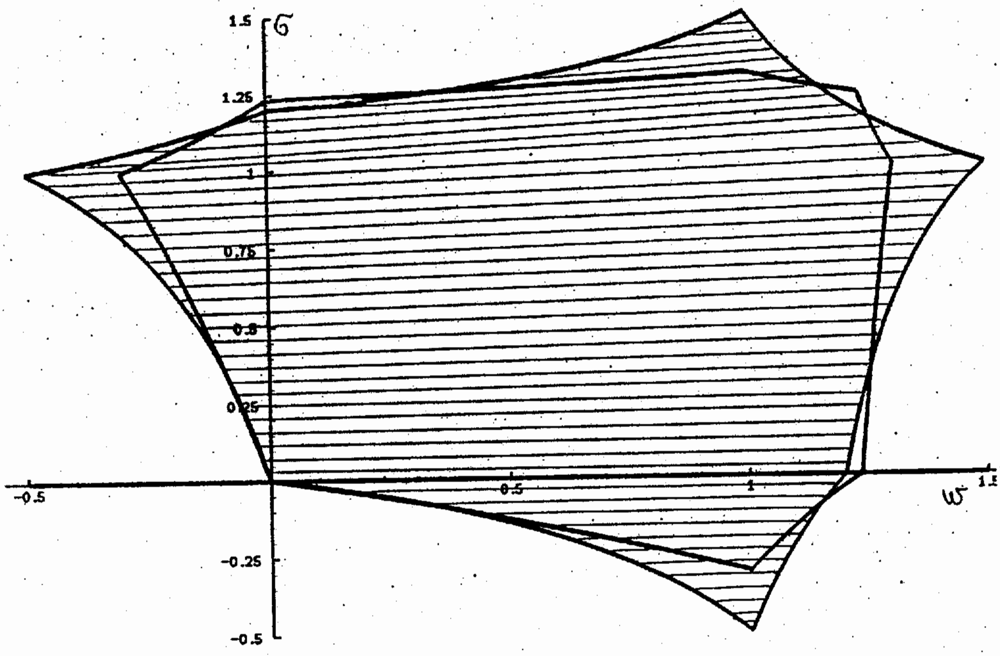

Figure 2.

References

[1] Alefeld,G., Varga,R.S.: Zur konvergenz des symmetrishen Relaxationsverfahrens, Numer. Math. 25(1976), 291-295

[2] Cvetković, Lj.: Some inequalities for apectral radius in connection with relaxation methods convergence theory, Univ. u Novom Sadu Zb. Rad. Prirod. - Mat. Fak. Ser. Mat. 21,2 (1991), 111-118.

[3] Cvetković, Lj., Herceg, D.: Some results on M- and H-matrices, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 17,1 (1987), 121-129.

[4] Herceg, D., Cvetković, Lj.: Some results on H – matrices in connection with the AOR convergence theory, Bull. Appl. Math. 720(1991), (LVIII), 7-13.

[5] Herceg, D., Krejić, N.: On the Convergence of the Unsymmetric successive Overrelaxation (USSOR) Method, Linear Algebra Appl. (in print).

[6] Li, X., Varga, R.S.: A Note on the SSOR and USSOR Iterative Methods Applied to p - Cyclic Matrices, in Iterative Methods for Large Linear Systems, D.R. Kincaid, L.J. Hayes, eds., Academic Press 1990, 235-250.

[7] Krishna,L.B.: Some new results on the unsymmetric successive overrelaxation, Numer. Math. 42(1983), 155-160

[8] Martins,M.M., Krishna,L.B.: Some New Results on the Convergence of the SSOR and USSOR Methods, Linear Algebra Appl.106 (1988), 185-193.

[9] Neumaier, A., Varga, R.S.: Exact convergence and divergence domains for the symmetric successive overrelaxation iterative (SSOR) method applied to the H - matrices, Linear Algebra Appl. 58,(1984) 261-272.

[10] Neumann, M.: On bounds for the convergence of the SSOR method for H - matrices, Linear Multilinear Algebra 15(1984), 13-21.

[11] Neumann, M., Varga, R.S.: On the sharpness of some upper bounds for the spectral radii of S.O.R. iteration matrices, Numer. Math. 35(1980), 69-79.

[12] Saridakis, Y.G.: Domains of Divergence of the USSOR Method Applied on p - Cyclic Matrices, Numer. Math. 57(1990), 405-412.

[13] Young,D.: Iterative Solution of Large Linear Systems, New York, Academic Press 1971.

[14] Varga,R.S.: Matrix Iterative Analysis, Prentice-Hall, Englewood Cliffs, N.J., 1962.

REZIME

KONVERGENCIJA NESIMETRIČNOG SOR POSTUPKA ZA SDD MATRICE

Klasa SDD matrica se razmatra iz dva razloga. Prvo, za tu potklasu H - matrica odredjuje se oblast konvergencije koja zavisi samo od elemenata matrice, tako da je jednostavna za izračunavanje. Ta oblast može biti i šira od poznate oblasti za H - matrice. Drugo, rezultati o konvergenciji za SDD matrice mogu se generalizovati na čitavu klasu H - matrica, koristeći činjenicu da za svaku H - matricu A postoji regularna dijagonalna matrica W tako da je AW SDD matrica.

Received by the editors December 4, 1992