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THE MODIFIED MELLIN TRANSFORM AND CONVOLUTION

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Abstract

The modified Mellin transform and its inverse on the spaces LG'_α, $\alpha > -1$, as well as the modified Mellin convolution and its properties over these spaces are investigated. The spaces $LG'_\alpha, \alpha > -1$, are inspected through the spaces of Newton's series using an isomorphism between them. Remarks are given on the domain of convergence of some Dirichlet's series. Finally, the modified Mellin convolution is applied in solving an integro-differential equation.

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1. Introduction

In the theory of integral transforms of generalised functions the monograph of Zemanian [11] takes a remarkable place. In this monograph are presented methods for constructing spaces of generalized functions which correspond to the appropriate differential operators and integral transforms. The investigation of transforms is the most effective procedure for solving many problems concerning partial differential equations and functional equations.

Usually, it is the case that inversion formula cannot be performed analytically and then suitable numerical techniques have to be found. There are a number of related methods in use, such as [1], [5].

The Laplace and the Mellin transform are studied and applied in [6] on the space of tempered distributions through the Laguerre expansions of its elements. Some useful results are obtained. The properties of the Mellin transform which concern the differentiation, and multiplication by x are examined. Also given are two inversion formulas for the $\mathcal{M}$-transform which are important in operational calculus. The first inversion formula is a new technique of inverting the Mellin transform using series of Laguerre polynomials. This approach is different from that given in [11] which can be considered as a distributional approach. It needs less operational calculus than the generalised version of Zemanian.

In this paper the generalization of results of [6] for the spaces $LG'_\alpha, \alpha > -1$, are given. This paper is organised as follows:

In the first part we shall give definitions of the spaces involved. Then, we shall define the modified Mellin transform $\mathcal{M}_\alpha$ in LG'_α - spaces for later use. In Section 4 we shall include the definitions of the modified Mellin convolution in the spaces LG'_0 and LG'_α, $\alpha > -1$. A new numerical method for inverting modified Mellin transform is then detailed in Section 5 where the generalised function version of the inverse of the modified Mellin transform is also given. In Section 6 is given the characterization of spaces $LG'_\alpha, \alpha > -1$, by using their isomorphisms with the spaces of Newton's series N_α. Remarks on the convergence of Dirichlet's series are given in Section 7. At the end, we shall give an algorithm for solving an integro-differential equation which uses the theoretical predictions given in Sections 3, 4 and 5.

2. Basic spaces

We shall consider the expansions of the spaces of generalised functions $LG'_\alpha, \alpha > -1$, with respect to the Laguerre orthonormal systems $\ell_{n,\alpha}, \alpha > -1, n \in \mathrm{N}_0 (\mathrm{N}_0 = \mathrm{N} \cup \{0\})$. When $\alpha = 0$ we have the space LG'_0, the space of tempered distributions with supports in $\bar{\mathbf{R}}_+ = [0, \infty)$ ([6]), known as the space S'_+. Basic references for the space LG'_0 are [2], [6], [7], [11]. For the properties of the spaces LG'_α we refer to [2], [7], [8], [9], [11].

We shall repeat some basic properties of the spaces LG'_α, and its most important case LG'_0 ([11]).

Let $\alpha > -1$. The generalised Laguerre orthonormal system in $L^2(\mathbf{R}_+)$ is given by

$$\ell_{n,\alpha}(t) = \tau_n t^{\alpha/2} L_n^{\alpha}(t) e^{-t/2},\ t \in \mathbf{R}_+,$$

where $\tau_n = (\Gamma(n+1)/\Gamma(n+\alpha+1))^{1/2}$, and $L_n^{\alpha}(t)$ are generalised Laguerre polynomials, defined by

$$L_n^{\alpha}(t) = \sum_{m=0}^{n} \begin{pmatrix} n+\alpha \\ n-m \end{pmatrix} \frac{(-t)^m}{m!}, t > 0, n \in \mathbf{N}_0.$$

The functions $\ell_{n,\alpha}(t)$ are eigenfunctions for the operator $\mathcal{R}_\alpha = t^{-\alpha/2} e^{t/2} D t^{\alpha+1} e^{-t} D t^{-\alpha/2} e^{t/2}, \mathcal{R}_\alpha^k = \mathcal{R}_\alpha^k = \mathcal{R}_\alpha(\mathcal{R}_\alpha^{k-1}),\ k \in \mathbf{N}_0, \mathcal{R}_\alpha^0$ is the identity operator, and the corresponding eigenvalues are $\lambda_n = -n, n \in \mathbf{N}_0$.

The spaces LG_α are the spaces of all the smooth functions $\phi \in C^\infty(\mathbf{R}_+)$, such that for every $k \in \mathbf{N}_0$, the seminorms $\| \phi \|_k = \| \mathcal{R}_\alpha^k \phi \|_{L^2} = \left(\int_0^\infty |\mathcal{R}_\alpha^k \phi(t)|^2 dt\right)^{1/2}$, are finite, and for every $n \in \mathbf{N}_0$ and every $k \in \mathbf{N}_0$, $(\mathcal{R}_\alpha^k \phi, \ell_{n,\alpha}) = (\phi, \mathcal{R}_\alpha^k \ell_{n,\alpha})$, where $(\phi, \psi) = \langle \phi, \bar{\psi} \rangle = \int_0^\infty \phi(t)\bar{\psi}(t) dt, \phi, \psi \in L^2(\mathbf{R}_+)$.

The following relation between the spaces LG_α is given in [7].

$$LG_\alpha = x^{\alpha/2} LG_0 = \left\{\psi \in C^\infty(\mathbf{R}_+); \psi = x^{\alpha/2}\phi \text{ for some } \phi \in LG_0\right\}.$$

We shall repeat some equivalent definitions for LG_0:

(i) $$LG_0 = \left\{\psi \in C^\infty(\mathbf{R}_+);\ \sup_{\substack{x \in \mathbf{R}_+ \\ j,k \le r}} |x^k \psi^{(j)}(x)|, r \in \mathbf{N}_0\right\} < \infty \quad [10].$$

(ii) $$LG_0 = \left\{\psi \in C^\infty(\bar{\mathbf{R}}_+);\ \sup_{\substack{x \in \bar{\mathbf{R}}_+ \\ k,j \le r}} |x^k \psi^{(j)}(x)|, r \in \mathbf{N}_0\right\} < \infty \quad [7].$$

Note that $L^2(\mathbf{R}_+) = LG_0'$.

Let $k \in \mathbf{N}_0, \alpha > -1. L_{k,\alpha}$ is the space defined as follows:

$$L_{k,\alpha} = \left\{ \phi = \sum_{n=0}^{\infty} a_{n,\alpha}\ell_{n,\alpha} \in L^2(\mathbf{R}_+);\ \|\psi\|_{k,\alpha} < \infty \right\},$$

where

$$\|\phi\|_{k,\alpha} = \left(|a_{0,\alpha}|^2 + \sum_{n=1}^{\infty} |a_{n,\alpha}|^2 n^{2k} \right)^{1/2};$$

$$L'_{k,\alpha} = \left\{ f = \sum_{n=0}^{\infty} b_{n,\alpha}\ell_{n,\alpha} \ \text{– formal series},\ \|f\|'_k < \infty \right\},$$

where $\|f\|'_k = \left(|b_{0,\alpha}|^2 + \sum_{n=1}^{\infty} |b_{n,\alpha}|^2 n^{-2k} \right)^{1/2}$.

Obviously, spaces $L_{k,\alpha}$ and $L'_{k,\alpha}$ can be defined for $k \in \mathbf{R}$. Note, for $k > 0, L'_{k,\alpha} = L_{-k,\alpha}$.

$$LG_\alpha = \operatorname{proj}\lim_{k\to\infty} L_{k,\alpha},\ LG'_\alpha = \operatorname{ind}\lim_{k\to\infty} L'_{k,\alpha}.$$

The connections between the spaces $L_{k,\alpha}$ and $L_{k,0}, k \in \mathbf{R}$, and consequently between LG_α and LG_0 are given in the next proposition.

Proposition 1. *Let $k \in \mathbf{R}, \alpha > -1$. Then*

$$L_{k,\alpha} = x^{\alpha/2} L_{k,0} = \left\{ \varphi \in C^\infty(\mathbf{R}_+); \varphi = x^{\alpha/2}\phi \text{ for some } \phi \in L_{k,0} \right\}.$$

Proof. Let $\varphi = \sum_{n=0}^{\infty} a_{n,\alpha}\ell_{n,\alpha} \in L_{k,\alpha}$. From ([3], p. 192(39)) $L_n^\alpha(x) = \sum_{m=0}^{n}(1/m!)(\Gamma(m+\alpha)/\Gamma(\alpha))L_{n-m}(x)$, $x \in \bar{\mathbf{R}}_+$, we have

$$x^{-\alpha/2}\varphi = \sum_{n=0}^{\infty} a_{n,\alpha}\tau_n e^{-x/2} L_n^\alpha(x) = \sum_{n=0}^{\infty} a_{n,\alpha}\tau_n e^{-x/2}.$$

$$\cdot \sum_{m=0}^{n} 1/m!(\Gamma(m+\alpha)/\Gamma(\alpha))L_{n-m}(x) =$$

$$= \sum_{m=0}^{\infty} \left(\sum_{n=0}^{\infty} \tau_n (1/n!)(\Gamma(n+\alpha)/\Gamma(\alpha)) a_{n+m,\alpha} \right) \ell_{n,0}(x).$$

Let $b_m = \sum_{n=0}^{\infty} \tau_n(1/n!)(\Gamma(n+\alpha)/\Gamma(\alpha))a_{n+m,\alpha}, n \in \mathbf{N}_0$. We have to prove that $\sum_{m=0}^{\infty} |b_m|^2 m^{2k} < \infty$. By Cauchy's inequality

$$\sum_{m=0}^{\infty} m^{2k} \left(\sum_{n=0}^{\infty} \tau_n(1/n!)(\Gamma(n+\alpha)/\Gamma(\alpha))|a_{n+m,\alpha}|^2 \right) \leq C \sum_{m=0}^{\infty} m^{2k} \sum_{n=0}^{\infty} \tau_n(1/n!)\cdot$$

$$\cdot|a_{n+m,\alpha}|^2 = C \sum_{n=0}^{\infty} \tau_n(1/n!) \sum_{m=0}^{\infty} |a_{n+m,\alpha}|^2 (n+m)^{2k} (m/(n+m))^{2k} < \infty.$$

So we prove if $\varphi \in L_{k,\alpha}$, then $x^{-\alpha/2}\varphi \in L_{k,0}$. By using the formula $L_n(x) = \sum_{m=0}^{n} \frac{1}{m!}(-\alpha)_m L_{n-m}^{\alpha}(x)$, ([3], p.192(4)) similarly as above we can prove if $\phi \in L_{k,0}$ then $x^{\alpha/2}\phi \in L_{k,\alpha}$.
$((-\alpha)_0 = 1, (-\alpha)_m = (-\alpha)(-\alpha+1)1\ldots(-\alpha+m-1), m \in \mathbf{N})$. □

Define the space $\mathcal{H}_\alpha$ as follows:

$$\mathcal{H}_\alpha = \left\{ \phi; \phi = e^{t/2}\psi(t) \text{ for some } \psi \in LG_\alpha \right\}$$

with the topology generalised by the norms $\|\cdot\|_{k,\alpha}, k \in \mathbf{N}_0, \alpha > -1$, defined by

$$\|\phi\|_{k,\alpha} = \sup_{t \in \mathbf{R}_+} t^k |(e^{-t/2} t^{-\alpha/2} \phi(t))^{(\ell)}|.$$

$$l \leq k$$

This sequence of norms defines the convergence structure in $\mathcal{H}_\alpha$.

Let the sequence φ_n belong to $\mathcal{H}_\alpha$. Then $\varphi_n \to 0$ in $\mathcal{H}_\alpha$ iff there exists a sequence ψ_n in LG_α such that $\psi_n \to 0$ in LG_α and $\phi_n = e^{t/2}\psi_n, n \in \mathbf{N}_0$.

Thus, $(\mathcal{H}_\alpha, \|\cdot\|_{k,\alpha}, k \in \mathbf{N}_0)$ is an F-space.

Put $\mathcal{H}_{k,\alpha} = \left\{ e^{t/2}\psi; \psi \in L_{k,\alpha} \right\}, k \in \mathbf{R}$, and transport the convergence structure from $L_{k,\alpha}$ to $\mathcal{H}_{k,\alpha}$.

We have:

$$\mathcal{H}_\alpha = \operatorname{proj} \lim_{k \to \infty} \mathcal{H}_{k,\alpha}, \ \mathcal{H}'_\alpha = \operatorname{ind} \lim_{k \to \infty} \mathcal{H}'_\alpha,$$

$$LG_\alpha = \left\{ e^{-t/2}\varphi; \varphi \in \mathcal{H}_\alpha \right\},$$

A duality argument gives the following proposition:

Proposition 2. *We have* $\mathcal{H}'_\alpha = \left\{e^{-t/2}f; f \in LG'_\alpha\right\}$;

$$L'_{k,\alpha} = \left\{e^{-t/2}f; f \in \mathcal{H}'_{k,\alpha}\right\}, k \in \mathbf{N}_0.$$

Proof. If $f \in LG'_\alpha$ then $e^{-t/2}f \in \mathcal{H}'_\alpha$ and conversely, via the dual pairing

$$_{\mathcal{H}'_\alpha}\langle e^{-t/2}f(t), \phi(t)\rangle_{\mathcal{H}_\alpha} = _{LG'_\alpha}\langle f(t), e^{-t/2}\phi(t)\rangle_{LG_\alpha} =$$

$$= _{LG'_0}\langle t^{\alpha/2}f(t), e^{-t/2}t^{-\alpha/2}\phi(t)\rangle_{LG_0}, \phi \in \mathcal{H}_\alpha.$$

The second assertion follows similarly. □.

Note that for $Res > k, t \to t^{s-1}, t \in \mathbf{R}_+$, belongs to $\mathcal{H}_{k,0}$, and $t \to t^{\alpha/2+s-1}$, $t \in \mathbf{R}_+$, belongs to $\mathcal{H}_{k,\alpha}, k \in \mathbf{R}, (\mathcal{H}_{k,\alpha} = x^{\alpha/2}\mathcal{H}_{k,0})$.

3. Modified Mellin transform

We present the modified Mellin transform in $LG'_\alpha, \alpha > -1$.

Let $\varphi_{s,\alpha}(t) = (1/\Gamma(s+\alpha))t^{\alpha/2}e^{-t/2}t^{s-1}, t \in \mathbf{R}_+$,
and

$$D_{k,\alpha/2} = \{s \in \mathbf{C}; Res > k + (1-\alpha)/2\},$$

$k \in \mathbf{R}$. Then the mapping

(1) $$s \to (\mathcal{M}_\alpha f)(s) = \langle f(t), \varphi_{s,\alpha}(t)\rangle, s \in D_{k,\alpha/2},$$

is the modified Mellin transform (hereafter refered to as MMT) of an $f \in L'_{k,\alpha}, k \in \mathbf{R}$.

The expansion of $\varphi_{s,\alpha}$ with respect to the generalised Laguerre orthonormal system is given by $\varphi_{s,\alpha} = \sum_{n=0}^{\infty} a_{n,\alpha}\ell_{n,\alpha}$, where $a_{n,\alpha} = \langle\varphi_{s,\alpha}(t), \ell_{n,\alpha}(t)\rangle = (-1)^n \binom{s-1}{n} \tau_n, n \in \mathbf{N}_0$ ([1], p.32).

Since $|a_{n,\alpha}| < Cn^{-(Res+\alpha/2)}, n > n_0(s,\alpha), C = C(s,\alpha)$ [6], we have $\varphi_{s,\alpha} \in L^2(\mathbf{R}_+)$ if $Res > (1-\alpha)/2$ and $\varphi_{s,\alpha} \in L_{k,\alpha}$ if $Res > (1-\alpha)/2 + k$.

Proposition 3. *The mapping* $s \to \varphi_{s,\alpha}$ *is a holomorphic mapping from* $D_{k,\alpha/2}$ *into* $L_{k,\alpha}, k \in \mathbf{N}_0$, $\alpha > -1$.

Proof. Since $\mathcal{R}_\alpha\varphi_{s,\alpha} = (s-1)(\varphi_{s-1,\alpha}, -\varphi_{s,\alpha})$, by similar arguments as in [6] it is easy to show the assertion of Proposition 3. □

Similarly as in [6] we have

Proposition 4. *The mapping* $s \to \varphi_{s,\alpha}$ *from* $D_{k,\alpha/2}$ *into* $L_{k,\alpha}, k \in \mathbf{R}, \alpha > -1$, *is holomorphic.*

Now we shall introduce the Null-Mellin transform in $\mathcal{H}'_\alpha$ by following the method of Zemanian. ([11], Ch.4.).

Since for $Res > k$,

$$\mathbf{R} \ni t \to t^{\alpha/2}t^{s-1} \in \mathcal{H}_{k,\alpha}, t \in \mathbf{R}_+, (k \in \mathbf{R})$$

the definition which is to follow is correct. The mapping

$$s \to (\tilde{\mathcal{M}}_\alpha f)(s) = \langle f(t), t^{\alpha/2}t^{s-1}\rangle, Res > k,$$

for an $f \in \mathcal{H}_{k,\alpha}$ is Null-Mellin transform. When $\alpha = 0$ we obtain the Zemanian Mellin transform $(\tilde{\mathcal{M}}_0 f)(s)$ (with our modification).

The relation between these two transforms is given in

Proposition 5. *For an* $f \in LG'_\alpha$, Res $> k$,

$$(\tilde{\mathcal{M}}_\alpha(e^{-t/2}f))(s) = \Gamma(s+\alpha)(\tilde{\mathcal{M}}_\alpha(f))(s)$$

Proof. If $f \in LG'_\alpha$ then $e^{-t/2}f(t) \in \mathcal{H}'_{k,\alpha}$, for some $k \in \mathbf{R}$, and for $Res > k$

$$(\tilde{\mathcal{M}}_\alpha(e^{-t/2}f))(s) = \langle e^{-t/2}f(t), t^{\alpha/2}t^{s-1}\rangle =$$
$$= \langle f(t), e^{-t/2}t^{\alpha/2}t^{s-1}\rangle = \Gamma(s+\alpha)(\tilde{\mathcal{M}}_\alpha(f))(s). \quad \square$$

The properties of MMT are given in

Proposition 6. *Let* $s \in D_{k,\alpha/2}, k \in \mathbf{R}$, *and* $f = \sum_{n=0}^\infty b_{n,\alpha}\ell_{n,\alpha} \in$ $\in L'_{k,\alpha}$. *Then*

$$\text{(i)}(\mathcal{M}_\alpha(\ell_{n,\alpha}))(s) = (-1)^n \binom{s-1}{n} \tau_n, n \in \mathbf{N}_0, (\textit{we take} \binom{s-1}{0} = 1);$$

(ii) $$(\mathcal{M}_\alpha f)(s) = \sum_{n=0}^{\infty} b_{n,\alpha}(-1)^n \binom{s-1}{n} \tau_n.$$

This series converges absolutely and uniformly in $D_{k,\alpha/2}$.

(iii) *The following holds :*

$$(\mathcal{M}_\alpha(Df))(s+1) = -(s+\alpha/2)/(s+\alpha)(\mathcal{M}_\alpha f)(s) + \tfrac{1}{2}(\mathcal{M}_\alpha f)(s+1);$$
$$(\mathcal{M}_\alpha(tf))(s-1) = (s+\alpha-1)(\mathcal{M}_\alpha f)(s);$$
$$\left(\mathcal{M}_\alpha(te^{-t/2}(e^{-t/2}f)')\right)(s) = -(s+\alpha/2)(\mathcal{M}_\alpha f)(s);$$
$$\left(\mathcal{M}_\alpha(te^{-t/2}(e^{-t/2}f)')\right)(s) = -(s-1+\alpha/2)/(s+\alpha-1)(\mathcal{M}_\alpha f)(s-1);$$
$$(\mathcal{M}_\alpha(\mathcal{R}_\alpha f))(s) = (s-1)((\mathcal{M}_\alpha f)(s-1) - (\mathcal{M}_\alpha f)(s)).$$

Proof. (i), (ii) folows from ([1].p.32). Since

$$\left|\binom{s-1}{n}\right| < cn^{-Res}, \tau_n < n^{-\alpha/2}, n \in \mathbf{N}_0,$$

this series converges in $D_{k,\alpha/2}$. (iii) follows from the definition of the MMT. □

Remark. Let us note that $(\mathcal{M}_\alpha f)$ is also defined for $s \in \mathbf{N}_0$ and $s \leq k + (1-\alpha)/2$ by

$$(\mathcal{M}_\alpha f)(s) = \sum_{n=0}^{s-1} b_{n,\alpha} \binom{s-1}{n} (-1)^n \tau_n \left(\text{we take } \binom{0}{0}, \binom{-1}{0} = 1\right)$$

4. Modified Mellin convolution (MMC)

For generalized functions which are Mellin's transformable ([11]) there exists the Mellin convolution ([11]).

In this section we shall give for MMT the analogues for the theorems of Zemanian ([11], p.151-156).

First, we consider the modified convolution over the elements in the spaces LG_0' and LG_α' separately.

Proposition 7. *Let* $\Theta \in \mathcal{H}_0$. *Then for* $g \in \mathcal{H}'_0$,

$$\phi(x) = \langle g(y), \Theta(x \cdot y)\rangle, \quad x \in \mathbf{R}_+,$$

belongs to $\mathcal{H}_0$.

The modified Mellin convolution is defined as follows.
Let $f, g \in LG'_0$. Then, by

$$(2) \qquad \langle f \vee g, \Theta\rangle = \langle f(x)e^{-x/2}, \langle g(y)e^{-y/2}, \Theta(x \cdot y)\rangle\rangle, \Theta \in \mathcal{H}_0,$$

the modified Mellin convolution (MMC) of f and g is defined.

Proposition 8. *For* $f, g \in LG'_0, f \vee g \in \mathcal{H}'_0$.

Proof. It follows from Proposition 7, because $e^{-x/2}f, e^{-x/2}g \in \mathcal{H}'_0$. □

Proposition 9. *Let* $f \in LG'_0, \ g \in \mathcal{D}(\mathbf{R}_+)$. *Then*

$$(3) \qquad f \vee g = \langle f(x)e^{-x/2}, e^{-y/2x}(1/x)g(y/x)\rangle, y \in \mathbf{R}_+,$$

in $\mathcal{D}'(\mathbf{R}_+)$. *(This means that for every* $\Theta \in \mathcal{D}(\mathbf{R}_+)$,

$$\langle f \vee g, \Theta\rangle = \langle\langle f(x)e^{-x/2}, e^{-y/2x}(1/x)g(y/x)\rangle, \Theta(x)\rangle.$$

Proof. By substitution of variables (2) can be expressed as

$$\langle f \vee g, \Theta\rangle = \langle f(x)e^{-x/2}, \ \langle e^{-y/2x}\Theta(y), (1/x)g(y/x)\rangle\rangle.$$

The supports of $e^{-y/2x}\Theta(y)$ and $(1/x)g(y/x)$ have the intersection in a compact subset of the open quadrant $\mathbf{R}_+ \times \mathbf{R}_+$.

Then (2) has the form

$$\langle f(x)e^{-x/2}, \langle\Theta(y)e^{-y/2x}, (1/x)g(y/x)\rangle\rangle =$$
$$= \langle f(x)e^{-x/2} \otimes \Theta(y), (1/x)g(y/x)e^{-y/2x}\rangle,$$

where $\otimes$ denotes the direct product.

From the commutativity of the direct product we have

$$\langle f \vee g, \Theta\rangle = \langle\Theta(y), \langle f(x)e^{-x/2}, e^{-y/2x}(1/x)g(y/x)\rangle\rangle,$$

which implies the assertion. □

Proposition 10. *Let $f, g \in LG_0'$. Then, (for s belonging to the domain which depends on f and g)*

$$\tilde{\mathcal{M}}_0(f \vee g)(s) = (\Gamma(s))^2(\mathcal{M}_0 f)(s)(\mathcal{M}_0 g)(s). \tag{4}$$

$$\mathcal{M}_0(e^{-x/2}(f \vee g))(s) = \Gamma(s)(\mathcal{M}_0 f)(s)(\mathcal{M}_0 g)(s)$$

Proof. $\tilde{\mathcal{M}}_0(f \vee g)(s) = \langle f \vee g, x^{s-1}\rangle = \langle f(x)e^{-x/2},$

$$\langle g(y)e^{-y/2}, (xy)^{s-1}\rangle\rangle = \langle f(x)e^{-x/2}, x^{s-1}\rangle\langle g(y)e^{-y/2}, y^{s-1}\rangle =$$

$$= (\Gamma(s))^2(\mathcal{M}_0 f)(s)(\mathcal{M}_0 g)(s). \ \square$$

Proposition 11. *Let $f, g \in LG_0'$, then*

$$\tilde{\mathcal{M}}_0(f \vee g)(s) = \tilde{\mathcal{M}}_0(e^{-t/2}f)(s)\tilde{\mathcal{M}}_0(ge^{-t/2})(s). \tag{5}$$

Proof. It follows from (4) and Proposition 5. $\square$

The MMC has the following properties over the generalized functions in LG_0':

i. Commutativity: $\tilde{\mathcal{M}}_0(f \vee g)(s) = \tilde{\mathcal{M}}_0(g \vee f)(s)$.
ii. Associativity: $\tilde{\mathcal{M}}_0((f \vee g) \vee h)(s) = \tilde{\mathcal{M}}_0(f \vee (g \vee h))(s)$.

We shall introduce the MMC in the spaces $LG_\alpha', \alpha > -1$. The arguments are very similar to those for the space LG_0' given in the previous part. Here we shall only quote the necessary changes to be made in the arguments given there.

Using the relations between the spaces: $LG_\alpha' = x^{-\alpha/2}LG_0'$ [7], and $\mathcal{H}_\alpha = x^{\alpha/2}\mathcal{H}_0$ we have

Proposition 12. *Let $\Theta \in \mathcal{H}_\alpha$. Then for $g \in \mathcal{H}_\alpha'$,*

$$\phi(x) = \langle g(y), \Theta(xy)x^{-\alpha}\rangle, \ x \in \mathbf{R}_+,$$

belongs to $\mathcal{H}_0$.

Let $f, g \in LG_\alpha'$. Then for $\Theta \in \mathcal{H}_\alpha$,

$$\langle f \vee_\alpha g, \Theta\rangle = \langle f(x)e^{-x/2}x^{\alpha/2}, \langle g(y)e^{-y/2}y^{\alpha/2}, \Theta(xy)(xy)^{-\alpha/2}\rangle\rangle \tag{6}$$

is the MMC over the functions in LG_α'.

Proposition 13. *$f \vee_\alpha g \in \mathcal{H}'_\alpha$, for $f, g \in LG'_\alpha$.*

Proposition 14. *Let $f \in LG'_\alpha, g \in \mathcal{D}(\mathbf{R}_+)$. Then*

$$f \vee_\alpha g = \langle f(x)e^{-x/2}, g(y/x)(1/x)e^{-y/2x}\rangle \tag{7}$$

$y \in \mathbf{R}_+$, in $\mathcal{D}'(\mathbf{R}_+)$, i. e. for all $\Theta \in \mathcal{D}(\mathbf{R}_+)$,

$$\langle f \vee_\alpha g, \Theta\rangle = \langle f(x)e^{-x/2}, \langle g(y/x)(1/x)e^{-y/2}, \Theta(y)\rangle\rangle.$$

Proposition 15. *For $f, g \in LG'_\alpha$ (and s in a suitable domain)*

$$\tilde{\mathcal{M}}_\alpha(f \vee_\alpha g)(s) = (\Gamma(s+\alpha))^2(\mathcal{M}_\alpha f)(s)(\mathcal{M}_\alpha g)(s).$$

$$\mathcal{M}_\alpha(e^{x/2}(f \vee_\alpha g))(s) = \Gamma(s+\alpha)(\mathcal{M}_\alpha f)(s)(\mathcal{M}_\alpha g)(s)$$

In particular when $\alpha = 0$ we obtain (4).

Proposition 16. *Let $f, g \in LG'_\alpha$. Then,*

$$\tilde{\mathcal{M}}_\alpha(f \vee_\alpha g)(s) = \tilde{\mathcal{M}}_\alpha(fe^{-t/2})(s)\tilde{\mathcal{M}}_\alpha(ge^{-t/2})(s).$$

For $\alpha = 0$ we have (5).

The MMC over the elements in LG'_α has the properties of commutativity and associativity.

5. Inversion of Transforms

We shall give two inversion formulas for MMT. Firstly, we give the generalised function version of the inverse of MMT. Secondly, we give a numerical inversion of MMT using the series of generalised Laguerre polinomials appropriate for applications.

If $f \in LG'_\alpha$, then there are $m \in \mathbf{N}_0$ and a continuous function of slow growth F with supp$F \subset \bar{\mathbf{R}}_+$ such that

$$f = x^{-\alpha/2}F^{(m)}$$

in the sense of dual pairing

$$ {}_{LG'_\alpha}\langle f,\varphi\rangle_{LG_\alpha} = \langle x^{-\alpha/2}F^{(m)},\varphi\rangle = {}_{LG'_0}\langle F^{(m)}, x^{-\alpha/2}\varphi\rangle_{LG_0}, $$

where $\varphi \in LG_\alpha$ and $x^{-\alpha/2}\varphi \in LG_0$. ([7]).

Let $F, F', \dots F^{(m)} \in L'_{r,0} r \geq 0$. Then, $f \in L'_{r,\alpha}$ for $s \in D_{r+m,\alpha/2}$.

$$ \varphi_{s-m,\alpha} = t^{\alpha/2}\frac{\Gamma(s-m)}{\Gamma(s-m+\alpha)}\varphi_{s-m,0} \in L_{r,\alpha}. $$

We choose $s \in D_{r+m,\alpha/2}$.

$$ (\mathcal{M}_\alpha f)(s) = \langle f(t), \varphi_{s,\alpha}(t)\rangle = \langle F^{(m)}, t^{-\alpha/2}\varphi_{s,\alpha}\rangle = $$

$$ = \frac{(-1)^m}{\Gamma(s+\alpha)}\langle F, (e^{-t/2}t^{s-1})^{(m)}\rangle = $$

$$ = \frac{(-1)^m}{\Gamma(s+\alpha)}\langle F(t), \sum_{j=0}^{m}\binom{m}{j}\left(-\frac{1}{2}\right)^{m-j}(s-1)\dots(s-j)t^{s-j-1}e^{-t/2}\rangle = $$

$$ = \frac{(-1)^m}{\Gamma(s+\alpha)}(-2)^{-m}\sum_{j=0}^{m}\binom{m}{j}(-2)^j(s-1)\dots(s-j)\int_0^\infty t^{s-j-1}e^{-t/2}F(t)dt. $$

Thus, we have

$$ (\mathcal{M}_\alpha f)(s) = 2^{-m}\frac{\Gamma(s)}{\Gamma(s+\alpha)}\sum_{j=0}^{m}\binom{m}{j}(-2)^j\frac{\tilde{\mathcal{M}}_0(Fe^{-x/2})(s-j)}{\Gamma(s-j)}. $$

Proposition 17. *Let $f, \in LG'_\alpha$ and $(\mathcal{M}_\alpha f)$ be defined on $D_{k,\alpha/2}, k \geq 0$, then there is $\sigma_0 > k + (1-\alpha)/2$ such that for all $\sigma > \sigma_0$,*

$$ \langle\frac{e^{x/2}}{2\pi i}\int_{\sigma-ir}^{\sigma+ir}\Gamma(s+\alpha)(\mathcal{M}_\alpha f)(s)x^{-s}ds, x^{-\alpha/2}\phi(x)\rangle \to $$

$$ \to \langle f,\phi\rangle, r\to\infty, \phi \in \mathcal{D}(0,\infty). $$

Proof. Since $f \in LG'_\alpha$ then $f = x^{-\alpha/2}F^{(m)}$, where supp $F \subset \bar{\mathbf{R}}_+$ and F is slowly increasing.

Take $\sigma_0 \in \mathbf{R}_+$ such that for some $k \geq 0$, $Res > \sigma_0, \varphi_{s-m,0} \in L_{k,0}, F, F', F'',$ $\ldots F^{(m)} \in L'_{k,0}$; then $\varphi_{s-m,\alpha/2} \in L_{k,\alpha}, f \in L'_{k,\alpha}$. If $Res > \sigma_0$ then for $\varepsilon > 0$

$$\frac{1}{2\pi i}\int_{\sigma-ir}^{\sigma+ir} \tilde{\mathcal{M}}_0(F(x)e^{-x/2})(s-j)x^{-(s-j)}ds \overset{L^2(\varepsilon,\infty)}{\to} F(x)e^{-x/2}, r \to \infty$$

(see [6]). We have

$$\lim_{r\to\infty}\langle\frac{e^{x/2}}{2\pi i}\int_{\sigma-ir}^{\sigma+ir}\Gamma(s+\alpha)(\mathcal{M}_\alpha f)(s)x^{-s}ds, x^{-\alpha/2}\phi(x)\rangle =$$

$$\lim_{r\to\infty}\langle 2^{-m}\sum_{j=0}^{m}\left(\begin{array}{c} m \\ j \end{array}\right)\frac{(-2)^j}{2\pi i}\int_{\sigma-ir}^{\sigma+ir}(s-1)\ldots(s-j)\tilde{\mathcal{M}}_0(Fe^{-x/2})(s-j)x^{-s}ds,$$

$$e^{x/2}x^{-\alpha/2}\phi(x)\rangle =$$

$$= \lim_{r\to\infty} 2^{-m}\sum_{j=0}^{m}\left(\begin{array}{c} m \\ j \end{array}\right)\frac{(-2)^j}{2\pi i}(-1)^j < \int_{\sigma-ir}^{\sigma+ir}\tilde{\mathcal{M}}_0(Fe^{-x/2}(s-j))x^{-(s-j)}ds)^{(j)},$$

$$e^{x/2}x^{-\alpha/2}\phi(x)\rangle =$$

$$= \lim_{r\to\infty} 2^{-m}\sum_{j=0}^{m}\left(\begin{array}{c} m \\ j \end{array}\right)\frac{(-2)^j}{2\pi i} < \int_{\sigma-ir}^{\sigma+ir}\tilde{\mathcal{M}}_0(Fe^{-x/2})(s-j)x^{-(s-j)}ds,$$

$$(e^{x/2}x^{-\alpha/2}\phi(x))^{(j)}\rangle = \lim_{r\to\infty} 2^{-m}\sum_{j=0}^{m}\left(\begin{array}{c} m \\ j \end{array}\right)(-2)^j < F(x)e^{-x/2},$$

$$(e^{x/2}x^{-\alpha/2}\phi(x))^{(j)}\rangle = \sum_{j=0}^{m}\left(\begin{array}{c} m \\ j \end{array}\right)\left(\frac{1}{2}\right)^{m-j}\langle(e^{-x/2}F(x)e^{x/2})^{(m)}, x^{-\alpha/2}\phi(x)\rangle$$

$$= \langle F^m(x), x^{-\alpha/2}\phi(x)\rangle = \langle x^{-\alpha/2}F^{(m)}(x), \phi(x)\rangle = \langle f, \phi\rangle.$$

and so the Proposition is proved. □

Now, we shall give the numerical inversion formula.

Proposition 18 *Let* $s \in D_{k,\alpha/2}, k \in \mathbf{R}_+$, *and* $f = \sum_{n=0}^{\infty} b_{n,\alpha}\ell_{n,\alpha} \in L'_{k,\alpha}$. *Then the inverse of MMT is given by*

$$(8) \qquad f(t) = \sum_{n=0}^{\infty}(-1)^n\Delta^n(\mathcal{M}_\alpha f)(1)(1/\tau_n)\ell_{n,\alpha},$$

where Δ^n *are the finite differences,* $n \in \mathbf{N}_0$, *where, for* $s \in \mathbf{N}_0, s \leq k + (1-\alpha)/2$ *the values of* $(\mathcal{M}_\alpha f)(s)$ *are determined in the Remark at the end of Section 3.*

Proof. From Proposition 6 (ii) we have

$$(-1)^n b_{n,\alpha}\tau_n = \Delta^n(\mathcal{M}_\alpha f)(1) =$$

$$= \sum_{m=0}^{n}(-1)^{n-m}\begin{pmatrix} n \\ m \end{pmatrix}(\mathcal{M}_\alpha f)(m+1), n \in \mathbf{N}_0, \ ([4], p.325)$$

and

$$b_{n,\alpha} = (-1)^n(1/\tau_n)\Delta^n(\mathcal{M}_\alpha f)(1). \ \square$$

6. The space of N_α-transforms, $\alpha > -1$.

Denote by N_α the space of all the Newton series of the form

$$F_\alpha \in N_\alpha \Leftrightarrow F_\alpha = \sum_{n=0}^{\infty}(-1)^n b_{n,\alpha}\tau_n \begin{pmatrix} s-1 \\ n \end{pmatrix},$$

where $\sum_{n=1}^{\infty}|b_{n.\alpha}|^2 n^{-2k} < \infty$ for some $k \in \mathbf{R}$. The abscisa of the common and absolute convergence of F are denoted by λ_F and μ_F respectively.

Proposition 19. *$\mathcal{M}_\alpha$ transform is an algebraic isomorphism between the spaces LG'_α and N_α.*

Proof. It follows from Proposition 6. $\square$

In particular, when $\alpha = 0$ we have the mapping $LG'_0 \leftrightarrow N_0$. (See [6]).

By using the connections between Newton's and Dirichlet's series we obtain:

$$\mu_{F\alpha} = \overline{\lim_{n\to\infty}} \, ln \sum_{k=0}^{n}|b_{k,\alpha}\tau_k|/ln\, n, \ \ \mu_{F\alpha} \geq 0,$$

and

$$\mu_{F\alpha} = \overline{\lim_{n\to\infty}} \, ln \sum_{k=0}^{n}|b_{k,\alpha}\tau_k|/ln\, n, \ \ \mu_{F\alpha} < 0.$$

All of the above enables us to inspect the spaces LG'_α through the spaces of all the Newton series.

Proposition 20. *If $f \in LG'_\alpha$ and $F = (\mathcal{M}_\alpha f)$ is defined by (1), then the following holds:*

(i) $$f \in L'_{k,\alpha} \Rightarrow k \geq \mu_{F\alpha} + (\alpha - 1)/2;$$

(ii) $$f \notin L'_{k,\alpha} \Rightarrow k \leq \mu_{F\alpha} + \alpha/2;$$

(iii) $$k > \mu_{F\alpha} + \alpha/2 \Rightarrow f \in L'_{k,\alpha};$$

(iv) $$k < \mu_{F\alpha} + (\alpha - 1)/2 \Rightarrow f \notin L'_{k,\alpha}.$$

Proof. All the above follows from the fact that if $f \in L'_{k,\alpha}$ then $s \in D_{k,\alpha/2}$ and $(\mathcal{M}_\alpha f)(s)$ converges when $Res > k + (1-\alpha)/2$. □

For more details see [6].

7. Remarks on Dirichlet's series

Let

(9) $$f(z) = \sum_{p=0}^{\infty} \frac{a_p}{(p+1)^z}, Re z > \mu,$$

be Dirichlet's series whose abscisa of absolute convergence is $\mu \in \mathbf{R}$ and let $\sigma > \mu$.

Consider the formal series

(10) $$g(x) = \sum_{p=0}^{\infty} \frac{a_p e^{-(p+1/2)x} x^{\alpha/2}}{(p+1)^{\sigma-1-\alpha}}, x > 0.$$

Put $b_{n,\alpha} = \langle g(x), \ell_{n,\alpha}(x) \rangle$. Then by using ([1], p.9) we obtain

$$b_{n,\alpha} = \tau_n \sum_{p=0}^{\infty} \frac{a_p}{(p+1)^{\sigma-1}} \int_0^{+\infty} e^{-(p+1)x} x^\alpha L_n^\alpha(x) dx =$$

$$= \sum_{p=0}^{\infty} \frac{a_p}{(p+1)^\sigma} (1/\tau_n) \left(\frac{p}{p+1} \right)^n.$$

Since, $1/\tau_n \sim n^{\alpha/2}, n \to \infty$,we have

$$|b_{n,\alpha}| < C n^{\alpha/2} \text{ and } g(x) \in L'_{r,\alpha}, r > (1+\alpha)/2.$$

MMT is then

$$(11) \qquad (\mathcal{M}_\alpha g)(s) = \sum_{p=0}^{\infty} \frac{a_p}{(p+1)^{\sigma-1+s}}, Res > r + ((1-\alpha)/2)$$

So with $z = \sigma + s - 1$ we get that $f(z) = (\mathcal{M}_\alpha g)(z - \sigma + 1)$ converges absolutely for $Rez{-}\sigma + 1 > r + (1-\alpha)/2$ i.e. for $Rez > \mu + r - (1+\alpha)/2$. If $g \in L'_{r,\alpha}, r \leq (1+\alpha)/2$, and $f(z)$ converges absolutely for $Rez > \mu$, then $f(z)$ converges absolutely for $Rez > \mu + r - (1+\alpha)/2$. This result generalizes the corresponding one from [1].

8. Solving integro-differential equations via MMC

In this Section we shall give an application of the inversion formula given in Proposition 18. and the operational calculus from Sections 3. and 4. in order to show its possibilities from the numerical point of view.

Consider the equation

$$(12) \qquad xf'(x) + 1/x\, e^{x/2} \int_{x/2}^{x} e^{-t/2} f(t)dt = h(x)$$

along the initial condition $\int_0^\infty f(t)e^{-t/2}dt = A$, where $h \in LG'_0$ is known, and A is a given constant.

For appropriate h the solution of (12) belongs to $L^2(\mathbf{R}_+)$.

Assume that this holds. In solving this equation we use the following

Proposition 21. *If $f, g \in L^2(\mathbf{R}_+)$ and suppg is a compact subset of $\mathbf{R}_+$, then*

$$(13) \qquad (f \vee g)(x) = \int_0^\infty \frac{f(t)e^{-t/2}e^{-x/2t}}{t} g(x/t)dt, x \in \mathbf{R}_+$$

Proof. (13) follows from Lebesque's theorem and Proposition 9. □

Setting

$$g(t) = \frac{e^{t/2}}{t} H(t-1/2)H(1-t), t \in \mathbf{R},$$

(H is Heaviside's function) equation (12) gets the equivalent convolution form

$$xf'(x) + e^{x/2}(f \vee g) = h(x), x \in \mathbf{R}_+. \tag{14}$$

Applying MMT($\mathcal{M}_0$) from Proposition 10. and 6. we obtain the difference equation

$$-sF(s) + F(s+1)/2 + \Gamma(s)F(s)G(s) = H(s), \tag{15}$$

where

$$(\mathcal{M}_0 f)(s) = F(s), (\mathcal{M}_0 g)(s) = G(s), (\mathcal{M}_0 h)(s) = H(s).$$

Since $(\mathcal{M}_0 f)(1) = F(1) = A$, one can find all the coefficients of $f(x), n \geq 2$, in the simplest case of the numerical inversion formula (8):

$$f(x) = \sum_{n=0}^{\infty}(-1)^n \Delta^n F(1)\ell_{n,0} \tag{16}$$

Then, (16) is the solution of our equation (12).

Note, if $f \overset{L^2}{=} \sum_{n=0}^{\infty} a_{n,0}\ell_{n,0}$ then

$$\left\| f - \sum_{n=0}^{n_0} a_{n,0}\ell_{n,0} \right\|_{L_2} = \sum_{n=n_0+1}^{\infty} |a_{n,0}|^2.$$

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REZIME

MODIFIKOVANA MELINOVA TRANSFORMACIJA I KONVOLUCIJA

Uvedeni su pojmovi modifikovane Melinove transformacije i konvolucije na prostorima $LG'_\alpha, \alpha > -1$. Osobine ovih prostora u odnosu na navedene pojmove omogućavaju odgovarajući operacioni račun koji je na kraju rada i primenjen na rešavanje jedne integro-diferencijalne jednačine.

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