Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 22, 2 (1992), 23-37

Review of Research
Faculty of Science
Mathematics Series

ON A DIFFERENCE SCHEME FOR SINGULAR PERTURBATION PROBLEMS

Nenad Petrović
University of Novi Sad, Advanced Technical School
Školska 1, 21000 Novi Sad, Yugoslavia

Abstract

The numerical solution of nonlinear two-point singular perturbation boundary value problem is studied. The developed method is based on a combination of numerical solutions of boundary value problems which approximate the problem in boundary layers and the solution of reduced problem. Numerical examples are presented.

AMS Mathematics Subject Classification (1991): 65L10
Key words and phrases: singular perturbation, numerical solution, boundary value problem.

1. Introduction

In this paper we shall consider the problem

$$T_\epsilon u := -\epsilon^2 u'' + c(x,u) = 0, \quad x \in I = [0,1], \tag{1}$$

$$u(0) = u(1) = 0,$$

where $\epsilon \in (0,\epsilon_0), \epsilon_0 \ll 1$, is a small perturbation parameter. We assume that the following conditions are satisfied:

$$c \in C^k(I \times \mathrm{R}), \quad k \in \mathrm{N}, \tag{2}$$

$$0 < \gamma^2 < c_u(x,u) \leq \Gamma, \quad (x,u) \in I \times \mathrm{R}.$$

Numerical treatment of problem (1) was considered among the others, in [1], [2], [9], [12], [13], [20], [23], [26] and [28]. This problem occurs in the study of chemical catalisys, fluid mechanics (boundary value problems), elasticity, quantum mechanics and fluid dynamics.

It is well known that there exists a unique solution $u_\epsilon \in C^{k+2}(I)$ to (1) which in general displays boundary layers at $x = 0$ and $x = 1$ for small ϵ, [1], [2], [9], [19]. The corresponding reduced problem $c(x, u) = 0$ has also a unique solution $u_0 \in C^k(I)$ which in general does not satisfy the boundary conditions. For the solution u_ϵ to (1) it holds, [23], [25]:

$$(3) | u_\epsilon^{(i)}(x) |= \begin{cases} M(1 + \epsilon^{-i} \exp(-\gamma x/\epsilon), 0 \le x \le 0.5, \\ M(1 + \epsilon^{-i} \exp(-\gamma(1 - x)/\epsilon), 0.5 \le x \le 1. \end{cases} \quad i = 0, 1, ..., k.$$

Here and througoht the paper M denotes any positive constant that may take a different values in different formulas, but that are always independent of ϵ and of discterization mesh.

In this paper we shall talk about a numerical solution of problem (1) which consists of $u_0(x)$ for $x \in [s, 1 - s_0], s, s_0 \in (0, 0.5)$ and numerical solutions to problems

$$(4) \qquad -\epsilon^2 v'' + c(x, v) = 0, \quad x \in [0, s],$$

$$v(0) = 0, \quad v(s) = u_0(s),$$

$$(5) \qquad -\epsilon^2 w'' + c(x, w) = 0, \quad x \in [1 - s_0, 1],$$

$$w(1 - s_0) = u_0(1 - s_0), \; w(1) = 0.$$

A choice of s and s_0 is described in the section 2.

From now on we consider a numerical solution to (1) on $[0, 0.5]$. A numerical solution on $[0, 0.5]$ can be constructed in a similar way.

Let

$$(6) \qquad n \in \mathbf{N}, \; h = \frac{1}{2n}, \; I_h = \{x_i = \lambda(ih) : i = 0, 1, ..., n\},$$

be a special discretization mesh with mesh genetaring function

$$(7) \qquad \lambda(t) = \frac{a\epsilon t}{q - t}, \quad t \in [0, 0.5],$$

where

$$q = a\epsilon + 0.5,$$

and a satisfies

(8) $$0 < 2a\epsilon < 1.$$

Let $I_s = \{x \in I_h : x \leq s\}$. On I_s we solve (4) using finite difference. It we denote this solution by $v_h = [v_0, v_1, ..., v_m]^T$, $m \leq n$, then we prove that

$$| v(x_i) - v_i | \leq Mh^4, \ x_i \in I_s,$$

where v is exact solution to (4). The existence of v and the following estimates

(9) $$| u_\epsilon(x) - v(x) | \leq M(\exp(-\gamma s/\epsilon) + \epsilon^2), \ x \in [0, s],$$

(10) $$| v^{(i)}(x) | \leq M(1 + \epsilon^{-i} \exp(-\gamma x/\epsilon)), \ x \in [0, 0.5], \ i = 0, 1, ..., k,$$

follow from the inverse monotonicity of (4) under assumption (2), see [19], [23], [25]. For $s \leq x \leq -0.5$ we approximate $u_\epsilon(x)$ by $u_0(x)$ and it holds

(11) $$| u_\epsilon(x) - u_0(x) | \leq M(\exp(-s\gamma/\epsilon) + \epsilon^2), \ x \in [s, 0.5].$$

Using (9) and (10) we prove

$$| u_\epsilon(x) - u(x) | \leq M(h^4 + \epsilon^2), \ x \in I_s \cup [s, 0.5],$$

where

(12) $$u(x) = \begin{cases} v_i & \text{for } x = x_i \in I_s, \\ u_0(x) & \text{for } x \in [s, 0.5] \end{cases}$$

For the mesh generating function on $[0.5, 1]$ we can take $\mu(t) = 1 - \lambda(1 - t)$, $t \in [0.5, 1]$. In this case one can obtain so in a similar way as s.

Our numerical results are obtained by solving boundary value problems which were considered in many papers: [2], [5-11], [15-17], [20], [24], [27-28]. These results show that the theoretical order of convergence is also established numerically.

2. The numerical method

From now on we shall take ϵ and a such that (8) holds. Our discretization mesh is of the form (6) with the mesh generating function λ given by (7).

In order to form a discretization of the problem (4) we approximate the differential equation of (4) by the following difference formulas:

$$
\begin{aligned}
F_1 v_h &:= -\epsilon^2(A_1(1)v_0 + A_0(1)v_1 + A_2(1)v_2) + c(x_1, v_1) = 0,\\
F_i v_h &:= -\epsilon^2(a_3(i)v_{i-2} + a_1(i)v_{i-1} + a_0(i)v_i + a_2(i)v_{i+1} + a_4(i)v_{i+2})\\
&\quad + C(c_i, v_i) = 0, \quad i = 2, 3, \ldots, m-1, \qquad (13)\\
F_m v_h &:= -\epsilon^2(A_1(m)v_{m-1} + A_0(m)v_m + A_2(m)v_{m+1}) + C(x_m, v_m) = 0,
\end{aligned}
$$

where

$$A_1(i) = \frac{2}{b_1,(b_1 - b_2)}, \quad A_0(i) + \frac{2}{b_1 b_2}, \quad A_2(i) = \frac{2}{b_2(b_2 - b_1)}$$

$$a_1(i) = \frac{-2(b_2b_3 + b_2b_4 + b_3b_4)}{b_1(b_2 - b_1)(b_3 - b_1)(b_4 - b_1)},$$

$$a_2(i) = \frac{-2(b_1b_3 + b_1b_4 + b_3b_4)}{b_2(b_1 - b_2)(b_3 - b_2)(b_4 - b_2)},$$

$$a_3(i) = \frac{-2(b_1b_2 + b_1b_4 + b_2b_4)}{b_3(b_1 - b_3)(b_2 - b_3)(b_4 - b_3)},$$

$$a_4(i) = \frac{-2(b_1b_2 + b_1b_3 + b_2b_3)}{b_4(b_1 - b_4)(b_2 - b_4)(b_3 - b_4)},$$

$$a_0(i) = \frac{+2(b_1b_2 + b_1b_3 + b_1b_4 + b_2b_3 + b_2b_4 + b_3b_4)}{b_1b_2b_3b_4},$$

with

$$b_1 = x_{i-1} - x_i, \quad b_2 = x_{i+1} - x_i,$$

$$b_3 = x_{i-2} - x_i, \quad b_4 = x_{i+2} - x_i.$$

Let us define

$$d_j(i) = -\epsilon^2 a_j(i), \; j = 0, 1, 2, 3, 4, \; i = 2, 3, \ldots, m-1,$$

$$D_j(i) = -\epsilon^2 A_j(i), \quad j = 0, 1, 2, \quad i = 1, \ldots, m.$$

One can obtain

$$d_1(i) = \frac{4(-3h - hi + q)(h - hi + q)^4}{3a^2h^2q^2(q - hi)},$$

$$d_2(i) = \frac{4(-h - hi + q)^4(3h - hi + q)}{3a^2h^2q^2(q - hi)},$$

$$d_3(i) + \frac{(6h + hi - q)(2h - hi + q)^4}{12a^2h^2q^2(-(hi) + q)},$$

$$d_4(i) = \frac{(-6h + hi - q)(-2h - hi + q)^4}{12a^2h^2q^2(-(hi) + q)},$$

$$d_0(i) = \frac{(q - ih)^2(24h^2 - 5h^2i^2 + 10hiq - 5q^2)}{2a^2h^2q^2},$$

$$D_1(i) = \frac{-(h + hi - q)(-hi) + q)(h - hi + q)^2}{a^2h^2q^2},$$

$$D_2(i) = \frac{(hi - q)(-h + hi - q)(h + hi - q)^2}{a^2h^2q^2},$$

$$D_0(i) = \frac{2(hi - q)^2(h + hi - q)(h - hi + q)}{a^2h^2q^2}.$$

It is easy to see that for $i = 1, 2, ..., p$, where $p = n - 7$, it holds

$$d_4(i > 0,\ d_3(i) > 0,\ d_0(i) > 0,$$

$$d_1(i) < 0,\ \ d_2(i) < 0,$$

$$D_1(i) < 0,\ \ D_0(i) > 0,\ \ D_2(i) < 0.$$

Let

$$\phi(i) = \frac{d_2(i)D_2(i+1)}{4d_4(i)} - D_0(i+1).$$

The value s we choose as $s = x_m$, where $m \in \{2, 3, ..., p\}$ is determined so that

(14) $$\phi(m) < \Gamma \leq \phi(m-1).$$

Since $\phi(i)$ is decreasing function in i, we can obtain m if $\Gamma < \phi(2)$. It we consider $\phi(2)$ as a function of n, then $\phi(2)$ is increasing. So, for sufficiently large n holds $\Gamma < \phi(2)$.

If $\Gamma > \phi(i)$ for all $i = 2, 3, ..., p$, our method can not be applied. If $\Gamma \leq \phi(i)$, $i = 2, 3, ..., p$ then we define $m = p$.

Now, using (13) and

$$f_0 v_h := v_0 = 0,$$

$$F_{m+1}v_h := v_{m+1} - u_0(x_{m+1}) = 0,$$

we form discrete analogoue of problem (1):

$$F_i v_h = 0, \quad i = 0, 1, ..., m+1. \tag{15}$$

The system (15) can be writen in the form $Fv_h = d$, where $F = (F_0, F_1, ..., F_{m+1})$, $d = [0, ..., 0, u_0(x_{m+1})]^T$.

Theorem 1. *Suppose that the conditions (2) are satisfied. Then the equation $Fv_h = d$ has a unique solution $v_j = [v_0, v_1, ..., v_{m+1}]^T$ which is point of atraction of Newton method.*

Proof. The Fresheet - derivative $F'(z)$ of F for arbitrary $z = [z_0, z_1, ..., z_{m+1}]^T$ is a five-diagonal matrix $F'(z)$ of the following form

$$\begin{bmatrix}
1 & & & & & \\
D_1(1) & D_0(1)+c_1 & D_2(1) & & & \\
d_3(2) & d_1(2) & d_0(2)+c_2 & d_2(2) & d_4(2) & \\
 & d_3(3) & d_1(3) & d_0(3)+c_3 & d_2(3) & d_4(3) \\
 & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & d_3(m-1) & d_1(m-1) & d_0(m-1)+c_{m-1} & d_2(m-1) & d_4(m-1) \\
 & & & D_1(m) & D_0(m)+c_m & D_2(m) \\
 & & & & & 1
\end{bmatrix}$$

where $c_i = c_u(x_i, z_i)$, $i = 2, 3, ..., m$. For the coefficients of this matrix it holds

$$d_0(i) + c_i > 0,\ d_1(i) < 0,\ d_2(i) < 0,\ d_3(i) \geq 0,\ d_4(i) > 0$$

$$D_0(i) + c_i > 0,\ D_1(i) < 0.\ D_2(i) < 0, \quad i = 1, ..., m.$$

To prove $F'(z)$ is inverse monotone, i.e. $(F'(x))^{-1} \geq 0$, we shall apply $ML-$criterion, see [4]. It is sufficient to prove

$$d_2(i)d_2(i+1) \geq 4(d_0(i+1) + c_{i+1})d_4(i), i = 2, ..., m-2, \tag{16}$$

$$d_2(m-1)D_2(m) \geq 4(D_0(m) + c_m)d_4(m-1), \tag{17}$$

$$(18)\qquad d_2(m-1)D_2(m) \geq 4(D_0(m)+c_m)d_4(m-1),$$

$$(19)\qquad D_1(1)d_1(2) \geq 4(D_0(1)+c_1)d_3(2),$$

$$(20)\qquad d_1(i-1)d_1(i) \geq 4(d_0(i-1)+c_{i-1})d_3(i),\ i=3,...,m-1.$$

Obviously, (17) is equivalent to

$$\phi(i) \geq c_{i+1},\ \ i=2,3,...,m-1.$$

Since $\Gamma > c_{i+1}, i = 1,2,...,m+1$, from (14) it follows $\phi_i \geq c_{i+1},\ i = 2,3,...,m-1$. Similarly, from

$$\psi(i) \geq \Gamma,\ \ i=3,...,m-1,$$

$$H(1) \geq \Gamma,$$

$$H(i) \geq \Gamma,\ \ i=2,...,m-2,$$

where

$$\psi(i) = \frac{d_1(i-1)d_1(i)}{4d_3(i)} - d_0(i-1),$$

$$H(1) = \frac{D_1(1)d_1(2)}{4d_3(2)} - D_0(1),$$

$$H(i) = \frac{d_2(i)d_2(i+1)}{4d_4(i)} - d_0(i+1),$$

we obtain (17), (18) and (19).

After some elementary calculation one can obtain that

$$\phi(i) < H(i) < \psi(i),\ \ i=2,3,...,m-1,$$

and

$$\phi(i) < H(i),\ \ i=2,3,...,m-1.$$

So, if m is determined from (14) we have (16)-(19) and $(F'(z))^{-1} \geq 0$.

Let $F'_\gamma(z)$ be a matrix which is obtained from $F'(z)$ if we replace c_i with γ^2. Thus matrix is also inverse monotone and it holds $F'(z) \geq F'_\gamma(z)$. It is easily seen that

$$(F'_\gamma(z))^{-1} \geq (F'(z))^{-1} \geq 0,$$

$$F'_\gamma(z)\delta \geq M^{-1}\delta,\ \ \delta = [1,1,...,1]^T,\ M^{-1} = \min\{1,\gamma^2\},$$

$$\|F'(z)^{-1}\|_{\alpha} \leq \|F'_\gamma(z)^{-1}\|_\infty = M.$$

Now, by Hodamard Theorem [21], it follows that the equation $Fv_h = d$ has a unique solution v_h. Convergence of the Newton method follows by well known theorems from [4], [21]. □

Theorem 2. *Suppose the conditions (2) are satisfied. Let u_ϵ be the solution to problem (1) and let u be defined by (12). Then for sufficiently large n holds*

$$| u_\epsilon(x) - u(x) | \leq M(h^4 + \epsilon^2),\ x \in I_s \cup [s, 0.5].$$

Proof. For $x \in [s, 0.5]$ from (11) it follows

$$| u_\epsilon(x) - u(x) | \leq M(\exp(-s\gamma/\epsilon) + \epsilon^2).$$

For $x = x_i \in I_s$ we have

$$| u_\epsilon(x) - u(x) | \leq | u_\epsilon(x) - v(x) | + | v(x) - v_i | . \tag{21}$$

From (9) it follows

$$| u_\epsilon(x) - u(x) | \leq M(\exp(-\gamma s/\epsilon) + \epsilon^2) + | v(x) - v_i | .$$

We see that for proof of theorem it is sufficient to prove

$$\exp(-\gamma s/\epsilon) \leq M\epsilon^2 \tag{22}$$

and

$$| v(x) - v_i | \leq Mh^4. \tag{23}$$

The inequality (22) can be proved by using technique from [27], [28]. We note here only that the truncation error of our discretization can be written as

$$r_1 = \frac{(x_2 - x_1) - (x_1 - x_0)}{3} v^{(3)}(x_1) + \frac{1}{12(x_2 - x_0)}[(x_1 - x_0)^3 v^{(4)}(\alpha) + (x_2 - x_1)^3 v^{(4)}(\beta)],$$

$$\alpha, \beta \in (x_0, x_2),$$

$$r_i = \frac{v^{(5)}(x_i)}{120} \sum_{p=1}^{4} a_p(i) b_p^5 + \frac{1}{720} \sum_{p=1}^{4} a_p(i) b_p^6 v^{(6)}(\tau_i),$$

$$\tau_i \in (x_{i-1}, x_{i+2}),\ i = 2, 3, ..., p,$$

where b_p depends on i, as we have seen. The truncation error r_1 is $\mathcal{O}(h^2)$, but using technique from [4] we can transform $F_1 v_h$ so that corresponding error is $\mathcal{O}(h^4)$.

So, to finish the proof we have to prove (21). To obtain $s = \lambda(mh)$ we solve equation $\Gamma = \phi(i)$ in i. Let ρ be solution , e.i. $\Gamma = \phi(\rho)$. If ρ is not integer we define $m = [\rho]$, and if ρ is integer we define $m = \rho - 1$. Since $\phi(i)$ is decreasing function in i, (14) is satisfied. The equation $\Gamma = \phi(i)$ is equivalent with

$$2(h + hi - q)(q - hi)^2(h - hi + q)$$
$$((-7h + hi - q)(-h - hi + q)^3 + 2(q - hi)^3(4h - hi + q))$$
$$-a^2h^2(-7h + hi - q)q^2(-h - hi + q)^3 = 0.$$

Since $h = \frac{1}{2n}$, with $s = ih$ one can obtain the following equation

$$p_n(s) = 2(q - s)^2(\frac{1}{2n} + q - s)(\frac{1}{2n} - q + s)$$
$$(2(q - s)^3(\frac{2}{n} + q - s) + (\frac{-1}{2n} + q - s)^3(\frac{-7}{2n} - q + s)) -$$
$$\frac{a^2q^2(\frac{-1}{2n} + q - s)^3(\frac{-7}{2n} - q + s)y}{4n^2} = 0.$$

It is easy to see that

$$\lim_{n\to\infty} p_n(s) = -2(q - s)^8.$$

The solution of $(q - s)^8 = 0$ is $s = q$. Let we denote by $s_k(n), k = 1, 2, ..., 8$, solutions of $p_n(s) = 0$. Then, see [29],

$$\lim_{n\to\infty} s_k(n) = q,\quad k = 1, 2, ..., 8.$$

Using this we conclude that $\rho h \to q$. Now, from $\rho - 1 \le m < \alpha$ it follows

$$mh \to q, ;\quad \text{for} \quad n \to \infty.$$

Now, we have $q - mh > 0$ and

$$\exp(-\gamma s/\epsilon) = \exp(-\gamma\frac{amh}{q - mh}) \le M\epsilon^2$$

for n enough big, since $q - mh \to 0$. □

3. The numerical example

In this section we present the results of numerical experiments using the scheme described in previous section. Our example

$$-\epsilon^2 u'' + u + \cos^2(\pi x) + 2(\epsilon\pi)^2 \cos(2\pi x) = 0,$$

$$u(0) = u(1) = 0,$$

is often used in the literature , [6-9], [11], [15-17], [24], [27-28], to compare different codes. We also give the numerical validation of the theoretical order of convergence for the scheme discused in section 2.

The solution of our problem is

$$u_\epsilon(x) = \frac{\exp(-x/\epsilon) + \exp(-(1-x)/\epsilon)}{1 + \exp(-1/\epsilon)} - \cos^2(\pi x),$$

with $\gamma = 1$ and $r = 1$.

We denote by E_n the maximum of $| u_\epsilon(x) - u(x) |$, $x \in I_h$, i.e.

$$E_n = \max\{| u_\epsilon(x) - u(x) |: \;\; i = 0, 1, ..., n\}.$$

Also, we define in the usual way the order of convergence Ord for two successive values of n with respective errors E_n and E_{2n} :

$$Ord = \frac{\log E_n - \log E_{2n}}{\log 2}.$$

We expect that $Ord = 4$ for small ϵ. In our calculation $a = 1$. Table 1 contains m, E_n and Ord for $n = 2^k$, $k = 4, 5, ..., q$. For ϵ we take 2^{-8} and $\epsilon \in [2^{-64}, 2^{-16}]$.

$n\backslash\epsilon$	2^{-8}	2^{-16}	2^{-64}	
16	9 1.954 (-1) -		9 1.889 (-1) -	m E_n Ord
32	25 1.561 (-2) 3.646		25 1.313 (-2) 3.846	
64	57 1.333 (-4) 6.872		57 6.356 (-5) 7.691	
128	120 1.240 (-6) 6.748		119 1.279 (-6) 5.636	
256	245 7.759 (-8) 3.998		243 8.003 (-8) 3.998	
512	497 4.852 (-9) 3.9995		493 5.004 (-9) 3.9994	

Table 1

In the Table 2 we present $s_k(n)$. It is obviously that $\lim_{n\to\infty} s_k(n) = q$.

$n\backslash\epsilon$	2^{-8}	2^{-16}
16	0.43682	0.43353
64	0.45710	0.45342
256	0.47855	0.47477
1024	0.49076	0.48692
8192	0.49928	0.49486
65536	0.503908	0.500016
524288	0.503907	0.500016
2097152	0.503906	0.500015
q	0.503906	0.500015 25

Table 2

References

[1] Bakhvalov,N.S.: K optimizacii metodov resheniya kraevykh zadach pri nalichii pogranichnogo sloya, Zh. vychisl. mat. i mat. fiz. 9 (4), 841-859 (1969).

[2] Boglaev,I.P.: Variacionno - raznostnaya skhema dlya kraevoi zadachi s malym parametrom pri starshei proizvodnoi, Zh. vychisl. mat. i mat. fiz. 21 (4), 887-896 (1981).

[3] Boglaev,I.P.: Priblizhennoe reshenie nelineinoi kraevoi zadachi s malym parametrom pri starshei proizvodnoi, Zh. vychisl. mat. i mat. fiz. 24 (11), 1649-1656 (1984).

[4] Bohl,E., Lorenz,J.: Inverse monotonicity and difference schemes of higher order. A summary for two-point boundary valu problems, Aeg. Math. 19, 1-36 (1979).

[5] Bohl,E.: Finite Modelle gewöhnlicher Randwertaufgaben. Stuttgart: Teubner 1981.

[6] Cash,J.R.: Adaptive Runge-Kutta methods for nonlinear two-point boundary value problems with mild boundary layers. Comp. & Maths. with Appls. Vol. 11 (8), 605-619 (1985).

[7] Deuflhard,P., Bader,G.: Multiple shooting techniques revised. Preprint No. 163, Univ. Heidelberg, Inst. Für Angewandte Informatik (1982).

[8] Dieckhoff,H.-J., Lory,P., Oberle,H.J., Pesch,H.J., Rentrop,P., Seidel,R.: Comparing routines for the numerical solution of initial value problems of ordinary differential equations in multiple shooting. TU Münhen, Inst. für Math. Report 7520 (1975).

[9] Doolan,E.P., Miller,J,J,H., Schilders,W.H.A.: Uniform numerical methods for problems with initial and boundary layers. Dublin: Boole Press 1980.

[10] Hemker,P.W.: A numerical study of stiff two-point boundary problems. Mathematical Centre Tracts 80, Mathematisch Centrum Amsterdam (1977).

[11] Hemker,P.W., Schippers,H., de Zeeuw,P.M.: Comparing some aspect of two codes for two-point boundary value problems. Mathematisch Centrum Amsterdam, Report NW 98 / 80 (1980).

[12] Herceg,D.: A uniformly convergent scheme with quasiconstant fitting factors, Zb. rad. Prir. - mat. Fak. Univ. u Novom Sadu, Ser. Mat. 11, 105-115 (1981).

[13] Herceg,D., Vulanović,R.: Some finite-difference schemes for a singular perturbation problem on a non-uniform mesch, Zb. rad. Prir. - mat. Fak. Univ. u Novom Sadu, Ser. Mat. 11, 117-134 (1981).

[14] Herceg,D.: On numerical solution of singularly perturbed boundary value problem. In: Conference on Applied Mathematics (Z.Bohte, ed.), 59-66. Ljubljana: University of Ljubljana, Institute of Mathematics, Physics and Mechanics 1986.

[15] Hermann,M., Berndt,H.: RWPM, a multiple shooting code for nonlinear two-point boundary value problems: Version 4, part I-II. Preprint Nr. 67, 68, 69, Fr. -Schiller - Univ. Jena (1982).

[16] Lentini,M., Osborne,M.R., Russel,R.D.: The close relationships between methods for solving two-point boundary value problems. SIAM J. Numer. Anal. 22, 280-309 (1985).

[17] Lentini,M., Pereyra,V.: An adaptive finite difference solver for nonlinear two-point boundary problems with mild boundary layers. SIAM J. Numer. Anal. 14, 91-111 (1977).

[18] Lorenz,J.: Combinations of initial and boundary value methods for a class of singular perturbation problems. In: Numerical Analysis of Singular Perturbation Problems (P.W. Hemker, J.J.H. Miller, eds.). London: Academic Press 1979.

[19] Lorenz,J.: Zur Theorie and Numerik von Differenzenverfahren für singuläre Störungen. Habilitationsschritft, Konstanz (1980).

[20] Marchuk,G.O., Schaidurov,V.V.: Povishenie tochnosti reshenii raznostnykh skhem. Moskva: Nauka 1979.

[21] Ortega,J.M., Rheinboldt,W.C.: Iterative solution of nonlinear equations in several variables. New York and London. Academic Press 1970.

[22] Pearson,C.E.: On a differential equation of boundary layer type. J. Math. Phys. 47, 134-154 (1968).

[23] Shishkin,G.I.: Raznostnaya skhema na neravnomernoi setke dlya differencial'nogo uravneniya s malym parametrom pri starshei proizvodnoi, Zh. vychisl. mat. i mat. fiz. 23 (3), 609-619 (1983).

[24] Stoer,J., Bulirsch,R.: Einführung in die Numerische Mathematik II. Berlin - Heidelberg - New York: Springer 1973.

[25] Vulanović,R.: On a numerical solution of a type of singularly perturbed boundary value problem by using a special discretization mesh. Zb. rad. Prir. - mat. Fak. Univ. u Novom Sadu, Ser. Mat. 13, 187-201 (1983).

[26] Vulanović,R.: Exponential fitting and special mesches for solving singularly perturbed problems. In IV Conference on Applied Mathematics (B. Vrdoljak, ed.), 59-64. Split: Faculty of Civil Engineering 1985.

[27] Vulanović,R., Herceg,D., Petrović,N.: On the extrapolation for a singularly perturbed boundary value problem. Computing 36, 1-2, 69-79 (1986).

[28] Vulanović,R.: Mesh construction for discretization of singularly perturbed boundary value problems. Doctoral Dissertation, Faculty of Sciences, University of Novi Sad (1986).

[29] Voevodin,V.V.: Vychislitel'nye osnovy lineinoi algebri, Nauka, Moskva, 1977.

REZIME

O JEDNOJ DIFERENCNOJ ŠEMI ZA SINGULARNO PERTURBOVANE KONTURNE PROBLEME

Posmatra se numeričko rešavanje nelinearnog singularno perturbovanog konturnog problema pomoću kombinacije rešenja odgovarajućeg redukovanog problema i numeričkog rešenja polaznog problema na onom delu intervala koji sadrži granični sloj. Pri tom se za aproksimaciju diferencijalne jednačine koriste diferencne formule četvrtog reda na specijalnoj neekvidistantnoj mreži. Numerički primeri ilistruju efikasnost predloženog postupka i potvrdjuju teoretske rezultate.

Received by the editors June 15, 1990