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GENERALIZED CONTRACTIONS FOR MULTIVALUED MAPPINGS IN PROBABILISTIC METRIC SPACES

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Abstract

In this paper a generalization of Hicks result from [3] for multivalued mappings is proved.

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1. Introduction

There are many fixed point theorems for singlevalued and multivalued mappings in probabilistic metric spaces([1],[2],[3],[4],[5],[7],[8]). In [3] T.Hicks proved a fixed point theorem for the C-contractions.

Definition 1. *Let $(S,\mathcal{F})$ be a probabilistic semimetric space and $f : S \to S$. The mapping f is a C-contraction if and only if there is a $k \in (0,1)$ such that for every $p, q \in S$ and $x > 0$ the following implication holds:*

$$F_{p,q}(x) > 1 - x \Rightarrow F_{fp,fq}(kx) > 1 - kx.$$

If $(S,\mathcal{F},t)$ is a probabilistic metric space with $t \geq t_m$, where $t_m(x,y) = \max\{x+y-1,0\}$, then V.Radu proved that f is a $C-$ contraction if and only if f is a metric contraction on the metric space (S,β). Here β is defined by

$$\beta(p,q) = \inf\{h; F_{p,q}(h^+) > 1-h\}.$$

Hence, if $t \geq t_m$ and $(S,\mathcal{F},t)$ is a complete Menger space the existence of a fixed point of a $C-$ contraction follows from the Banach fixed point theorem .

V.Radu proved also the existence of a fixed point of a $C-$contraction $f : S \to S$ if $(S,\mathcal{F},t)$ is a complete Menger space and $T-$ norm t is such that $\sup_{a<1} t(a,a) = 1$.

In this paper we shall prove a multivalued version of the fixed point theorem of Radu, using some additional conditions for the mapping f.

2. Preliminaries

Definition 2. *Let $(S,\mathcal{F})$ be a probabilistic semimetric space and $f : S \to \mathcal{P}(S)\backslash\emptyset$. The mapping f is weakly demicompact if and only if for every sequence $\{x_n\}_{n\in\mathrm{N}}$ from S such that $x_{n+1} \in fx_n (n \in \mathrm{N})$ and $\lim_{n\to\infty} F_{x_{n+1},x_n}(u) = 1$, for every $u > 0$, there exists a convergent subsequence $\{x_{n_k}\}_{k\in\mathrm{N}}$.*

If t is a $T-$ norm and $x \in [0,1]$ then $T_p(x)$ is defined by the relations:

$$T_1(x) = t(x,x), T_p(x) = t(T_{p-1}(x),x), p \geq 2.$$

A nontrivial example of a $T-$norm t such that the family of mappings $\{T_p(x)\}_{p\in\mathrm{N}}$ is equicontinuous at the point $x=1$ can be constructed easily. For example, let t_1 be a continuous $T-$ norm and let for every $m \in \mathrm{N}\cup\{0\}$, $I_m = [1-2^{-m}, 1-2^{-m-1}]$. Further, let for $(x,y) \in I_m \times I_m$

$$t_2(x,y) = 1-2^{-m} + 2^{-m-1} t_1(2^{m+1}(x-1+2^{-m}), 2^{m+1}(y-1+2^{-m}))$$

and for $(x,y) \notin \cup_{m\in\mathrm{N}\cup\{0\}} I_m \times I_m$, $t_2(x,y) = \min\{x,y\}$.

It is easy to see that the family $\{T_p(x)\}_{p\in\mathrm{N}}$ is equicontinuous at the point $x=1$ for $t=t_2$.

It is well known that every $B-$ contraction (in the sense of Sehgal and Bharucha-Reid) $f : S \to S$ has a fixed point if the family $\{T_p(x)\}_{p\in\mathbb{N}}$ is equicontinuous at the point $x = 1$.

3. Fixed point theorem

Theorem 1. *Let $(S, \mathcal{F}, t)$ be a complete Menger space, t a continuous $T-$norm, M a closed subset of S, $f : M \to \mathrm{Cl}(M)\backslash\emptyset$ a closed mapping such that the following condition holds:*
There exists $k \in (0,1)$ such that for every $p, q \in M$ and every $x > 0$:

(1) $F_{p,q}(x) > 1 - x \Rightarrow$ *for every $u \in fp$ there exists $v \in fq$ such that*

$$F_{u,v}(kx) > 1 - kx.$$

If one of the conditions (i) and (ii) holds then there exists $x \in M$ such that $x \in fx$ where

(i) f is weakly demicompact.

(ii) The family $\{T_p(x)\}_{p\in\mathbb{N}}$ is equicontinuous at the point $x = 1$.

Proof. Let $x_0 \in M$ and $x_1 \in fx_0$. If $h > 1$ then we have that $F_{x_1,x_0}(h) > 1-h$, since $F_{x_1,x_0}(h) \geq 0$. From (1) , taking that $p = x_0, q = x_1$ and $u = x_1 \in fx_0$, we obtain that there exists $v = x_2 \in fx_1$ such that $F_{x_2,x_1}(kh) > 1-kh$.

If we proceed in this way we obtain that there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ from M such that

(2) $$F_{x_{n+1},x_n}(k^n h) > 1 - k^n h, \text{for every } n \in \mathbb{N}.$$

From (2) it follows that for every $r > 0$ and $s \in (0,1)$ there exists $n_0(r,s) \in \mathbb{N}$ such that for every $n \geq n_0(r,s)$

(3) $$F_{x_{n+1},x_n}(r) > 1 - s.$$

Indeed, since $k < 1$ there exists $n_0(r,s) \in \mathbb{N}$ such that $k^n h \leq r$ and $k^n h \leq s$, for every $n \geq n_0(r,s)$ and so

$$F_{x_{n+1},x_n}(r) \geq F_{x_{n+1},x_n}(k^n h) > 1 - k^n h > 1 - s.$$

Inequality (3) implies that

$$\lim_{n\to\infty} F_{x_{n+1},x_n}(r) = 1, \text{for every } r > 0. \tag{4}$$

If f is weakly demicompact from (4) it follows that there exists a convergent subsequence $\{x_{n_k}\}_{k\in\mathbf{N}}$ of the sequence $\{x_n\}_{n\in\mathbf{N}}$. We shall prove that if (ii) is satisfied then we can take that $\{x_n\}_{n\in\mathbf{N}} = \{x_{n_k}\}_{k\in\mathbf{N}}$ is convergent.

First, we shall prove that $\{x_n\}_{n\in\mathbf{N}}$ is a Cauchy sequence, which means that for every $r > 0$ and $s \in (0,1)$ there exists $n_1(r,s) \in \mathbf{N}$ such that

$$F_{x_{n+p},x_n}(r) > 1 - s, \text{for every } n \geq n_1(r,s) \text{ and } p \in \mathbf{N}.$$

Since the inequality

$$1 > \frac{1}{2} + \frac{1}{6} + \dots + \frac{1}{(p+1)(p+2)}$$

holds for every $p \in \mathbf{N}$ it follows that for every $r > 0$, every $n \in \mathbf{N}$ and every $p \in \mathbf{N}$ we have that

$$F_{x_{n+p+1},x_n}(r) \geq t(F_{x_{n+p+1},x_{n+1}}(\sum_{i=2}^{p+1} \frac{r}{i(i+1)}),$$

$$F_{x_{n+1},x_n}(\frac{r}{2})) \geq \dots \geq t(t(\dots t(F_{x_{n+p+1},x_{n+p}}(\frac{r}{(p+1)(p+2)}),$$

$$F_{x_{n+p},x_{n+p-1}}(\frac{r}{p(p+1)})), \dots, F_{x_{n+2},x_{n+1}}(\frac{r}{6})), F_{x_{n+1},x_n}(\frac{r}{2})).$$

We shall prove that for every $n \geq n(r,h)$ and every $m \in \mathbf{N} \cup \{0\}$

$$\frac{r}{(m+1)(m+2)} \geq k^{n+m}h. \tag{5}$$

Since $\lim_{p\to\infty} \frac{r}{k^p(p+1)(p+2)} = \infty$, for every $r > 0$ there exists $p_0(r,h) \in \mathbf{N}$ such that

$$\frac{r}{k^m(m+1)(m+2)} \geq h, \text{ for every } m \geq p_0(r,h).$$

Then for every $n \in \mathbf{N}$ and every $m \geq p_0(r,h)$

$$\frac{r}{(m+1)(m+2)} \geq k^m h \geq k^{n+m}h.$$

For every $m \in \{0, 1, ..., p_0(r,h) - 1\}$ there exists $n_m(r,h)$ such that

$$k^n \leq \frac{r}{hk^m(m+1)(m+2)},$$

for every $n \geq n_m(r,h)$.

Then for every $n \geq \max\{n_m(r,h); 0 \leq m \leq p_0(r,h) - 1\} = n(r,h)$ we have that $k^n \leq \frac{r}{hk^m(m+1)(m+2)}$, for every $m \in \{0, 1, ..., p_0(r,h) - 1\}$ and so $\frac{r}{(m+1)(m+2)} \geq k^{n+m}h$, for every $n \geq n(r,h)$ and every $m \in \mathbf{N}$.

If $n \geq n(r,h)$ we have that

$$F_{x_{n+p+1},x_n}(r) \geq t(t(...t(F_{x_{n+p+1},x_{n+p}}(k^{n+p}h),$$

$$F_{x_{n+p},x_{n+p-1}}(k^{n+p-1}h)), ...), F_{x_{n+2},x_{n+1}}(k^{n+1}h)), F_{x_{n+1},x_x}(k^n h)).$$

From this inequality we obtain for $n \geq \max\{n(h), n(r,h)\}$ and $p \in \mathbf{N}$ that

$$F_{x_{n+p+1},x_n}(r) \geq t(t(...t(1 - k^{n+p}h, 1 - k^{n+p-1}h), ...,$$

$$1 - k^n h)),$$

where $n(h)$ is such that $k^n h < 1$, for $n \geq n(h)$. Hence

$$F_{x_{n+p+1},x_n}(r) \geq T_p(1 - k^n h)$$

for every $p \in \mathbf{N}$ and $n \geq \max\{n(h), n(r,h)\}$.

Since the family $\{T_p(x)\}_{p \in \mathbf{N}}$ is equicontinuous at the point $x = 1$ for every $s \in (0,1)$ there exists $\delta(s) \in (0,1)$, such that $T_p(1 - \delta(s)) > 1 - s$, for every $p \in \mathbf{N}$. Hence, if $k^n h < \delta(s)$ for $n \geq n'(h,s)$ we have for

$$n \geq \max\{n(r,h), n(h), n'(h,s)\} = n(r,h,s)$$

that

$$F_{x_{n+p+1},x_n}(r) > 1 - s, \text{ for every } p \in \mathbf{N}.$$

This means that $\{x_n\}_{n \in \mathbf{N}}$ is a Cauchy sequence and since S is complete it follows that there exists $\lim_{n \to \infty} x_n = z$.

It remains to be proved that if $\lim_{k \to \infty} x_{n_k} = u$ then $u \in fu$.

We have from (4) that $\lim_{k \to \infty} x_{n_k + 1} = u$ and since $x_{n_k} \in f x_{n_k+1} (k \in \mathbf{N})$ and f is closed we conclude that $u \in fu$.

Remark: If $t(a,b) \geq ab$, for every $a, b \in [0,1]$ then for every $n \geq \max\{n(h), n(r,h)\}$ we obtain that

$$F_{x_{n+p+1}, x_n}(r) \geq \prod_{m=n}^{\infty} (1 - k^m h)$$

which implies that

$$F_{z, x_n}(r) \geq \prod_{m=n}^{\infty} (1 - k^m h).$$

This gives an exit criteria for the sequence $\{x_n\}_{n \in \mathbb{N}}$ since $F_{z,x_n}(r) > 1 - s$ if $\prod_{m=n}^{\infty}(1 - k^m h) > 1 - s$ ($n \geq \max\{n(h), n(r,h)\}$).

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REZIME

UOPŠTENA KONTRAKCIJA ZA VIŠEZNAČNA PRESLIKAVANJA U VEROVATNOSNIM METRIČKIM PROSTORIMA

U ovom radu dokazano je uopštenje rezultata Hicksa iz [3] za višeznačna preslikavanja.

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