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THE APPROXIMATE SOLUTION OF A NONHOMOGENEOUS PARTIAL DIFFERENTIAL-DIFFERENCE EQUATION

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Abstract

In the paper the approximate solution of a partial differential-difference equation is constructed using the two dimensional operational calculus introduced by T. Ogata ([4]) and the results of J. Wloka [8] for ordinary differential-difference equations.

In section 4, an estimate is given for the error of approximation in the space $\mathcal{F}_0$ of the Mikusiński operator field $\mathcal{F}$ and also obtained is that the sequence of approximate solutions $\{x_n\}$ converges to the exact solution in the convergence type I'(observed by T. Boehme [1] and J.Burzyk [2]). If the exact and the approximate solutions belong to the space $\mathcal{L}$ of locally integrable functions, then it is obtained that the sequence of the approximate solutions converges to the exact solution in $\mathcal{L}$.

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1. The form of the exact solution

We consider the linear partial nonhomogeneous differential- difference equation with constant coefficients

$$(1)\qquad \sum_{\mu=0}^{m}\sum_{\nu=0}^{n}\alpha_{\mu,\nu}\frac{\partial^{\mu+\nu}x(\lambda,t)}{\partial\lambda^{\mu}\partial t^{\nu}}-\sum_{\mu=0}^{m_1}\sum_{\nu=0}^{n_1}\beta_{\mu,\nu}\frac{\partial^{\mu+\nu}x(\lambda,t-\tau)}{\partial\lambda^{\mu}\partial t^{\nu}}=\phi(\lambda,t),$$

where $0 \leq \lambda \leq \lambda_0$, $0 \leq t < \infty$, $\tau > 0$, $x(\lambda,t) = 0$, $t < 0$, and $\phi(\lambda,t)$ is a continuous function. Equation (1) corresponds in the field of Mikusiński operators, $\mathcal{F}$, to the equation

$$\sum_{\mu=0}^{m} a_\mu(s)x^{(\mu)}(\lambda) - e^{-s\tau}\sum_{\mu=0}^{m_1} b_\mu(s)x^{(\mu)}(\lambda) = f(\lambda), \tag{2}$$

where

$$a_\mu(s) = \sum_{\nu=0}^{n} \alpha_{\mu,\nu}s^\nu, \qquad \mu = 0,1,2,...,m, \tag{3}$$

$$b_\mu(s) = \sum_{\nu=0}^{n_1} \beta_{\mu,\nu}s^\nu, \qquad \mu = 0,1,2,...,m_1, \tag{4}$$

and

$$f(\lambda) = \{\phi(\lambda,t)\} + \sum_{k=0}^{n-1} s^{n-k-1}\sum_{\mu=0}^{m}\sum_{\nu=0}^{k}\alpha_{\mu,n-k+\nu}\frac{\partial^{\mu+\nu}x(\lambda,0)}{\partial\lambda^\mu\partial t^\nu} -$$

$$-e^{-s\tau}\sum_{k=0}^{n_1-1} s^{n_1-k-1}\sum_{\mu=0}^{m_1}\sum_{\nu=0}^{k}\beta_{\mu,n_1-k+\nu}\frac{\partial^{\mu+\nu}x(\lambda,0)}{\partial\lambda^\mu\partial t^\nu}.$$

In the field of Mikusiński operators, $\mathcal{F}$, s is the differential operator, l is the integral operator, it holds $l = s^{-1}$, and $e^{-s\tau}$ is the translation operator defined by

$$e^{-s\tau}f = \left\{ \begin{array}{cc} 0, & t \leq \tau \\ f(t-\tau), & t > \tau \end{array} \right\}.$$

The expressions

$$\frac{\partial^{\mu+\nu}x(\lambda,0)}{\partial\lambda^\mu\partial t^\nu}, \qquad \mu = 0,...M, \quad \nu = 0,...N-1,$$

where $M = \max(m, m_1)$ and $N = \max(n, n_1)$ are given as the appropriate conditions for the equation (1). Let us take the conditions of equation (2) as

$$x^{(\mu)}(0) = d_\mu, \quad \mu = 0,...M-1, \tag{5}$$

(which certainly have the corresponding form as the conditions for equation (1)).

Let $\mathcal{K}$ be the algebraically closed subfield of the field $\mathcal{F}$ which consists of elements of the form

$$p = e^{-s\tau} \sum_{i=i_0}^{\infty} h_i l^{\alpha i - \beta}, \qquad i_0 > -\infty, \quad \tau \geq 0,$$

and $h_i, \quad i = i_0, i_0 + 1, \ldots$, are complex numbers while α and β are rational numbers, such that $\alpha > 0, \quad \beta \leq -1$.

Let $\mathcal{A}$ ([4]) be the module of the formal power series of a variable λ with coefficients in $\mathcal{K}$, with the elements given by

$$P(\lambda) = \sum_{\nu=0}^{\infty} p_\nu \lambda^\nu, \tag{6}$$

where $p_\nu, \quad \nu = 0, 1, 2, \ldots$, belong to $\mathcal{K}$. Multiplication in $\mathcal{A}$ can be defined as

$$PQ(\lambda) = \sum_{\rho=0}^{\infty} \Big(\sum_{\rho=\mu+\nu} \frac{\nu!\mu!}{(\rho+1)!} p_\nu q_\mu \Big) \lambda^{\rho+1}, \tag{7}$$

where $P(\lambda)$ is given by (6) and

$$Q(\lambda) = \sum_{\mu=0}^{\infty} q_\mu \lambda^\mu, \quad q_\mu \in \mathcal{K}.$$

By the usual addition and multiplication given by (7), the set $\mathcal{A}$ forms an integral domain without a unit element.

The quotient field of the ring $\mathcal{A}$ introduced by T. Ogata in [4] is the space of operators $Q(\mathcal{A})$. The following operators will be used mostly in the sequel:

- the integral operators $\mathrm{L} = \{1\}$;
- the differential operators $\mathrm{S} = 1/\mathrm{L}$;
- numerical operators, or $\mathcal{K}$ operators, which are in fact the elements of $\mathcal{K}$.

Similarly, as in the field $\mathcal{F}$, in the field $Q(\lambda)$ we have:

$$a) \quad S^n P = \partial^n P + \partial^{n-1} P(0) + S\partial^{n-2} P(0) + \ldots + S^{n-1} P(0),$$

for $P = \{P(\lambda)\} = \{\sum_{\nu=0}^{\infty} p_\nu \lambda^\nu\}$.

$$b) \qquad \frac{1}{(S-a)^n} = \{\frac{\lambda^{n-1}}{(n-1)!} e^{a\lambda}\}, \quad n = 1, 2, ..., \quad a \in \mathcal{K};$$

c) the rational operator is the fractional expression

$$R(S) = \frac{a_m S^m + a_{m-1} S^{m-1} + ... + a_0}{b_n S^n + b_{n-1} S^{n-1} + ... + b_0},$$

where $a_i, b_j \in \mathcal{K} \quad (i = 1, ..., m \quad j = 1, ...n)$.

If in the last expression it holds that $m < n$, then $R(S)$ belongs to $\mathcal{A}$.

If we suppose that $f(\lambda) \in \mathcal{A}$, then equation (2) with conditions (5) corresponds in the field $Q(\lambda)$ to the equation

$$(8) \qquad X(\sum_{\mu=0}^{m} a_\mu(s) S^\mu - e^{-s\tau} \sum_{\mu=0}^{m_1} b_\mu(s) S^\mu) =$$

$$= \sum_{\mu=0}^{m-1} q_\mu S^\mu - e^{-s\tau} \sum_{\mu=0}^{m_1-1} t_\mu(s) S^\mu + F,$$

where $F = \{f(\lambda)\}$, $a_\mu(s)$, for $\mu = 0, ..., m$, and $b_\mu(s)$, for $\mu = 0, 1, ..., m_1$, are given by relation (3) and (4) respectively, and

$$(9) \qquad q_\mu = a_{\mu+1}(s) d_0 + a_{\mu+2}(s) d_1 + ... + a_m(s) d_{m-\mu-1}$$

for $\mu = 0, 1, ..., m-1$,

$$t_\mu = b_{\mu+1}(s) d_0 + b_{\mu+2}(s) d_1 + ... + b_{m_1}(s) d_{m_1-\mu-1}, \quad \mu = 0, 1, ..., m_1 - 1,$$

(the operators d_μ, $\mu = 0, 1, ..., M-1$, are given by relation (5)).

The solution of equation (8) has the form

$$X = \frac{\sum_{\mu=0}^{m-1} q_\mu(s) S^\mu - e^{-s\tau} \sum_{\mu=0}^{m_1-1} t_\mu(s) S^\mu + F}{\sum_{\mu=0}^{m} a_\mu(s) S^\mu - e^{-s\tau} \sum_{\mu=0}^{m_1} b_\mu(s) S^\mu}.$$

By using the simple-fraction-decomposition method, we obtain

Lemma 1. *The solution of equation (8) can be written as*

$$X = \frac{\sum_{\mu=0}^{m-1} q_\mu(s)S^\mu - e^{-s\tau}\sum_{\mu=0}^{m_1-1} t_\mu(s)S^\mu}{\sum_{\mu=0}^{m} a_\mu(s)S^\mu} + \tag{10}$$

$$+\frac{(\sum_{\mu=0}^{m-1} q_\mu(s)S^\mu)(\sum_{\mu=0}^{m_1} b_\mu(s)S^\mu) - (e^{-s\tau}\sum_{\mu=0}^{m_1-1} t_\mu(s)S^\mu)(\sum_{\mu=0}^{m_1} a_\mu(s)S^\mu)}{(\sum_{\mu=0}^{m} a_\mu(s)S^\mu)^2}\cdot$$

$$\cdot\sum_{k=1}^{\infty}\Big(\frac{\sum_{\mu=0}^{m_1} b_\mu(s)S^\mu}{\sum_{\mu=0}^{m} a_\mu(s)S^\mu}\Big)^{k-1}(e^{-s\tau})^k + \frac{F}{\sum_{\mu=0}^{m} a_\mu(s)S^\mu}\cdot\sum_{k=0}^{\infty}\Big(\frac{\sum_{\mu=0}^{m_1} b_\mu(s)S^\mu}{\sum_{\mu=0}^{m} a_\mu(s)S^\mu}\Big)^{k}(e^{-s\tau})^k.$$

From relation (10) follow Lemma 2 and Lemma 3:

Lemma 2. *The polynomial*

$$A(S) = (\sum_{\mu=0}^{m-1} q_\mu(s)S^\mu)(\sum_{\mu=0}^{m_1} b_\mu(s)S^\mu) - (\sum_{\mu=0}^{m_1-1} t_\mu(s)S^\mu)(\sum_{\mu=0}^{m} a_\mu(s)S^\mu)$$

has the degree less or equal to M, where $M = 2\max(m, m_1) - 1$.

Lemma 3. *If $m_1 < m$, then the solution X, given by (10), of equation (8) belongs to space $\mathcal{A}$ (provided that $f(\lambda)$ belongs to $\mathcal{A}$).*

The proof follows from c).

Since the space $\mathcal{K}$ is algebraically closed, we can write

$$P(S) = \sum_{\mu=0}^{m} a_\mu(s)S^\mu = a_m\prod_{j=0}^{m}(S-\omega_j), \tag{11}$$

where

$$\omega_j = \sum_{i=0}^{\infty} c_{i,j} l^{i\alpha_j - \beta_j}, \tag{12}$$

where α_j, and β_j are rational numbers, $\alpha_j > 0$, $j = 1, 2, ...,$ and $c_{i,j}$ are complex numbers.

In this paper we shall consider those equations in which $\beta_j < -1, j = 1, 2, ..., m$, and additionally, satisfy condition $m_1 \leq m$, because of Lemma 3.

Let us suppose that ω_j, $j = 1, 2, ..., m$, are simple solutions of equation $P(S) = 0$. Then the first term in relation (10) can be decomposed as (see [7]):

$$\frac{\sum_{\mu=0}^{m-1} q_\mu S^\mu}{\sum_{\mu=0}^{m} a_\mu(s) S^\mu} = \tag{13}$$

$$= \sum_{j=1}^{m} \frac{\sum_{\mu=0}^{m-1} q_\mu \omega_j^\mu}{P'(\omega_j)(S-\omega_j)} = \sum_{j=0}^{m} \frac{A_j}{S-\omega_j} = X_1.$$

Similarly as in the previous case, we can decompose the other terms in relation (10), and obtain the following form of equation (8):

$$X = \sum_{j=1}^{m} (\frac{A_j + e^{-s\tau} C_j}{(S-\omega_j)} + \sum_{k=1}^{\infty} \sum_{i=0}^{k-1} (\frac{G_{i,j}}{(S-\omega_j)^i} e^{-sk\tau}) + \tag{14}$$

$$+ F \sum_{k=0}^{\infty} \sum_{i=0}^{k-1} (\frac{D_{i,j}}{(S-\omega_j)^i} e^{-sk\tau}.$$

The coefficients $C_j, G_{i,j}$ and $D_{i,j}$ can be found in a similar way as A_j,

$$A_j = \frac{\sum_{\mu=0}^{m-1} q_\mu \omega_j^\mu}{P'(\omega_j)}, \qquad P'(\omega_j) = \sum_{\mu=1}^{m-1} \mu a_\mu(s) \omega_j^{\mu-1},$$

and we shall omit these somewhat cumbersome expressions.

It can be remarked that $A_j \in \mathcal{K}$, $j = 1, ..., m$, and, also, all coefficients $C_j, G_{i,j}$ and $D_{i,j}$ belong to $\mathcal{K}$.

From relation (14) it follows that the solution of equation (2) with conditions (5) in the space $\mathcal{F}$ can be written in the form

$$x(\lambda) = \sum_{j=1}^{m} e^{\lambda \omega_j} (A_j + e^{-s\tau} C_j + \sum_{k=1}^{\infty} \sum_{i=0}^{k+1} \frac{G_{i,j}}{i!} \lambda^i e^{-sk\tau}) + \tag{15}$$

$$+\int_0^\lambda f(\lambda-u)(\sum_{k=0}^{\infty}(\sum_{i=0}^{k-1}\frac{D_{i,j}}{i!}u^i)e^{-sk\tau}e^{u\omega_j})du.$$

Lemma 4. *If in equation (2) (or (1)) $m_1 < m$, then the equation is logarithmic ([3]) when the equation (its first part)*

$$\sum_{\mu=0}^{m}\sum_{\nu=0}^{n}\alpha_{\mu,\nu}\frac{\partial^{\mu+\nu}x(\lambda,t)}{\partial\lambda^\mu\partial t^\nu}=0 \tag{16}$$

is logarithimic.

Proof. From the form of the exact solution given by relation (15) it follows that there appear only Mikusiński operator functions $e^{u\omega_j}$ for $0 \le u \le \lambda$ and ω_j are, in fact, solutions of the characteristic eqation of the equation

$$\sum_{\mu=0}^{m} a_\mu x^\mu(\lambda)=0, \tag{17}$$

(see [3] page 18). Equation (17), in the field $\mathcal{F}$, corresponds to equation (16).□

2. The form of the approximate solution

Let us consider the approxima solution of the characteristic equation of equation (17) in the space $\mathcal{F}$

$$\omega_{j,n}=\sum_{i=0}^{n}c_{i,j}l^{i\alpha_j-\beta_j}, \tag{18}$$

where $l \in \mathcal{F}$ and α_j, β_j, $j = 1, ...m$, satisfy the same conditions as in (12).

The approximation of the first part of the solution of equation (8) in the space $Q(\mathcal{A})$ can be written as

$$X_{1,n}=\sum_{j=1}^{m}\frac{A_{j,n}}{(S-\omega_{j,n})}=\sum_{j=1}^{m}\frac{\sum_{\mu=0}^{m-1}q_\mu\omega_{j,n}^\mu}{P'(\omega_{j,n})(S-\omega_{j,n})}. \tag{19}$$

Similarly, by replacing ω_j with $\omega_{j,n}$, we obtain the approximate solution of eqution (8) in the space $Q(\mathcal{A})$:

$$(20)\qquad X_n = \sum_{j=1}^{m}\left(\frac{A_{j,n} + e^{-s\tau}C_{j,n}}{(S-\omega_{j,n})} + \sum_{k=1}^{\infty}\sum_{i=0}^{k+1}\left(\frac{G_{i,j,n}}{(S-\omega_{j,n})^i}e^{-sk\tau}\right) + \right.$$

$$\left. + F\sum_{k=0}^{\infty}\sum_{i=0}^{k+1}\left(\frac{D_{i,j,n}}{(S-\omega_{j,n})^i}e^{-sk\tau}\right)\right).$$

The coefficients $C_{j,n}, G_{i,j,n}, D_{i,j,n}$ are also, obtained as an approximations $C_j, G_{i,j}$ and $D_{i,j}$ by replacing ω_j with $\omega_{j,n}$.

In the space of Mikusiński operators the approximate solution of equation (2) with conditions (5) has the form

$$(21)\qquad x_n(\lambda) = \sum_{j=1}^{m}\left(e^{\lambda\omega_{j,n}}(A_{j,n} + e^{-s\tau}C_{j,n} + \sum_{k=1}^{\infty}\sum_{i=0}^{k+1}\frac{G_{i,j,n}}{i!}\lambda^i e^{-sk\tau}\right) +$$

$$+ \int_0^\lambda f(\lambda - u)\sum_{k=0}^{\infty}\sum_{i=0}^{k+1}\frac{D_{i,j,n}}{i!}u^i e^{-sk\tau}e^{u\omega_{j,n}}du.$$

As usual, we denote by $\mathcal{L}$ the space of locally integrable functions on $[0,\infty)$, and by $\mathcal{L}_0$ the subspace of $\mathcal{L}$ which consists of all functions f such that

$$||f||_T = \int_0^T |f(t)|dt > 0,$$

for any $T > 0$. By $\mathcal{F}_0$ we denote the subspace of $\mathcal{F}$, of the elements of the form f/g, where $f \in \mathcal{L}$ and $g \in \mathcal{F}_0$.

The operator $g_k = \frac{kl}{l+k}$ for $k \in \mathbf{N}$ represents a continuous function from $\mathcal{L}_0$. Since, in the most interesting cases of equation (2), it holds that if $g_k x(\lambda) \in \mathcal{L}$ and $g_k x_n(\lambda) \in \mathcal{L}$ (for fixed $\lambda \in [0,\lambda_0]$), hence $x(\lambda) \in \mathcal{F}_0$ and $x_n(\lambda) \in \mathcal{F}_0$, we shall observe only the equations with such properties.

Therefore, we can estimate the error of approximation in the space $\mathcal{F}_0$, and in the space $\mathcal{L}$ if the exact and the approximate solutions belong to $\mathcal{L}$.

3. The measure of approximation

Let us introduce the functional ([5]):

$$A(x) = \sum_{i=0}^{\infty} \frac{B_{i,1/i}(x)}{e^{ie^{i^2}}(1 + B_{i,1/i}(x))}, \quad x \in \mathcal{F}_0, \tag{22}$$

where the functional $B_{T,\epsilon}(x)$ introduced by J. Burzyk in [1], has the form

$$B_{T,\epsilon}(x) = \inf\{||f||_T : x = \frac{f}{g},\ ||g||_T < 1,\ ||l - lg||_T < \epsilon\}, \quad x \in \mathcal{F}_0, \tag{23}$$

As in [5], we have

Definition 1. *The operator $y \in \mathcal{F}_0$ is the approximation of the operator $x \in \mathcal{F}_0$ according to the functional A (given by (22)) with the measure of approximation $\delta > 0$, if $A(x - y) < \delta$.*

The measure of approximation for the approximate solution given by (21) can be obtained by the estimation of the expression $A(x(\lambda) - x_n(\lambda))$, where $x(\lambda)$ is given by 15 and $x_n(\lambda)$ by (21). Therefore, first we have to estimate the functional

$$B_{T,1/k}(x(\lambda) - x_n(\lambda)), \quad k \in \mathbf{N}.$$

The operator $g_k = \frac{lk}{I+k}$ satisfies the inequalities $||g_k||_T < 1$ and $||l - lg_k|| < \frac{1}{k}$, for every $T > 0$.

Using (23) we can write

$$B_{T,1/k}(x(\lambda) - x_n(\lambda)) \leq ||\epsilon g_k(\lambda, t)||_T, \tag{24}$$

where

$$\frac{g_k(x(\lambda) - x_n(\lambda))}{g_k} =: \{\frac{\epsilon g_k(\lambda, t)}{g_k}\}.$$

Lemma 5. *If $x(\lambda)$ and $x_n(\lambda)$ are given by relations (15) and (21) respectively, then for fixed $\lambda = \lambda_1 \in [0, \lambda_0]$ we have*

$$B_{T,1/k}(x(\lambda) - x_n(\lambda)) = \sum_{j=1}^{\infty} \frac{1}{\Gamma(\frac{(n+1)\alpha_j - \beta_j}{2} + 1)} k_{1,j} e^{k_{2,j} e^{k_{3,j}}}, \tag{25}$$

where α_j and β_j appear in (12) and $k_{1,j}, k_{2,j}, k_{3,j}$ do not depend on T.

Proof. Since for the translation operator from $\mathcal{F}$ it holds that

$$e^{-s\tau} = sH_\tau, \quad H_\tau = \begin{cases} 0, & t \leq \tau, \\ 1, & t > \tau, \end{cases}$$

and $le^{s\tau} = H_\tau$, the infinite sums in relations (15) and (21) become finite on every interval $[0,T]$ for $T > 0$ (see [5]). Observe, that $\phi(\lambda, t)$ is a bounded function for $t \in [0,T]$ and $\lambda \in [0, \lambda_0]$.

So, similarly as in papers [5] and [7], we can find $k_{1,j}, k_{2,j}, k_{3,j}$ satisfying relation (25).□

From relation (22) and Lemma 5, one can find Q_j (depending only on $\lambda = \lambda_1 \in [0, \lambda])$ such that

$$A(x(\lambda) - x_n(\lambda)) \leq \sum_{j=1}^{\infty} \frac{1}{\Gamma(\frac{(n+1)\alpha_j - \beta_j}{2} + 1)} Q_j. \tag{26}$$

The convergence defined by the functional $A(x)$ from relation (22) is equovalent to the convergence type I'. Relation (26) implies

Proposition 1. *The sequence of the approximate solutions $\{x_n(\lambda)\}$ of equation (1), given by (21) converges in $\mathcal{F}_0$, when $n \to \infty$ to the exact solution given by (15) in the convergence type I'.*

Using definition (1) we can say that the measure of approximation in $\mathcal{F}_0$ for the approximate solution given by (21) can be expressed by relation (26).

4. The measure of approximation in $\mathcal{L}$

If the exact and approximate solution represent functions from $\mathcal{L}$, for fixed $\lambda = \lambda_1 \in [0, \lambda_0]$, then using the functional

$$F(f) = \sum_{i=0}^{\infty} \frac{||f||_i}{e^{ie^{i^2}}(1 + ||f||_i)}, \quad f \in \mathcal{L}, \tag{27}$$

we can find $Q_{1,j}$, depending on $\lambda = \lambda_1 \in [0, \lambda_0]$, such that

$$F(x(\lambda) - x_n(\lambda)) \leq \sum_{j=1}^{\infty} \frac{1}{\Gamma(\frac{(n+1)\alpha_j - \beta_j}{2} + 1)} Q_{1,j}, \tag{28}$$

where α_j and β_j appeared in relation (12).

The convergence defined by the functional $F(f)$ from (27) is equivalent to the convergence in $\mathcal{L}$. So, from relation (26) follows

Proposition 2. *The sequence of the approximate solutions $\{x_n(\lambda)\}$ of equation (1), given by (21), converges in $\mathcal{L}$ when $n \to \infty$, to the exact solution given by (15).*

The measure of approximation in $\mathcal{L}$ can be expressed by relation (25).

5. Example

Let us consider the differential-difference equation

$$\frac{\partial x(\lambda,t)}{\partial\lambda\partial t} - \frac{\partial x(\lambda,t)}{\partial\lambda} - x(\lambda,t) + x(\lambda,t-1) = 0 \tag{29}$$

with conditions

$$\frac{\partial x(\lambda,0)}{\partial\lambda} = 0, \quad x(\lambda,t) = 0, \quad t \le 0, \tag{30}$$

$$x(0,t) = e^t.$$

In the field of Mikusiński operators, this equation corresponds to the equation

$$(s-1)x'(\lambda) - x(\lambda) + e^{-s}x(\lambda) = 0, \tag{31}$$

with condition

$$x(0) = \frac{1}{s-1}. \tag{32}$$

In the space $Q(\mathcal{A})$, the corresponding equation is

$$(s-1)SX - X - e^{-s}X = 1 \tag{33}$$

The solution of equation (33) can be written as

$$X = \frac{1}{(s-1)S - 1 + e^{-s}} = \frac{1}{s-1}\cdot\frac{1}{S - \frac{1}{s-1}} +$$

$$+\sum_{k=1}^{\infty} \frac{(-1)^k e^{-sk}}{(s-1)^{k+1}(S - \frac{1}{s-1})^{k+1}}.$$

The approximate solution in the space $\mathcal{Q}(\mathcal{A})$ is

$$X_n = (\sum_{i=0}^{n} l^{i+1}) \frac{1}{S - (\sum_{i=0}^{n} l^{i+1})} +$$

$$+ \sum_{k=1}^{\infty} (\sum_{i=0}^{n} l^{i+1})^{k+1} \cdot \frac{(-1)^k}{(S - \sum_{i=0}^{n} l^{i+1})^{k+1}} e^{-sk}).$$

In the space $\mathcal{F}$ the corresponding solution can be written as

$$x_n(\lambda) = (\sum_{i=0}^{n} l^{i+1}) exp(\lambda \sum_{i=0}^{n} l^{i+1}) +$$

$$+ \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} (\sum_{i=0}^{n} l^{i+1})^{k+1} (exp(\lambda) \sum_{i=0}^{n} l^{i+1})) e^{-sk}.$$

The error of appoximation can be estimated as

$$A(x(\lambda) - x_n(\lambda)) \le \frac{Q}{\Gamma(\frac{n+2}{2})},$$

where

$$Q \ge 2e^{e^4} + \frac{1}{e^4 - 1}.$$

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REZIME

PRIBLIŽNO REŠENJE NEHOMOGENE PARCIJALNE DIFERENCIJNO-DIFERENTNE JEDNAČINE

U radu se konstruiše približno rešenje parcijalne diferencijalno-diferentne jednačine pomoću dvodimenzionalnog računa koji je uveo T. Ogata [4] kao i pomoću rezultata J. Wloke [8] za obične diferencijalno-diferentne jednačine. Takodje se daje i ocena greške.

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