Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 22, 2 (1992), 81-100

Review of Research
Faculty of Science
Mathematics Series

BORNOLOGICAL LINEAR FUZZY NEIGHBOURHOOD SPACES

A. K. Katsaras
Mathematics Department, University of Ioannina
Ioannina, Greece

Abstract

Some properties of the inductive limits of linear fuzzy neighbourhood spaces are investigated. Also the notion of a bornological linear fuzzy neighbourhood space is given and some of the properties of such a space are studied. Every probabilistic pseudometrizable linear fuzzy neighbourhood space is bornological.

AMS Mathematics Subject Classification (1991): 04A72
Key words and phrases: linear fuzzy neighbourhood system, fuzzy string, probabilistic pseudometric, probabilistic seminorm, bornological linear fuzzy neighbourhood space.

1. Introduction

The concept of fuzzy neighbourhood space was given by Lowen in [12] while the fuzzy topological vector spaces were introduced by the author in [5] and [6]. In [7] - [9] the author studied some of the properties of those fuzzy topological vector spaces whose fuzzy topology is given by some fuzzy neighbourhood system.

In this paper we continue with the investigation of linear fuzzy neighbourhood spaces. First we shall give the notion of a fuzzy string and it show that every linear fuzzy neighbourhood system is generated by a family of fuzzy strings. A linear fuzzy neighbourhood system is probabilistic pseudometrizable iff it is generated by a fuzzy string.

Next, we give the notion of a bornologocal linear fuzzy neighbourhood space, namely a linear fuzzy neighbourhood space $(E, \mathbb{N})$ with the property that every linear map, from $(E, \mathbb{N})$ to a linear fuzzy neighbourhood space $(F, \mathbb{N}')$, maps bounded fuzzy sets into bounded fuzzy sets, is continuous. It is shown that, to every linear fuzzy neighbourhood system $\mathbb{N}$, corresponds a bornological linear fuzzy neighbourhood system $\mathbb{N}^b$ which is the finest linear fuzzy neighbourhood system having the same with $\mathbb{N}$ bounded fuzzy sets. If $\mathbb{N}$ is probabilistic pseudometrizable or it is a countable product of bornological linear fuzzy neighbourhood systems, then $\mathbb{N}$ is bornological. We also introduce and study some of the properties of the inductive limits of families of linear fuzzy neighbourhood spaces.

2. Preliminaries

For the definitions of the fuzzy topological vector spaces and the fuzzy neighbourhood spaces we will refer to [5] and [2] respectively. For the notion of a *Hausdorff* fuzzy neighbourhood system we will refer to [17].

If $\{t_\alpha : \alpha \in A\}$ is a family of real numbers, then we will denote by $\bigvee_\alpha t_\alpha$ and $\bigwedge_\alpha t_\alpha$ the $\sup_\alpha t_\alpha$ and $\inf_\alpha t_\alpha$, respectively. Let now E be a vector space over $\mathbb{K}$, where $\mathbb{K}$ is the space of either the real or complex numbers. For μ_1, μ_2 fuzzy sets in E, the fuzzy set $\mu = \mu_1 \oplus \mu_2$ is defined by

$$\mu(x) = \bigvee\{\mu_1(x_1) \wedge \mu_2(x_2) : x = x_1 + x_2\}.$$

Also, for $t \in \mathbb{K}$ and μ a fuzzy set in E, the fuzzy set $t\mu$ is defined as follows (see [10]): If $t \neq 0$, then $(t\mu)(x) = \mu(t^{-1}x)$. For $t = 0$, we define $(t\mu)(x) = 0$ $x \neq 0$, and $(t\mu)(0) = \sup_{y \in E} \mu(y)$.
For $x \in E$, the fuzzy set $x \oplus \mu$ is defined by $(x \oplus \mu)(y) = \mu(y - x)$. A fuzzy set μ in E is called:

(1) convex if $t\mu \oplus (1-t)\mu \leq \mu$ for $t \in [0,1]$;
(2) balanced if $t\mu \leq \mu$ if $|t| \leq 1$;
(3) apsolutely convex if it is both convex and balanced;
(4) apsorbing if $\sup\{\mu(tx) : t > 0\} = 1$, for all $x \in E$.

A linear fuzzy neighbourhood space (see [7]) is a fuzzy topological vector space whose fuzzy topology is given by a fuzzy neighbourhood system. If $(E, \mathbb{N})$ is a linear fuzzy neighbourhood space, then each of $\mathbb{N}(0)$ is absorbing and there is a base for in $\mathbb{N}(0)$ consisting of balanced fuzzy sets (see [7]). If $\mathbb{N}(0)$ has a base consisting of convex fuzzy sets, then $(E, \mathbb{N})$ is said to be locally n-convex. In this case, $\mathbb{N}(0)$ has a base consisting of absolutely convex fuzzy sets. A fuzzy set μ in $(E, \mathbb{N})$ is called n-bounded, or just bounded in this paper, if for each $\sigma \in \mathbb{N}(0)$ and each $\epsilon > 0$, there exists $t > 0$ with $t\mu - \epsilon \leq \sigma$.

Next, we recall definition of a probabilistic pseudometric and a probabilistic seminorm on a vector space E over $\mathbb{K}$. Let $\mathcal{D}(\mathbb{R}^+)$ denote the family of all the fuzzy sets α of $\mathbb{R}^+ = [0, \infty)$ such that:

(1) α is increasing and left continuous;

(2) $\alpha(0) = 0$ and $\lim_{t\to\infty} \alpha(t) = 1$.

By ϵ_0 we will denote the element of $\mathcal{D}(\mathbb{R}^+)$ defined by: $\epsilon_0(0) = 0$ and $\epsilon_0(t) = 1$ if $t > 0$. For $\alpha_1, \alpha_2 \in \mathcal{D}(\mathbb{R}^+)$, we will denote by $\alpha_1 \oplus \alpha_2$ the element α of $\mathcal{D}(\mathbb{R}^+)$ defined by

$$\alpha(x) = \bigvee\{\alpha_1(t_1) \wedge \alpha_2(t_2) \; : \; t = t_1 + t_2\}.$$

A probabilistic seminorm on E is a mapping $p : E \to \mathcal{D}(\mathbb{R}^+)$ with the following properties:

(i) $p(0) = \epsilon_0$;

(ii) $p(\lambda x)(t) = p(x)(|\lambda|^{-1}t)$ if $\lambda \neq 0$;

(iii) $p(x) \oplus p(y) \leq p(x + y)$.

Every probabilistic seminorm p generates a linear fuzzy neighbourhood system $\mathbb{N}_p$ (see [7]) having as a base at zero the family $\{b_{p,t} : t > 0\}$, where $b_{p,t}(x) = p(x)(t)$.

A mapping $F : X \times X \to \mathcal{D}(\mathbb{R}^+)$ is called a probabilistic pseudometric on X (see [2]) if for all x, y, z in X we have:

(1) $F(x, x) = \epsilon_0$;

(2) $F(x, y) = F(y, x)$;

(3) $F(x, y) \oplus F(y, z) \leq F(x, z)$.

Every probabilistic pseudometric F on X includes a fuzzy neighbourhood system $\mathbb{N}_F$, where for each $x \in X$, $\mathbb{N}_F(x)$ has a base the family $\{\sigma_{F,x,t} : t >$

$> 0\} \sigma_{F,x,t}(y) = F(x,y)(t)$. The mapping $D : \mathcal{D}(\mathbb{R}^+)\times\mathcal{D}(\mathbb{R}^+) \to \mathcal{D}(\mathbb{R}^+)$,

$$D(\alpha,\beta) = \bigvee\{\xi \, ; \, \xi \in \mathcal{D}(\mathbb{R}^+), \, \alpha \oplus \xi \le \beta, \, \beta \oplus \xi \le \alpha\}$$

is a probabilistic pseudometric on $\mathcal{D}(\mathbb{R}^+)$.
We will refer to $\mathbb{N}_D$ as the usual fuzzy neighbourhood structure on $\mathcal{D}(\mathbb{R}^+)$.

Finally, a fuzzy set μ in a vector space E, is said to absorb another one p if for each $\epsilon > 0$ such that $tp - \epsilon \le \mu$.

3. Fuzzy strings

Definition 3.1. *A fuzzy string in a vector space E is a decreasing sequence (b_n) of balanced absorbing fuzzy sets in E for which there exists a sequence (ϵ_n) of positive numbers, with $\sum_{n=1}^{\infty} \epsilon_n < \infty$, such that $b_{n+1} \oplus b_{n+1} - \epsilon_n \le b_n$ for all n .*

Example 3.2. If b is an absolutely convex absorbing fuzzy set in E, then the sequence (b_n), $b_n = 2^{-n}b$, is a fuzzy string since $b_{n+1} \oplus b_{n+1} \le b_n$. In particular if p is a probabilistic seminorm on E, then the sequence (d_n), $d_n(x) = p(x)(2^{-n})$, is a fuzzy string.

Lemma 3.3. *If $\mathcal{U} = (b_n)$, $\mathcal{V} = (d_n)$ are fuzzy strings in E and λ a non-zero scalar, then $\mathcal{U} \oplus \mathcal{V} = \{b_n \oplus d_n\}$, $\mathcal{U} \wedge \mathcal{V} = \{b_n \wedge d_n\}$ and $\lambda\mathcal{U} = \{\lambda b_n\}$ are fuzzy strings.*

Recall that a fuzzy subset b of E is called a fuzzy subspace (see [10]) if $b \oplus b \le b$ and $\lambda b \le b$ for all $\lambda \in \mathbb{K}$.

Proposition 3.4. *If $\mathcal{U} = (b_n)$ is a fuzzy string in E then $b = \wedge\mathcal{U} = \bigwedge_{n=1}^{\infty} b_n$ is a fuzzy subspace.*

Proof. Let (ϵ_n) be as in definition 3.1 and let $b(x)$, $b(y) > \theta$. There exists n_0 such that $b(x), b(y) > \theta + \epsilon_n$ if $n \ge n_0$. If now $n \ge n_0$, then $b_n(x+y) \ge b_{n+1}(x) \wedge b_{n+1}(y) - \epsilon_n > \theta$. Also, $b_n(x+y) \ge b_{n_0}(x+y) > \theta$ if $n < n_0$. Hence $b(x+y) \ge \theta$. This proves that $b \oplus b \le b$. To show that $\lambda b \le b$, for $\lambda \in \mathbb{K}$, we first notice that

$$\underbrace{b_{n+m} \oplus b_{n+m} \oplus \cdots \oplus b_{n+m}}_{2^m \text{ terms}} \le \epsilon_n + \epsilon_{n+1} \cdots + \epsilon_{n+m-1} + b_n.$$

Let now $\epsilon > 0$ and choose n_0 so that $\sum_{n_0}^{\infty} \epsilon_k < \epsilon$. Let $n \ge n_0$. If $2^m \ge |\lambda|$, then

$$\lambda b_{n+m} \leq 2^m b_{n+m} \leq \underbrace{b_{n+m} \oplus \cdots \oplus b_{n+m}}_{2^m \text{ terms}} \leq b_n + \sum_{k=0}^{m-1} \epsilon_{n+k} \leq b_n + \epsilon$$

and so $\lambda b \leq \epsilon + b_n$. Since (b_n) is decreasing, we get $\lambda b \leq \epsilon + b$. Hence $\lambda b \leq b$ because $\epsilon > 0$ was arbitrary. This completes the proof.

Recall that a non-empty family $\mathbb{B}$ of fuzzy subsets of X is a base for a fuzzy filter if $0 \notin \mathbb{B}$ and for $\mu_1, \mu_2 \in \mathbb{B}$ there exists $\mu \in \mathbb{B}$ with $\mu \leq \mu_1 \wedge \mu_2$.

Proposition 3.5. *Let $\mathbb{B}$ be a base for a fuzzy filter on a vector space E and suppose that each member of $\mathbb{B}$ is absorbing and balanced. Then, $\mathbb{B}$ is a base at zero for a linear fuzzy neighbourhood system iff for each $b \in \mathbb{B}$ and each $\epsilon > 0$ there exists $b_1 \in \mathbb{B}$ such that $b_1 \oplus b_1 - \epsilon \leq b$.*

Proof. The necessity of the condition is clear from [7]. For the sufficiency, we will show that $\mathbb{B}$ satisfies the hypotheses of [7, Theorem 3.6]. Let $b \in \mathbb{B}$ and $\epsilon > 0$. By hypothesis, there exists $b_1 \in \mathbb{B}$ such that $b_1 \oplus b_1 - \epsilon \leq b$. Taking $b_x^\epsilon = b_1$ for each $x \in E$, we have $b_0^\epsilon(y) \wedge b_y^\epsilon(x-y) - \epsilon \leq b(x)$, for all x, y in E, and so $\mathbb{B}$ satisfies (i). To prove (iii), let $t \neq 0$ nad $\epsilon > 0$. Our hypothesis implies that there exists $b_1 \in \mathbb{B}$ sush that

$$\underbrace{b_1 \oplus b_1 \cdots \oplus b_1}_{2^m \text{ terms}} - \epsilon \leq b.$$

If now $2^m \geq |t|^{-1}$, then

$$t^{-1} b_1 \leq 2^m b_1 \leq \underbrace{b_1 \oplus \cdots b_1}_{2^m \text{ terms}} \leq \epsilon + b$$

and so $b_1 - \epsilon \leq tb$. Thus $\mathbb{B}$ satisfies also (iii) and the result follows.

Corollary 3.6. *If* $\mathcal{U} = (b_n)$ is a fuzzy string in E, then $\mathcal{U}$ is a base at zero for a linear fuzzy neighbourhood system $\mathbb{N}_{\mathcal{U}}$ on E.

Proof. Let (ϵ_n) be as in definition 3.1. Let $b_n \in \mathcal{U}$ and $\epsilon > 0$. There exists $n_0 \geq n$ such that $\epsilon_k < \epsilon$ if $kgeqn_0$. If $m \geq n_0$, then $b_{m+1} \oplus b_{m+1} \leq \epsilon_m + b_m \leq \epsilon + b_n$. The result now follows from the preceding Proposition.

Lemma 3.7. *Let $\mathcal{U} = (b_n)$ be a fuzzy string in E and let (δ_n) be a sequence of positive numbers. Then there exists a subsequence $\{b_{n_k}\} = (d_k) = \mathcal{V}$ of (b_n) such that $d_{k+1} \oplus d_{k+1} - \delta_k \leq d_k$. Moreover, $\mathbb{N}_{\mathcal{U}} = \mathbb{N}_{\mathcal{V}}$.*

Proof. Let (ϵ_n) be a sequence of positive numbers such that $b_{n+1} \oplus b_{n+1} - \epsilon_n \leq b_n$. There exists a strictly increasing sequence (n_k) of positive integers such that $\epsilon_m \leq \delta_k$ if $m \geq n_k$. Now (b_{n_k}) satisfies the requirements.

Definition 3.8. *For $\mathcal{U}_1 = (b_n)$, $\mathcal{U}_2 = (d_n)$ fuzzy strings in E we say that $\mathcal{U}_1$ is finer than $\mathcal{U}_2$, and write $\mathcal{U}_1 \leq \mathcal{U}_2$, if $b_n \leq d_n$ for all n. A family $\mathcal{F}$ of fuzzy strings in E is called directed if given $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{F}$ there exists $\mathcal{U}$ in $\mathcal{F}$ finer than $\mathcal{U}_1 \wedge \mathcal{U}_2$.*

Using Proposition 3.5, we get easily the following

Proposition 3.9. *If $\mathcal{F}$ is a family of fuzzy strings in E, then $\mathbb{B} = \cup\mathcal{F}$ is a subbase at zero for a linear fuzzy neighbourhood system $\mathbb{N}_{\mathcal{F}}$. If $\mathcal{F}$ is directed, then $\mathbb{B}$ is a base at zero for $\mathbb{N}_{\mathcal{F}}$. Moreover, every linear fuzzy neighbourhood system is of the form $\mathbb{N}_{\mathcal{F}}$ for some family $\mathcal{F}$ of fuzzy strings.*

Definition 3.10. *A fuzzy string $\mathcal{U} = (b_n)$ in a linear fuzzy neighbourhood space $(E, \mathbb{N})$ is called topological if $\mathcal{U} \subset \mathbb{N}(0)$.*

Clearly for every balanced $b \in \mathbb{N}(0)$ there exists a topological fuzzy string containing b.

Every linear fuzzy neighbourhood system $\mathbb{N}$ on E is generated by the family $\mathcal{F}$ of all topological fuzzy strings.

Definition 3.11. ([8]).*A probabilistic quasi-seminorm on a vector space E is a function $p : E \to \mathcal{D}(\mathbb{R}^+)$ with the following properties:*

(i) $p(0) = \epsilon_0$;

(ii) $p(x) \oplus p(y) \leq p(x+y)$;

(iii) $p(\lambda x) \geq p(x)$ if $|\lambda| \leq 1$;

(iv) $\lim_{\lambda \to 0^+} p(\lambda x)(t) = 1$ for all $t > 0$.

If p is a probabilistic quasi-seminorm on E, then the family $\mathbb{B}_p = \{\sigma_t : t > 0\}$, $\sigma_t(x) = p(x)(t)$, is a base at zero for a linear fuzzy neighbourhood system $\mathbb{N}_p$ ([8, Proposition 4.2]). By ([8, Theorem 4.3]) we have

Proposition 3.12. *For every fuzzy string $\mathcal{U}$ on E there exists a probabilistic quasi-seminorm p on E with $\mathbb{N}_p = \mathbb{N}_{\mathcal{U}}$.*

Definition 3.13. *A linear fuzzy neighbourhood space $(E, \mathbb{N})$ is called locally n-bounded if $\mathbb{N}(0)$ contains an n-bounded element.*

Proposition 3.14. *If $(E : \mathbb{N})$ is locally n-bounded, then $\mathbb{N}$ is probabilistic pseudometrizable.*

Proof. Let $b_0 \in \mathbb{N}(0)$ be banded. For each positive integer n there exists a balanced member b_n of $\mathbb{N}(0)$ with $b_n - 1/n \leq b_0$. Now $b = \vee_n(b_n - 1/n)$ is a balanced member of $\mathbb{N}(0)$. Since $b \leq b_0$, it is n-bounded. The countable family $\mathbb{B} = \{n^{-1}b : n = 1, 2, \cdots\}$ is a base for $\mathbb{N}(0)$. In fact, given $\sigma \in \mathbb{N}(0)$ and $\epsilon > 0$ there exists $\lambda > 0$ with $\lambda b - \epsilon \leq \sigma$. If $1/n \leq \lambda$, then $n^{-1}b \leq \lambda b \leq \epsilon + \sigma$, and so $\mathbb{B}$ is a base for $\mathbb{N}(0)$. The result now follows from [7, Theorem 3.3].

4. Bornolgical linear fuzzy neighbourhood spaces

Definition 4.1. *A fuzzy string $\mathcal{U} = \{b_n\}$, in a linear fuzzy neighbourhood space $(E, \mathbb{N})$ is called bornivorous if every b_n absorbs n-bounded fuzzy sets.*

Definition 4.2. *A linear fuzzy neighbourhood space $(E, \mathbb{N})$ is called bornological if every bornivorous fuzzy string in E is topological. If $\mathbb{N}$ is locally n-convex, then $(E, \mathbb{N})$ is called locally convex bornological, abbreviated l.c. bornological, if every absolutely convex fuzzy set in E absorbing n-bounded fuzzy sets is in $\mathbb{N}(0)$.*

Note 4.3. By example 3.2, it is clear that if $(E, \mathbb{N})$ is locally n-convex and bornological, then it is l.c. bornological.

Theorem 4.4. *Let $(E, \mathbb{N})$ be a linear fuzzy neighbourhood space and let $\mathcal{F}$ be a family of all bornivorous fuzzy string in E. Then $\mathcal{F}$ is directed and $\mathbb{N}_{\mathcal{F}}$ is the finest linear fuzzy neighbourhood structure on E having the same with $\mathbb{N}$ bounded fuzzy sets. Moreover, $\mathbb{N}_{\mathcal{F}}$ is bornological.*

Proof. It is easy to see that $\mathcal{F}$ is directed and $\mathbb{N}$ and $\mathbb{N}_{\mathcal{F}}$have the same bounded fuzzy sets. If $\mathcal{U} = (b_n)$ is an $\mathbb{N}_{\mathcal{F}}$-bornivorous fuzzy string, then $\mathcal{U}$ is also $\mathbb{N}$-bornivorous and so $b_n \in \mathbb{N}_{\mathcal{F}}(0)$. Thus $\mathbb{N}_{\mathcal{F}}$ is bornological. Finally, let $\mathbb{N}'$ be a linear fuzzy neighbourhood system having the same with $\mathbb{N}$ bounded fuzzy sets. Given $b \in \mathbb{N}'(0)$ balanced, there exists an $\mathbb{N}'$-topological fuzzy string $\mathcal{U}$ containing b. Each member of $\mathcal{U}$ is $\mathbb{N}'$-bornivorous and hence $\mathbb{N}$-bornivorous; thus $\mathcal{U} \in \mathcal{F}$ and so $b \in \mathbb{N}_{\mathcal{F}}(0)$. This proves that $\mathbb{N}' \subset \mathbb{N}_{\mathcal{F}}$ and the result follows.

Notation 4.5. For a linear fuzzy neighbourhood system $\mathbb{N}$ on E, we will denote by $\mathbb{N}^b$ the finest linear fuzzy neighbourhood system on E having the same with $\mathbb{N}$ bounded fuzzy sets. We will refer to $\mathbb{N}^b$ as the bornological system associated with $\mathbb{N}$.

Corollary 4.6. *(1) $I\!N^b$ is the coarsest bornological linear fuzzy neighbourhood system finer than $I\!N$. (2) $I\!N$ is bornological iff $I\!N = I\!N^b$.*

Proof. (1) It is clear that $I\!N \subset I\!N^b$. On the other hand, if $I\!N' \supset I\!N$ is bornological, then every $I\!N$-bornivorous string $\mathcal{U}$ is also $I\!N'$-bornivorous, and so $\mathcal{U}$ is $I\!N'$-topological; hence $I\!N^b \subset I\!N'$.

(2) It follows from (1).

The proof of the next Theorem is analogous to the one of Theorem 4.4.

Theorem 4.7. *Let $(E, I\!N)$ be a locally n-convex linear fuzzy neighbourhood space. Then, the family $I\!B$, of all absolutely convex fuzzy sets in E which absorb n-bounded fuzzy sets, is a base at zero for an l.c. bornological linear fuzzy neighbourhood system $I\!N^{bc}$ on E.*

Moreover, $I\!N^{bc}$ is the finest locally n-convex linear fuzzy neighbourhood system on E having the same with $I\!N$ bounded fuzzy sets and $I\!N$ is l.c. bornological iff $I\!N = I\!N^{bc}$.

We will refer to the $I\!N^{bc}$ in the above Theorem as the l.c. bornological linear fuzzy neighbourhood system associated with $I\!N$. it is clear that $I\!N \subset I\!N^{bc} \subset I\!N^b$.

Definition 4.8. *A map f, from a linear fuzzy neighbourhood space $(E, I\!N)$ to another one (E' ,N'), is called bounded if it maps $I\!N$-bounded fuzzy sets into $I\!N'$ bounded fuzzy sets.*

Proposition 4.9. *If $f : (E, I\!N) \to (E', I\!N')$ is linear and continuous, then f is bounded.*

Proof. Let $b \in I\!N'(0)$ and μ be a bounded fuzzy set in E. Since $f^{-1}(b) \in I\!N(0)$, there exists $t > 0$ such that $t\mu - \epsilon \le f^{-1}(b)$ and so $tf(\mu) - \epsilon \le b$. Hence $f(\mu)$ is bounded.

Theorem 4.10. *A linear fuzzy neighbourhood space $(E, I\!N)$ is bornological iff every bounded linear map from $(E, I\!N)$ to any linear fuzzy neighbourhood space $(E', I\!N')$ is continuous.*

Proof. ($\Longrightarrow$) Let $b \in I\!N'(0)$ be balanced. There exists an $I\!N'$-topological fuzzy string $\mathcal{U}$ in E' containing b. Since f is bounded, it folows that $f^{-1}\mathcal{U}$ is $I\!N$-bornivorous and hence $f^{-1}(\mathcal{U})$ is $I\!N$-topological since $I\!N$ is bornological. Thus $f^{-1}(b) \in I\!N(0)$, which proves that f is continuous.

($\Longleftarrow$) The identity map $g : (E, I\!N) \to (E, I\!N^b)$ is bounded, hence $I\!N^b = g^{-1}(I\!N^b) \subset I\!N$ and so $I\!N = I\!N^b$.

The proof of the next Theorem is analogous to the one of the preceding

Theorem.

Theorem 4.11. *A locally n-convex linear fuzzy neighbourhood space* $(E, \mathbb{N})$ *is l.c. bornological iff every bounded linear map from* $(E, \mathbb{N})$ *to any locally n-convex linear fuzzy neighbourhood space* $(E', \mathbb{N}')$ *is continuous.*

Let (E, τ) be a topological vector space and let $\omega(\tau)$ denote the fuzzy topology consisting of all τ-lower semicontinuous fuzzy sets. By Lowen [12], there exists a unique fuzzy neighbourhood system $\mathbb{N}_\tau$ on E such that $\omega(\tau)$ coincides with the fuzzy topology induced by $\mathbb{N}_\tau$. For each $x \in X$, the family $\{\chi_A \ : \ A \ \tau - neighbourhood\, of x\}$ is base for $\mathbb{N}_\tau(x)$. Since $\omega(\tau)$ is linear topology (by [5, Theorem 3.2]), the system $\mathbb{N}_\tau$ is linear (see [7, Theorem 3.2]). If τ is locally convex, then $\mathbb{N}_\tau$ is locally n-convex. By [7, Proposition 5.6], a fuzzy set μ in E is $\mathbb{N}_\tau$-bounded iff each $\{x \ : \ \mu(x) > \theta\}$, $0 < \theta < 1$, is τ-bounded.

Theorem 4.12. *Let* (E, τ) *and* $\mathbb{N}_\tau$ *be as above. Then,* (E, τ) *is bornological (resp. l.c. bornological) iff* $(E, \mathbb{N}_\tau)$ *is bornological (resp. l.c. bornological).*
Proof. We will give the proof for the bornological case. The proof for the l.c. bornological case is analogous. Let $f \ : \ (E, \mathbb{N}_\tau) \to (E, \mathbb{N}_\tau^b)$ be the identity map and let $\tau' = i(t(\mathbb{N}_\tau^b))$ (The mapping i is as defined by Lowen in [13]. Let $A \subset E$ be τ-bounded. It is easy to see that the characteristic function χ_A of A is $\mathbb{N}_\tau^b$-bounded. Thus, given $b \in \mathbb{N}_\tau^b(0)$ and $0 < \theta < 1$, there exists $t > 0$ such that $t\chi_A + \theta - 1 \leq b$, which implies that $tA \subset \{x \ : \ b(x) \geq \theta\}$. Since the sets of the form $\{x \ : \ b(x) \geq \theta\}$, $b \in \mathbb{N}_\tau^b(0)$ $\quad 0 < \theta < 1$, from a τ'-base at zero, it follows that A is τ'-bounded. Note that τ' is linear. Since $\tau \subset \tau'$, it follows that τ and τ' have the same bounded sets. If τ is bornological, then $\tau = \tau'$. Now

$$t(\mathbb{N}_\tau^b) \leq \omega \circ i(t(\mathbb{N}_\tau^b)) = \omega(\tau') = \omega(\tau) = t(\mathbb{N}_\tau)$$

and so $\mathbb{N}_\tau^b \subset \mathbb{N}_\tau$ (by [12]), which implies that $\mathbb{N}_\tau = \mathbb{N}_\tau^b$ and thus is bornological.

Conversely, let $\mathbb{N}_\tau$ be bornological and let τ^b be the bornological topology associated with τ (see [1]). Let $\mathbb{N}' = \mathbb{N}_{\tau^b}$. A fuzzy set μ in E is $\mathbb{N}_\tau$-bounded (resp. $\mathbb{N}'$-bounded) iff each $\{x \ : \ \mu(x) > \theta\}$, $0 < \theta < 1$, is τ-bounded (resp. τ_b-bounded). Since τ and τ_b have the same bounded sets, it follows that $\mathbb{N}_\tau$ and $\mathbb{N}'$ have the same bounded sets and hence $\mathbb{N}' = \mathbb{N}_\tau$ since $\mathbb{N}_\tau \subset \mathbb{N}'$ and $\mathbb{N}_\tau$ is bornological. Therefore

$\omega(\tau) = \omega(\tau_b)$ and so $\tau = i \circ \omega(\tau) = i \circ \omega(\tau_b) = \tau_b$,

i.e. τ is bornological.

Definition 4.13. *A fuzzy set μ in $\mathcal{D}(\mathbb{R}^+)$ is called $\mathbb{N}_D$-bounded, or simply bounded, if for each $\epsilon > 0$ there exists $t > 0$ such that $\mu(x) - \epsilon \leq x(t)$ for all $x \in \mathcal{D}(\mathbb{R}^+)$.*

Definition 4.14. *A probabilistic seminorm p, on a linear fuzzy neighbourhood space $(E, \mathbb{N})$, is called bounded if it maps bounded fuzzy sets into bounded fuzzy sets.*

Proposition 4.15. *Every continuous probabilistic seminorm p on a linear fuzzy neighbourhood space $(E, \mathbb{N})$ is bounded.*

Proof. Let μ be bounded in E and let $\epsilon > 0$. The fuzzy set in σ in $\mathcal{D}(\mathbb{R}^+)$, $\sigma(x) = x(1)$ is in $\mathbb{N}_D(0)$ and so $p^{-1}(\sigma) \in \mathbb{N}(0)$. Thus, there exists $t > 0$ such that $t\mu - \epsilon \leq p^{-1}(\sigma)$ and so $\mu(x) - \epsilon \leq \sigma(p(tx)) = p(tx)\,(1) = p(x)(s)$, $s = t^{-1}$. Thus $p(\mu)(\xi) - \epsilon \leq \xi(s)$, for all $\xi \in \mathcal{D}(\mathbb{R}^+)$, and the result follows.

Theorem 4.16. *A locally n-convex linear fuzzy negibourhood space $(E, \mathbb{N})$ is l.c. bornological iff every bounded probabilistic pseudometric p on E is continuous.*

Proof. Let $(E, \mathbb{N})$ be l.c. bornological and let p be bounded. If $s > 0$, then the fuzzy set $\sigma(x) = p(x)(s)$ is absolutely convex. If μ is a bounded fuzzy set in E, then $p(\mu)$ is bounded and so, given $\epsilon > 0$, there exists $t > 0$ such that $\mu(x) - \epsilon \leq p(x)(t)$ for all $x \in E$. Thus

$$\mu(x) - \epsilon \leq p(\gamma^{-1}x)(s) = \sigma(\gamma^{-1}x), \quad \gamma = s^{-1}t. \text{ i.e.}$$

$\gamma^{-1}\mu - \epsilon \leq \sigma$. This proves that σ absorbs $\mathbb{N}$-bounded fuzzy sets and so $\sigma \in \mathbb{N}(0)$ since $\mathbb{N}$ is l.c. bornological. It follows that p is continuous. Conversely, let the condition be satisfied and let b be an absolutely convex fuzzy set in E absorbing bounded fuzzy sets. The Minkowski functional $p = p_b$ of b is bounded. In fact, if μ is $\mathbb{N}$-bounded and $\epsilon > 0$, there exists $t > 0$ such that $t\mu - \epsilon \leq b$. Therefore

$$\mu(x) - \epsilon \leq b(tx) \leq p(tx)(1+) \leq p(tx)(s) = p(x)(st^{-1})$$

$(s > 1)$. Thus, $p(\mu)(\xi) - \epsilon \leq \xi(st^{-1})$, for all $\xi \in \mathcal{D}(\mathbb{R}^+)$, which proves that $p(\mu)$ is bounded. Hence, p is bounded and so p is (by hypothesis) continuous. This implies that $b \in \mathbb{N}(0)$ (by [11, Prop. 2.3]) and the result follows.

Theorem 4.17. *Every probabilistic pseudometrizable linear fuzzy neighbourhood space $(E, \mathbb{N})$ is bornological.*

Proof. Since $\mathbb{N}$ is probabilistic pseudometrizable, there exists (by [8, 4.2, 4.3]) a decreasing sequence (b_n) in $\mathbb{N}(0)$ consisting of balanced fuzzy sets which is a base at zero. Let now $d \in \mathbb{N}^b(0)$ be balanced and suppose that

$d \notin I\!N(0)$. Then, there exists $\epsilon > 0$ such that there is no member σ of $I\!N(0)$ with $\sigma - \epsilon \leq d$. In particular $n^{-2}b_n - \epsilon \not\leq d$ and hence there exists $y \in E$ with $b_n(ny) - \epsilon > d(n^{-1}y)$. Since d is balanced and absorbing, we choose an increasing sequence (n_k) of positive integers and a sequence (x_k) of distinct members of E such that $b_{n_k}(n_k x_k) - \epsilon > d(n_k^{-1}x_k)$.

Let σ be the fuzzy set in E defined by

$$\begin{aligned} \sigma(x) &= b_{n_k}(n_k x_k) && \text{if } x = x_k \\ &= 0 && \text{if } x \neq x_k \text{ for all } k. \end{aligned}$$

Then σ is $I\!N$-bounded. In fact, given a positive integer m and $\delta > 0$, there exists $0 < t < 1$ such that $b_m(tx_k) > 1 - \delta$ for $k = 1, 2, \cdots, m$. Now, $t\sigma - \delta \leq b_m$. Indeed,if $x = x_k$ for some $k \leq m$, then $\sigma(x_k) - \delta \leq b_m(tx)$. If $x = x_k$ with $k > m$, then $\sigma(x_k) = b_{n_k}(n_k x_k) \leq b_m(n_k x_k) \leq b_m(tx)$. Thus, $\sigma(x) - \delta \leq b_m(tx)$, for all x, and so $t\sigma - \delta \leq b_m$. This shows that σ is bounded and thus it is $I\!N^b$-bounded. Therefore, there exists $\lambda > 0$ with $\lambda\sigma - \epsilon \leq d$. If $n_k > \lambda^{-1}$, then

$$d(\lambda x_k) \leq d(n_k^{-1}x_k) < b_{n_k}(n_k x_k) - \epsilon = \sigma(x_k) - \epsilon \leq d(\lambda x_k).$$

This contradiction completes the proof.

Next we will show that the Cartesian product of at most countably many bornological (resp. l.c. bornological) spaces is bornological (resp. l.c. bornological). We will need first the following

Lemma 4.18. *Let $(E, I\!N)$ be a Cartesian product of at most countably many linear fuzzy neighbourhood spaces and let $f : (E, I\!N) \to (G, I\!N')$ be a linear mapping $I\!N$-bounded fuzzy sets into $I\!N'$-bounded fuzzy sets. Then, for each $b \in I\!N'$ and each $\epsilon > 0$, there exists a positive integer n_0 such that $b(f(x)) \geq 1 - \epsilon$ for all $x = (x_i)$ in E with $x_i = 0$ if $i \leq n_0$.*

Proof. Let $E = \Pi E_i$, $I\!N = \Pi I\!N_i$ and suppose that the statement of Lemma does not hold. Then there exists a sequence $\{x^{(k)}\}$ in E, $x_i^{(k)} = 0$ if $i \leq k$, such that $b(f(x^{(k)})) < 1 - \epsilon$. The characteristic function χ_A of the set $A = \{kx^{(k)} : k = 1, 2, \cdots\}$ is $I\!N$-bounded. In fact, let $b_n \in I\!N_n(0)$, $n = 1, 2, \cdots, m$, $d = \wedge_{n=1}^{m} \pi_n^{-1}(b_n)$($\pi_n : E \to E_n$ the projection map) and $\gamma > 0$. Since d is absorbing, there exists $t > 0$ such that $d(tkx^{(k)}) > 1 - \gamma$ if $k \leq m$. Since $d(tnx^{(n)}) = 1$ if $n > m$, it follows that $\chi_A - \gamma \leq d(tx)$, for all $x \in E$, which proves that χ_A is $I\!N$-bounded. Let now $\sigma \in I\!N'(0)$ be balanced with $\sigma - \epsilon/\ 2 \leq b$. By hypothesis, $f(\chi_A) = \chi_{f(A)}$ is $I\!N'$-bounded and so there exists $s > 0$ such that $\sigma(nsf(x^{(n)})) \geq 1 - \epsilon/2$ for all n. If $n > s^{-1}$, then

$$b(f(x^{(n)})) \geq \sigma(f(x^{(n)})) - \epsilon/2 \geq \sigma(nsf(x^{(n)})) - \epsilon/2 \geq 1 - \epsilon.$$

This contradiction completes the proof.

Theorem 4.19. *The product $(E, \mathbb{N}) = (\Pi E_i, \Pi \mathbb{N}_i)$ of at most countably many bornological (resp. l.c. bornological) linear fuzzy neighbourhood spaces is bornological (resp. l.c. bornological).*

Proof. Assume that each $\mathbb{N}_i$ is bornological and let $d \in \mathbb{N}^b(0)$. Given $\epsilon > 0$ there exists $d_1 \in \mathbb{N}^b(0)$ balanced with $d_1 \oplus d_1 - \epsilon/2 \leq d$. Since the identity map $f : (E, \mathbb{N}) \to (E, \mathbb{N}^b)$ is bounded, given $\epsilon > 0$ there exists (by the preceding Lemma) a positive integer m such that $d_1(f(x)) \geq 1 - \epsilon/2$ for each $x = (x_i)$ with $x_i = 0$ if $i \leq m$. The canonical mapping $h_k : (E_k, \mathbb{N}_k) \to (E, \mathbb{N})$, ($h_k(x) = y$ with $y_k = x_k$ and $y_i = 0$ if $i \neq k$) is linear and continuous. It follows that $h_k : (E_k, \mathbb{N}_k) \to (E, \mathbb{N}^b)$ is bounded and thus continuous since $\mathbb{N}_k$ is bornological. Therefore $f_k = h_k \circ \pi_k : (E, \mathbb{N}) \to (E, \mathbb{N}^b)$ is continuous (π_k is the projection of E onto E_k). Let now $g = f - \sum_{k=1}^m f_k$. If $y = x - \sum_{k=1}^m h_k(x_k)$, then $y_i = 0$ for $i = m$. Since $f(y) = g(x)$, we have $d_1(g(x)) \geq 1 - \epsilon/2$ for all $x \in E$. The function $h = \sum_{k=1}^m f_k$ is continuous and so $h^{-1}(d_1) \in \mathbb{N}(0)$. Now, for $x \in E$, we have

$$d(x) = d(g(x) + h(x)) \geq -\epsilon/2 + d_1(g(x)) \wedge d_1(h(x)) \geq d_1(h(x) - \epsilon.$$

Thus $h^{-1}(d_1) - \epsilon \leq d$, which proves that $d \in \mathbb{N}(0)$. It follows that $\mathbb{N}^b \subset \mathbb{N}$ and so $\mathbb{N}$ is bornological. This proves the result for the bornological case. The proof for the case when each $\mathbb{N}_i$ is l.c. bornological is analogous.

Lemma 4.20. *Let $(E, \mathbb{N})$ be the product of family $(E_\alpha, \mathbb{N}_\alpha)_{\alpha \in A}$ of linear fuzzy neighbourhood spaces. Then a fuzzy set σ in E is $\mathbb{N}$-bounded iff there exists a family (σ_α), where σ_α is an $\mathbb{N}_\alpha$-bounded in E_α, such that $\sigma \leq \pi\sigma_\alpha = \mu$, where $\mu(x) = \inf_\alpha \sigma_\alpha(x_\alpha)$.*

Proof. First of all we note that if μ is of the form $\pi\sigma_\alpha$, with σ_α $\mathbb{N}_\alpha$-bounded, then μ is $\mathbb{N}$-bounded. In fact given $b_k \in \mathbb{N}_{\alpha_k}(0)$ balanced, $k = 1, 2, \cdots, m$, and $\epsilon > 0$ there exists $t > 0$ such that $t\sigma_{\alpha_k} - \epsilon \leq b_k$. Thus

$$t\mu - \epsilon \leq \bigwedge_{k=1}^{m} \pi_{\alpha_k}^{-1}(b_k),$$

which proves that μ is $\mathbb{N}$-bounded. Conversely, if σ is $\mathbb{N}$-bounded, then $\sigma_\alpha = \pi_\alpha(\sigma)$ is $\mathbb{N}_\alpha$-bounded and $\sigma \leq \pi\sigma_\alpha$.

As a Corollary to Theorem 4.19, we have the following

Proposition 4.21. *Let $(E < \mathbb{N}) = (\Pi E_i, \Pi \mathbb{N}_i)$ be a product of at most countably many linear fuzzy neighbourhood spaces. Then $\mathbb{N}^b = \Pi \mathbb{N}_i^b$. If* *each $\mathbb{N}_i$ is locally n-convex, then $\mathbb{N}^{cb} = \Pi \mathbb{N}_i^{cb}$.*

Proof. By Theorem 4.19, $\mathbb{N}' = \Pi \mathbb{N}_i^b$ is bornological. Using the preceding Proposition we easily get that $\mathbb{N}'$ and $\mathbb{N}$ have the same bounded fuzzy sets. Hence $\mathbb{N}^b = \mathbb{N}'$ since $\mathbb{N}'$ is bornological and $\mathbb{N} \subset \mathbb{N}'$. The proof of the second part of the Proposition is analogous.

5. Inductive linear fuzzy neighbourhood systems

We have the following easy established

Theorem 5.1. *Let $(E_\alpha, \mathbb{N}_\alpha)_{\alpha \in J}$ be a family of linear fuzzy neighbourhood spaces, E a vector space and $f_\alpha : E_\alpha \to E$ a linear map for each $\alpha \in J$. Let $\mathcal{F}$ be the family of all fuzzy strings $\mathcal{U}$ in E such that $f_\alpha^{-1}(\mathcal{U})$ is topological in $(E_\alpha, \mathbb{N}_\alpha)$ for all α. Then:*

1) $\mathcal{F}$ is directed.

2) $\mathbb{N}_\mathcal{F}$ is the finest linear fuzzy neighbourhood system on E for which each f_α is continuous.

Definition 5.2. *The linear fuzzy neighbourhood system $\mathbb{N}_\mathcal{F}$ in the above Theorem is called the inductive linear fuzzy neighbourhood system on E with respect to the family $\{E_\alpha, \mathbb{N}_\alpha, f_\alpha\}_{\alpha \in J}$.*

Remark 5.3. Assume that each $\mathbb{N}_\alpha$ in Theorem 5.1 is locally n-convex. Then, the finest locally n-convex linear fuzzy neighbourhood system $\mathbb{N}$ on E for which each f_α is continuous is called the locally n-convex inductive linear fuzzy neighbourhood system on E with respect to the family $\{E_\alpha, \mathbb{N}_\alpha, f_\alpha\}_{\alpha \in J}$ (see [9]). It has as a base at zero all absolutely convex absorbing fuzzy sets d in E such that $f_\alpha^{-1}(d) \in \mathbb{N}_\alpha(0)$ for all $\alpha \in J$. If d is such a fuzzy set, then $\mathcal{U} = \{2^{-n} d\}_{n=0}^{\infty} \in \mathcal{F}$ and so the locally n-convex inductive limit is coarser than $\mathbb{N}_\mathcal{F}$.

Definition 5.4. *A family $\mathcal{F}$ of topological fuzzy strings in a linear fuzzy neighbourhood space $(E, \mathbb{N})$ is called a base for the set of all topological fuzzy strings if for each topological fuzzy string $\mathcal{U} = (b_n)$ in $(E, \mathbb{N})$ and each sequence (ϵ_n) of positive numbers with $\sum \epsilon_n < \infty$ there exists $\mathcal{V} = (d_n) \in \mathcal{F}$ with $d_n - \epsilon \leq b_n$ for all n.*

Theorem 5.5. *Let $(E_\alpha, \mathbb{N}_\alpha, f_\alpha)_{\alpha \in J}$ and E be as in Theorem 5.1 and suppose that E is the linear hull of $\cup_\alpha f_\alpha(E_\alpha)$. For each $\alpha \in J$ let $\mathcal{F}_\alpha$ be a base for the topological fuzzy strings in $(E_\alpha, \mathbb{N}_\alpha)$. Then:*

1) If $G = \{\mathcal{U}_\alpha\}$, with $\mathcal{U}_\alpha = \{d_n^\alpha\}$ a topological fuzzy string in $(E_\alpha, I\!N_\alpha)$ and if $d_n = \vee_m \oplus_{k=1}^m \vee_\alpha f_\alpha(d_{2^{n-1}k}^\alpha)$, then $V_G = \{d_n\}$ is an $I\!N$-topological fuzzy string in E where $I\!N$ is the inductive linear fuzzy neighbourhood system.

2) If $\mathcal{F}$ is the family of all fuzzy strings of the form V_G, where $G = \{\mathcal{U}_\alpha\}$ is such that $\mathcal{U}_\alpha \in \mathcal{F}_\alpha$, then $I\!N_{\mathcal{F}} = I\!N$.

Proof.

1) Let $V_G = (d_n)$ be as in the Theorem. Clearly each d_n is balanced. Also d_n is absorbing. In fact, given $z \in E$, there are $\alpha_1, \alpha_2, \cdots, \alpha_m \in J$ and $x_k \in E_{\alpha_k}$ such that $z = \sum_{k=1}^m f_{\alpha_k}(x_k)$. If $0 < \theta < 1$, there exists $t > 0$ such that $\wedge_{k=1}^m d_{2^{n-1}k}^{\alpha_k}(tx_k) > \theta$. Now

$$d_n(tz) \geq [\oplus_{k=1}^m f_{\alpha_k}(d_{2^{n-1}k}^{\alpha_k})](tz) \geq \wedge_{k=1}^m d_{2^{n-1}k}^{\alpha_k}(tx_k) > 0,$$

which proves that d_n is absorbing. Since

$$\begin{aligned}
&[\oplus_{k=1}^m \vee_\alpha f_\alpha(d_{2^n k}^{\alpha_k})] \oplus [\oplus_{k=1}^m \vee_\alpha f_\alpha(d_{2^n k}^\alpha)] \\
&\quad \leq [\oplus_{k=1}^m \vee_\alpha f_\alpha(d_{2^{n-1}(2k-1)}^\alpha)] \oplus [\oplus_{k=1}^m \vee_\alpha f_\alpha(d_{2^{n-1}(2k)}^\alpha)] \\
&\quad = \oplus_{k=1}^{2m} \vee_\alpha f_\alpha(d_{2^{n-1}k}^\alpha) \leq d_n,
\end{aligned}$$

we easily get that $d_{n+1} \oplus d_{n+1} \leq d_n$ and hence $\mathcal{U}$ is a fuzzy string. Moreover $\mathcal{U}$ is $I\!N$-topological since $f_\alpha^{-1}(d_n) \geq d_{2^{n-1}}^\alpha$.

2) Let $\mathcal{V} = (\sigma_n)$ be $I\!N$-topological and let (ϵ_n) be a sequence of positive numbers such that $\sigma_{n+1} \oplus \sigma_{n+1} - \epsilon_n \leq \sigma_n$. For each $\alpha \in J$, $f_\alpha^{-1}(\mathcal{V})$ is $I\!N_\alpha$-topological and hence the fuzzy string $\{f_\alpha^{-1}(\sigma_{n+1})\}$ is $I\!N_\alpha$-topological. Therefore there exists $\mathcal{U}_\alpha = \{d_n^\alpha\} \in \mathcal{F}$ such that $d_n^\alpha - \epsilon_n \leq f_\alpha^{-1}(\sigma_{n+1})$. Let $\mathcal{U} = (d_n)$, $d_n = \vee_m \oplus_{k=1}^m \vee_\alpha f_\alpha(d_{2^{n-1}k}^\alpha)$. Given a positive integer r and $\epsilon > 0$, choose n so that $2^{n-1} \geq r$ and $2\sum_{j=2^{n-1}}^\infty \epsilon_j < \epsilon$. Now

$$\begin{aligned}
d_n &\leq \vee_m((\sum_{k=1}^m \epsilon_{2^{n-1}k}) + \oplus_{k=1}^m \sigma_{2^{n-1}k+1}) \\
&\leq \vee_m(2\sum_{j=2^{n-1}}^\infty \epsilon_j + \sigma_{2^{n-1}}) \leq \epsilon + \sigma_r.
\end{aligned}$$

This shows that $\mathcal{V}$ is $I\!N_{\mathcal{F}}$-topological and the result follows.

Note 5.6. Let $(E, \mathbb{N})$ be a linear fuzzy neighbourhood space and let $\varphi = (\epsilon_n)$ be a sequence of positive numbers with $\sum \epsilon_n < \infty$. If $\mathcal{F}_\varphi$ is the family of all topological fuzzy strings b_n in E for which $b_{n+1} \oplus b_{n+1} - \epsilon_n \leq b_n$ for all n, then by Lemma 3.7 given any topological fuzzy string (b_n) there exists a member (d_n) of $\mathcal{F}_\varphi$ such that $d_n \leq b_n$ for all n. Hence $\mathcal{F}_\varphi$ is a base for the topological fuzzy strings.

Proposition 5.7. *Let the linear fuzzy neighbourhood system $(E, \mathbb{N})$ be the inductive limit with respect to a countable $\{E_n, \mathbb{N}_n, f_n\}$ and suppose that E is the linear hull of $\cup f_n(E_n)$. Let $\varphi = (\epsilon_n)$ be a sequence of positive numbers with $\sum \epsilon_n < \infty$ and let $\mathcal{F}_m$ be a base for the topological fuzzy strings in $(E_m, \mathbb{N}_m)$ such that for all $\mathcal{U}_m = (d_n^m)$ in $\mathcal{F}_m$ we have $d_{n+1}^m \oplus d_{n+1}^m - \epsilon_n \leq d_n^m$. For $\mathcal{U}_k = (d_n^k) \in \mathcal{F}_k$, let $d_n = \vee_m \oplus_{k=1}^m f_k(d_n^k)$. Then $\mathcal{U} = (d_n)$ is an $\mathbb{N}$-topological fuzzy strings in E and if $\mathcal{F}$ is the family of all such $\mathcal{U}$, then $\mathbb{N}_\mathcal{F} = \mathbb{N}$.*

Proof. Let $\mathcal{U} = (d_n)$ be as in the Proposition. Each d_n is clearly balanced and absorbing as it can be easily shown. Since

$$f_k(d_{n+1}^k) \oplus f_{n+1}^k \leq \epsilon_n + f_k(d_n^k),$$

we have $d_{n+1} \oplus d_{n+1} \leq \epsilon_n + d_n$ and so $\mathcal{U}$ is a fuzzy string which is $\mathbb{N}$-topological since $f_k^{-1}(d_n) \geq d_n^k$. Hence $\mathbb{N}_\mathcal{F} \subset \mathbb{N}$. On the other hand, given $\sigma \in \mathbb{N}(0)$ and $\epsilon > 0$ there exists (by [8, Theorem 4.4]) and $\mathbb{N}$-topological fuzzy string $\mathcal{V} = (\sigma_n)$ such that $\sigma_1 - \epsilon/2 \leq \sigma$ and $\sigma_{n+1} \oplus \sigma_{n+1} \leq \sigma_n$ for all n. Since $f_k^{-1}(\mathcal{V})$ is $\mathbb{N}_k$-topological, there exists $\mathcal{U}_k = (d_n^k) \in \mathcal{F}_n$ such that $d_n^k \leq \epsilon_n + f_k^{-1}(\sigma_{n+k})$. Let $\mathcal{U} = (d_n)$, $d_n = \vee_m \oplus_{k=1}^m f_k(d_m^k)$. Since $f_k(d_n^k) \leq \epsilon_n + \sigma_{k+n}$, it follows that

$$\oplus_{k=1}^m f_k(d_n^k) \leq \epsilon_n + \oplus_{k=1}^m \sigma_{n+k} \leq \epsilon_n \sigma_n \leq \epsilon_n + \epsilon/2 + \sigma.$$

Thus $d_n \leq \epsilon_n + \epsilon/2 + \sigma$. For n large enough, we have $d_n \leq \epsilon + \sigma$ which proves that $\sigma \in \mathbb{N}_\mathcal{F}(0)$. Thus $\mathbb{N} \subset \mathbb{N}_\mathcal{F}$ and the result follows.

Definition 5.8. *A fuzzy string d_n in a vector space E is called convex if each d_n is convex.*

Lemma 5.9. *Let $(E, \mathbb{N})$ ba a locally n-convex linear fuzzy neighbourhood space and let $\varphi = (\epsilon_n)$ be a sequence of positive numbers. Then the family $\mathcal{F}_\varphi$ of all convex topological fuzzy strings d_n in E, for which $d_{n+1} \oplus d_{n+1} - \epsilon_n \leq d_n$ for all n, is a base for the set of all topological fuzzy strings.*

Proof. Let $\mathcal{U} = (b_n)$ be $\mathbb{N}$-topological and let (δ_n) be a sequence of positive numbers with $\sum \delta_n < \infty$. Let $\epsilon_n' = min\{\epsilon_n, \delta_{n+1}\}$. We choose by induction a convex topological fuzzy string (d_n) as follows: Choose $d_1 \in \mathbb{N}(0)$

absolutely convex such that $d_1 - \delta_1 \leq b_1$. Suppose that we have already choosen $d_1, d_2, \cdots, d_n$. Choose $d_{n+1} \in I\!N(0)$ absolutely convex, $d_{n+1} \leq d_n$ and $d_{n+1} - \epsilon_n' \leq (1/\,2d_n) \wedge b_{n+1}$. Now $d_{n+1} \oplus d_{n+1} - \epsilon_n' \leq d_n$. We get in this way a convex topological fuzzy string $\mathcal{V} = (d_n)$ which belongs to $\mathcal{F}_\varphi$ since $\epsilon_n' \leq \epsilon_n$. Also $d_n \leq \epsilon_{n-1}' + b_n \leq \delta_n + b_n$. This completes the proof.

Theorem 5.10. *Let $(E_n, I\!N_n, f_n)$ and $(E, I\!N)$ be as in Proposition 5.7 and suppose that each $I\!N_n$ is locally n-convex. Then $I\!N$ coincides with the locally n-convex inductive linear fuzzy neigbourhood system $I\!N_c$*

Proof. Let $\mathcal{U}_k = (d_n^k)$ is a convex topological fuzzy string in $(E_k, I\!N_k)$ and if $d_n = \vee_m \oplus_{k=1}^m f_k(d_n^k)$, then d_n is convex. In fact, let $d_n(x)$, $d_n(y) > \theta$ and $0 < t < 1$. Since $\oplus_{k=1}^m f_k(d_n^k) \leq \oplus_{k=1}^{m'} f_k(d_n^k)$ when $m \leq m'$, there exists m such that

$$[\oplus_{k=1}^m f_k(d_n^k)](z) > \theta, \text{ for } z = x, y.$$

Let $x_k, y_k \in E_k$ be such that $x = \sum_{k=1}^m f_k(x_k)$, $y = \sum_{k=1}^m f_k(y_k)$, $d_n^k(x_k) > \theta$, $d_n^k(y_k) > \theta$. If $z_k = tx_k + (1-t)y_k$, then $\sum_{k=1}^m f(z_k) = tx + (1-t)y$ and $d_n^k(z_k) > \theta$ and so d_n^k is convex. Now the result follows from Proposition 5.7 and Lemma 5.9.

Theorem 5.11. *Let $(E, I\!N)$ be the inductive limit with respect to a family $(E_\alpha, I\!N_\alpha)_{\alpha \in J}$ of linear fuzzy neighbourhood spaces and linear maps f_α : $E_\alpha \to E$. If each $I\!N_\alpha$ is bornological, then $I\!N$ is bornological.*

Proof. Let $f : (E, I\!N) \to (G, I\!N')$ ba a linear mapping $I\!N$-bounded fuzzy sets into $I\!N'$-bounded fuzzy sets. For each $\alpha \in J$, the function $h_\alpha =$ - $f \circ f_\alpha : (E_\alpha, I\!N_\alpha) \to (G, I\!N')$ maps bounded fuzzy sets into bounded fuzzy sets. Since $I\!N_\alpha$ is bornological, h_α is continuous. It follows from this that f is continuous and this proves that $I\!N$ is bornological.

The proof of the next Theorem is analogous to the proof of the preceding one.

Theorem 5.12. *Let $(E_\alpha, I\!N_\alpha, f_\alpha)_{\alpha \in J}$ and E be as in the preceding Theorem and suppose that each $I\!N_\alpha$ is l.c. bornological. Then, the locally n-convex inductive linear fuzzy neighbourhood system on E is l.c. bornological.*

Corollary 5.13. *Let $(E, I\!N)$ be a linear fuzzy neighbourhood space and F a vector space of E. If $(E, I\!N)$ is bornological, then $E/\,F$ with the quotient linear fuzzy neighbourhood system is also bornological.*

Direct sums. *Let $E = \oplus E_\alpha$ be the algebraic direct sum of a family $(E_\alpha)_{\alpha \in J}$ of vector spaces. For each $\alpha \in J$, let $f_\alpha : E_\alpha \to E$ be the canonical imbedding. The inductive linear fuzzy neighbourhood system on E with respect* *to the family $(E_\alpha, \mathbb{N}_\alpha, f_\alpha)_{\alpha\in J}$ will be called the direct sum of the family $(\mathbb{N}_\alpha)_{\alpha\in J}$ and will be denoted by $\oplus_{\alpha\in J}\mathbb{N}_\alpha$.*

Remarks 5.14.

(1) $\oplus_\alpha \mathbb{N}_\alpha$ is finer than the linear fuzzy neighbourhood system $\mathbb{N}$ induced on E by the product $\Pi_\alpha \mathbb{N}_\alpha$. This follows from the fact that each $f_\alpha : (E_\alpha, \mathbb{N}_\alpha) \to (\Pi E_\beta, \Pi \mathbb{N}_\beta)$ is continuous since the composition of f_α with each projection $\pi_\beta (\beta \in J)$ is continuous.

(2) Suppose that $J = \{1, \cdots, m\}$ is finite. Then $\Pi_{k=1}^m E_k = \oplus_{k=1}^m E_k$ and (by (1)) $\oplus_{k=1}^m \mathbb{N}_k$ is finer than $\Pi_{k=1}^m \mathbb{N}_k$. On the other hand, let (d_n^k) be a topological fuzzy string in $(E_k, \mathbb{N}_k)$ with $d_{n+1}^k \oplus d_{n+1}^k - 2^{-n} \le d_n^k$. Let $d_n = \oplus_{k=1}^m f_k(d_n^k)$. Then $d_n = \wedge_{k=1}^m \pi_k^{-1}(d_n^k)$.It follows now from Proposition 5.7 that $\oplus_{k=1}^m \mathbb{N}_k \subset \Pi_{k=1}^m \mathbb{N}_k$, hence $\oplus_{k=1}^m \mathbb{N}_k = \Pi_{k=1}^m \mathbb{N}_k$.

Strict Inductive Limits. Let $\{E_n\}$ ba an increasing sequence of vector space and $E = \cup E_n$. For each n, let $\mathbb{N}_n$ be a linear fuzzy neighbourhood system on E_n and let $\mathbb{N}$ be the inductive linear fuzzy neighbourhood system on E with respect to the family $(E_n, \mathbb{N}_n, f_n)$ where $f_n : E_n \to E$ is the inclusion map. We say that the inductive limit $\mathbb{N}$ is strict if for each n $\mathbb{N}_n$ coincides with the linear fuzzy neighbourhood system $\mathbb{N}_{n+1}|E_n$ induced on E_n by $\mathbb{N}_{n+1}$.

Lemma 5.15. *Let $(E, \mathbb{N})$ be a linear fuzzy neighbourhood space, F a vector subspace of E and $\mathbb{N}' = \mathbb{N}|F$. Given an $\mathbb{N}'$-topological fuzzy string $\mathcal{U}' = (b_n)$ is F and a sequence (ϵ_n) of positive numbers with $\sum \epsilon_n < \infty$, there exists an $\mathbb{N}$-topological fuzzy string $\mathcal{U} = \{\sigma_n\}$ in E such that $(\sigma_n \oplus \sigma_n)(x) \le \epsilon_n + b_n(x)$ for all $x \in F$ and $\sigma_{n+1} \oplus \sigma_{n+1} - \epsilon \le \sigma_n$.*

Proof. There exists a sequence (σ_n') in $\mathbb{N}(0)$ such that $b_n = \sigma_n'|F$. Choose by induction a decreasing sequence (σ_n) of balanced members of $\mathbb{N}(0)$ such that $\sigma_1 \oplus \sigma_1 - \epsilon_1 \le \sigma_1'$ and $\sigma_{n+1} \oplus \sigma_{n+1} - \epsilon_n \le \sigma_n \wedge \sigma_{n+1}'$. Clearly $\mathcal{U} = (\sigma_n)$ satisfies the requirements.

Theorem 5.16. *Let $(E, \mathbb{N})$ be strict inductive limit of an increasing sequence $(E_n, \mathbb{N}_n)$ of linear fuzzy neighbourhood spaces. Then $\mathbb{N}_m = \mathbb{N}|E_m$ for all m.*

Proof. Clearly $\mathbb{N}' = \mathbb{N}|E_m$ is coarser than $\mathbb{N}_m$. Let (b_n) be an $\mathbb{N}_m$-topological fuzzy string on E_m and let $\epsilon > 0$. Let (δ_n) be a sequence of positive numbers, with $\sum \delta_n < \infty$, such that $b_{n+1} \oplus d_{n+1} - \delta_n \leq b_n$ for all n. Let (ϵ_n) be a decreasing sequence of positive numbers such that $\epsilon_n \leq \delta_n$ and $\sum \epsilon_n < \epsilon$. Let $\mathcal{U}_0 = (b_n^0)$, $b_n^0 = b_n$.

By induction and by the preceding Lemma, we get a sequence $(\mathcal{U}_k)$, where $\mathcal{U}_k = (b_n^k)$ is an $I\!N_{m+k}$-topological fuzzy string in E_{m+k} such that $b_{n+1}^k \oplus b_{n+1}^k - \epsilon_k \leq b_n^k$ and $(b_n^k \oplus b_k^n)(x) \leq \epsilon_k + b_n^{k-1}(x)$ for $x \in E_{m+k-1}$, $k = 1, 2, \cdots$. For a fuzzy subset σ of a subspace F of E we let $\hat{\sigma}$ be defined on E by $\hat{\sigma} = \sigma$ on F and $\hat{\sigma}(x) = 0$ if $x \notin F$. Let $\sigma_n = \vee_j \oplus_{k=1}^j \hat{b_n^k}$ and $\mathcal{U} = (\sigma_n)$. As in the proof of Proposition 5.7, we have show that $\mathcal{U}$ is $I\!N$-topological. Moreover $\sigma_n - \epsilon \leq b_n$ on E_m. In fact, let $x \in E_m$. If $\sigma_n(x) > \theta > 0$, there are x_k, $k = 1, 2, \cdots, j$, such that $x = x_1 + x_2 + \cdots + x_j$, $\hat{b_n^k}(x_k) = b_n^k(x_k) > \theta$. Since $x_j = x - (x_1 + x_2 + \cdots + x_{j-1}) \in E_{m+j-1}$, we have $\hat{b_n^j}(x_j) = \hat{b_n^j}(x_j) \wedge \hat{b_n^j}(0) \leq \epsilon_j + b_n^{j-1}(x_j)$. Hence $b_n^{j-1}(x_j) > \theta - \epsilon_j$. Now $x_{j-1} + x_j = x - \sum_{k=1}^{j-2} x_k \in E_{m+j-2}$. Thus

$$\theta - \epsilon_j < b_n^{j-1}(x_j) \wedge b_n^{j-1}(x_{j-1}) \leq$$
$$\leq (b_n^{j-1} \oplus b_n^{j-1})(x_j + x_{j-1}) \leq$$
$$\epsilon_{j-1} + d_n^{j-2}(x_j + x_{j-1}).$$

Continuing in this way, we get

$$\theta - (\epsilon_1 + \epsilon_2 + \cdots + \epsilon_j) \leq$$
$$\leq b_n^0(x_1 + \cdots + x_j) = b_n(x)$$

and so $b_n(x) > \theta - \epsilon$. This proves that $b_n(x) \geq b_n(x) - \epsilon$ for all $x \in E_m$. It follows that (b_n) is $I\!N'$-topological and this completes the proof.

Theorem 5.17. *Let $(E, I\!N)$ be the strict inductive limit of an increasing sequence $(E_n, I\!N_n)$ of linear fuzzy neighbourhood spaces. If each $(E, I\!N_n)$ is Hausdorff, then $(E, I\!N)$ is also Hausdorff.*

Proof. Let $x \neq 0$ in E and $\epsilon > 0$. There exists n such that $x \in E_n$. Since $(E_n, I\!N_n)$ is Hausdorff, there exists $b_0 \in I\!N_n(0)$ such that $b_0(x) < \epsilon$. By the preceding Theorem, there exists $b \in I\!N(0)$ with $b|E_n = b_0$. Now $b(x) < \epsilon$ and this completes the proof.

References

[1] Adash, N., Emst, B., Keim: Topological vector spaces, Lecture Notes in Mathematics 639, Springer-Vrelag 1978.

[2] Höle, U.: Probabilistic metrization of fuzzy unuformities, Fuzzy Sets and Systems 8(1982), 63-69.

[3] Höle, U.: Probabilistic topologien, Manuscripta Math. 26(1978), 223-245.

[4] Höle, U.: Probabilistic topologies induced by L-fuzzy uniformities, Manuscripta Math. 38(1982), 289-323.

[5] Katsaras, A.K.: Fuzzy topological vector spaces, I, Fuzzy Sets and Systems 6(1981), 85-95.

[6] Katsaras, A.K.: Fuzzy topological vector spaces, II, Fuzzy Sets and Systems 12(1984), 143-154.

[7] Katsaras, A.K.: Linear fuzzy neighbourhood spaces, Fuzzy Sets and Systems 16(1985), 25-40.

[8] Katsaras, A.K.: Linear fuzzy neighbourhood spaces, II, J. Math. Anal. Appl. 115, no 2(1986), 560-573.

[9] Katsaras, A.K.: On linear fuzzy neighbourhood spaces, Proc. Polish Symp. Interval and fuzzy Mathematics, Poznan (1987), 77-92.

[10] Katsaras,A.K., Liu, D.B.: Fuzzy vector space and fuzzy topological vector spaces, J. Math. Anal. Appl. 58(1977), 135-146.

[11] Katsaras, A.K.: Complete linear fuzzy neighbourhood space, Quastiones Mathematicae 14(1) (1991), 11-31.

[12] Lowen, R.: Fuzzy neighbourhood spaces, Fuzzy Sets and Systems 7 (1982), 165-189.

[13] Lowen, R.: Fuzzy topological spaces and fuzzy compactness, J. Math. Anal. Appl. 56(1976), 621-663.

[14] Lowen, R.: Compactness notions in fuzzy neighbourhood spaces. Manuscripta Math. 38(1982), 265-287.

[15] Morsi, N.N.: On fuzzy pseudo-normed vector spaces (preprint).

[16] Rodbaugh, E.: A categorical accomodation of various notions of fuzzy topology, Fuzzy Sets and Systems 9(1983), 241-265.

[17] Wuyts,P., Lowen, R.: On separation axioms in fuzzy topological spaces, fuzzy neighbourhood spaces and fuzzy uniform spaces, J. Math. Anal. Appl. 93(1983), 27-41.

REZIME

BORNOLOŠKI LINEARNI RASPLINUTI OKOLINSKI SKUPOVI

Ispitivane su neke osobine induktivnih limesa bornoloških linearnih rasplinutih okolinskih skupova. Pokazano je da je svaki verovatnosni pseudometrizabilan linearni rasplinuti okolinski skup bornološki.

Received by the editors December 20, 1989