

## THE APPROXIMATE SOLUTION OF A NONHOMOGENEOUS PARTIAL DIFFERENTIAL- DIFFERENCE EQUATION

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### Abstract

In the paper the approximate solution of a partial differential-difference equation is constructed using the two dimensional operational calculus introduced by T. Ogata ([4]) and the results of J. Wloka [8] for ordinary differential-difference equations.

In section 4, an estimate is given for the error of approximation in the space  $\mathcal{F}_0$  of the Mikusiński operator field  $\mathcal{F}$  and also obtained is that the sequence of approximate solutions  $\{x_n\}$  converges to the exact solution in the convergence type I' (observed by T. Boehme [1] and J. Burzyk [2]). If the exact and the approximate solutions belong to the space  $\mathcal{L}$  of locally integrable functions, then it is obtained that the sequence of the approximate solutions converges to the exact solution in  $\mathcal{L}$ .

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### 1. The form of the exact solution

We consider the linear partial nonhomogeneous differential-difference equation with constant coefficients

$$(1) \quad \sum_{\mu=0}^m \sum_{\nu=0}^n \alpha_{\mu,\nu} \frac{\partial^{\mu+\nu} x(\lambda, t)}{\partial \lambda^\mu \partial t^\nu} - \sum_{\mu=0}^{m_1} \sum_{\nu=0}^{n_1} \beta_{\mu,\nu} \frac{\partial^{\mu+\nu} x(\lambda, t-\tau)}{\partial \lambda^\mu \partial t^\nu} = \phi(\lambda, t),$$

where  $0 \leq \lambda \leq \lambda_0$ ,  $0 \leq t < \infty$ ,  $\tau > 0$ ,  $x(\lambda, t) = 0$ ,  $t < 0$ , and  $\phi(\lambda, t)$  is a continuous function. Equation (1) corresponds in the field of Mikusiński operators,  $\mathcal{F}$ , to the equation

$$(2) \quad \sum_{\mu=0}^m a_{\mu}(s)x^{(\mu)}(\lambda) - e^{-s\tau} \sum_{\mu=0}^{m_1} b_{\mu}(s)x^{(\mu)}(\lambda) = f(\lambda),$$

where

$$(3) \quad a_{\mu}(s) = \sum_{\nu=0}^n \alpha_{\mu,\nu} s^{\nu}, \quad \mu = 0, 1, 2, \dots, m,$$

$$(4) \quad b_{\mu}(s) = \sum_{\nu=0}^{n_1} \beta_{\mu,\nu} s^{\nu}, \quad \mu = 0, 1, 2, \dots, m_1,$$

and

$$f(\lambda) = \{\phi(\lambda, t)\} + \sum_{k=0}^{n-1} s^{n-k-1} \sum_{\mu=0}^m \sum_{\nu=0}^k \alpha_{\mu, n-k+\nu} \frac{\partial^{\mu+\nu} x(\lambda, 0)}{\partial \lambda^{\mu} \partial t^{\nu}} - e^{-s\tau} \sum_{k=0}^{n_1-1} s^{n_1-k-1} \sum_{\mu=0}^{m_1} \sum_{\nu=0}^k \beta_{\mu, n_1-k+\nu} \frac{\partial^{\mu+\nu} x(\lambda, 0)}{\partial \lambda^{\mu} \partial t^{\nu}}.$$

In the field of Mikusiński operators,  $\mathcal{F}$ ,  $s$  is the differential operator,  $l$  is the integral operator, it holds  $l = s^{-1}$ , and  $e^{-s\tau}$  is the translation operator defined by

$$e^{-s\tau} f = \left\{ \begin{array}{ll} 0, & t \leq \tau \\ f(t - \tau), & t > \tau \end{array} \right\}.$$

The expressions

$$\frac{\partial^{\mu+\nu} x(\lambda, 0)}{\partial \lambda^{\mu} \partial t^{\nu}}, \quad \mu = 0, \dots, M, \quad \nu = 0, \dots, N - 1,$$

where  $M = \max(m, m_1)$  and  $N = \max(n, n_1)$  are given as the appropriate conditions for the equation (1). Let us take the conditions of equation (2) as

$$(5) \quad x^{(\mu)}(0) = d_{\mu}, \quad \mu = 0, \dots, M - 1,$$

(which certainly have the corresponding form as the conditions for equation (1)).

Let  $\mathcal{K}$  be the algebraically closed subfield of the field  $\mathcal{F}$  which consists of elements of the form

$$p = e^{-s\tau} \sum_{i=i_0}^{\infty} h_i t^{\alpha i - \beta}, \quad i_0 > -\infty, \quad \tau \geq 0,$$

and  $h_i$ ,  $i = i_0, i_0 + 1, \dots$ , are complex numbers while  $\alpha$  and  $\beta$  are rational numbers, such that  $\alpha > 0$ ,  $\beta \leq -1$ .

Let  $\mathcal{A}$  ([4]) be the module of the formal power series of a variable  $\lambda$  with coefficients in  $\mathcal{K}$ , with the elements given by

$$(6) \quad P(\lambda) = \sum_{\nu=0}^{\infty} p_{\nu} \lambda^{\nu},$$

where  $p_{\nu}$ ,  $\nu = 0, 1, 2, \dots$ , belong to  $\mathcal{K}$ . Multiplication in  $\mathcal{A}$  can be defined as

$$(7) \quad PQ(\lambda) = \sum_{\rho=0}^{\infty} \left( \sum_{\rho=\mu+\nu} \frac{\nu! \mu!}{(\rho+1)!} p_{\nu} q_{\mu} \right) \lambda^{\rho+1},$$

where  $P(\lambda)$  is given by (6) and

$$Q(\lambda) = \sum_{\mu=0}^{\infty} q_{\mu} \lambda^{\mu}, \quad q_{\mu} \in \mathcal{K}.$$

By the usual addition and multiplication given by (7), the set  $\mathcal{A}$  forms an integral domain without a unit element.

The quotient field of the ring  $\mathcal{A}$  introduced by T. Ogata in [4] is the space of operators  $Q(\mathcal{A})$ . The following operators will be used mostly in the sequel:

- the integral operators  $L = \{1\}$ ;
- the differential operators  $S = 1/L$ ;
- numerical operators, or  $\mathcal{K}$  operators, which are in fact the elements of  $\mathcal{K}$ .

Similarly, as in the field  $\mathcal{F}$ , in the field  $Q(\lambda)$  we have:

$$a) \quad S^n P = \partial^n P + \partial^{n-1} P(0) + S \partial^{n-2} P(0) + \dots + S^{n-1} P(0),$$

for  $P = \{P(\lambda)\} = \{\sum_{\nu=0}^{\infty} p_{\nu} \lambda^{\nu}\}$ .

$$b) \quad \frac{1}{(S-a)^n} = \left\{ \frac{\lambda^{n-1}}{(n-1)!} e^{a\lambda} \right\}, \quad n = 1, 2, \dots, \quad a \in \mathcal{K};$$

c) the rational operator is the fractional expression

$$R(S) = \frac{a_m S^m + a_{m-1} S^{m-1} + \dots + a_0}{b_n S^n + b_{n-1} S^{n-1} + \dots + b_0},$$

where  $a_i, b_j \in \mathcal{K}$  ( $i = 1, \dots, m$   $j = 1, \dots, n$ ).

If in the last expression it holds that  $m < n$ , then  $R(S)$  belongs to  $\mathcal{A}$ .

If we suppose that  $f(\lambda) \in \mathcal{A}$ , then equation (2) with conditions (5) corresponds in the field  $\mathcal{Q}(\lambda)$  to the equation

$$(8) \quad X \left( \sum_{\mu=0}^m a_{\mu}(s) S^{\mu} - e^{-s\tau} \sum_{\mu=0}^{m_1} b_{\mu}(s) S^{\mu} \right) = \\ = \sum_{\mu=0}^{m-1} q_{\mu} S^{\mu} - e^{-s\tau} \sum_{\mu=0}^{m_1-1} t_{\mu}(s) S^{\mu} + F,$$

where  $F = \{f(\lambda)\}$ ,  $a_{\mu}(s)$ , for  $\mu = 0, \dots, m$ , and  $b_{\mu}(s)$ , for  $\mu = 0, 1, \dots, m_1$ , are given by relation (3) and (4) respectively, and

$$(9) \quad q_{\mu} = a_{\mu+1}(s)d_0 + a_{\mu+2}(s)d_1 + \dots + a_m(s)d_{m-\mu-1}$$

for  $\mu = 0, 1, \dots, m-1$ ,

$$t_{\mu} = b_{\mu+1}(s)d_0 + b_{\mu+2}(s)d_1 + \dots + b_{m_1}(s)d_{m_1-\mu-1}, \quad \mu = 0, 1, \dots, m_1-1,$$

(the operators  $d_{\mu}$ ,  $\mu = 0, 1, \dots, M-1$ , are given by relation (5)).

The solution of equation (8) has the form

$$X = \frac{\sum_{\mu=0}^{m-1} q_{\mu}(s) S^{\mu} - e^{-s\tau} \sum_{\mu=0}^{m_1-1} t_{\mu}(s) S^{\mu} + F}{\sum_{\mu=0}^m a_{\mu}(s) S^{\mu} - e^{-s\tau} \sum_{\mu=0}^{m_1} b_{\mu}(s) S^{\mu}}.$$

By using the simple-fraction-decomposition method, we obtain

**Lemma 1.** *The solution of equation (8) can be written as*

$$(10) \quad X = \frac{\sum_{\mu=0}^{m-1} q_{\mu}(s)S^{\mu} - e^{-s\tau} \sum_{\mu=0}^{m_1-1} t_{\mu}(s)S^{\mu}}{\sum_{\mu=0}^m a_{\mu}(s)S^{\mu}} + \frac{(\sum_{\mu=0}^{m-1} q_{\mu}(s)S^{\mu})(\sum_{\mu=0}^{m_1} b_{\mu}(s)S^{\mu}) - (e^{-s\tau} \sum_{\mu=0}^{m_1-1} t_{\mu}(s)S^{\mu})(\sum_{\mu=0}^m a_{\mu}(s)S^{\mu})}{(\sum_{\mu=0}^m a_{\mu}(s)S^{\mu})^2} \cdot \sum_{k=1}^{\infty} \left( \frac{\sum_{\mu=0}^{m_1} b_{\mu}(s)S^{\mu}}{\sum_{\mu=0}^m a_{\mu}(s)S^{\mu}} \right)^{k-1} (e^{-s\tau})^k + \frac{F}{\sum_{\mu=0}^m a_{\mu}(s)S^{\mu}} \cdot \sum_{k=0}^{\infty} \left( \frac{\sum_{\mu=0}^{m_1} b_{\mu}(s)S^{\mu}}{\sum_{\mu=0}^m a_{\mu}(s)S^{\mu}} \right)^k (e^{-s\tau})^k.$$

From relation (10) follow Lemma 2 and Lemma 3:

**Lemma 2.** *The polynomial*

$$A(S) = \left( \sum_{\mu=0}^{m-1} q_{\mu}(s)S^{\mu} \right) \left( \sum_{\mu=0}^{m_1} b_{\mu}(s)S^{\mu} \right) - \left( \sum_{\mu=0}^{m_1-1} t_{\mu}(s)S^{\mu} \right) \left( \sum_{\mu=0}^m a_{\mu}(s)S^{\mu} \right)$$

has the degree less or equal to  $M$ , where  $M = 2 \max(m, m_1) - 1$ .

**Lemma 3.** *If  $m_1 < m$ , then the solution  $X$ , given by (10), of equation (8) belongs to space  $\mathcal{A}$  (provided that  $f(\lambda)$  belongs to  $\mathcal{A}$ ).*

The proof follows from c).

Since the space  $\mathcal{K}$  is algebraically closed, we can write

$$(11) \quad P(S) = \sum_{\mu=0}^m a_{\mu}(s)S^{\mu} = a_m \prod_{j=0}^m (S - \omega_j),$$

where

$$(12) \quad \omega_j = \sum_{i=0}^{\infty} c_{i,j} l^{i\alpha_j - \beta_j},$$

where  $\alpha_j$ , and  $\beta_j$  are rational numbers,  $\alpha_j > 0$ ,  $j = 1, 2, \dots$ , and  $c_{i,j}$  are complex numbers.

In this paper we shall consider those equations in which  $\beta_j < -1$ ,  $j = 1, 2, \dots, m$ , and additionally, satisfy condition  $m_1 \leq m$ , because of Lemma 3.

Let us suppose that  $\omega_j$ ,  $j = 1, 2, \dots, m$ , are simple solutions of equation  $P(S) = 0$ . Then the first term in relation (10) can be decomposed as (see [7]):

$$(13) \quad \frac{\sum_{\mu=0}^{m-1} q_{\mu} S^{\mu}}{\sum_{\mu=0}^m a_{\mu}(s) S^{\mu}} = \\ = \sum_{j=1}^m \frac{\sum_{\mu=0}^{m-1} q_{\mu} \omega_j^{\mu}}{P'(\omega_j)(S - \omega_j)} = \sum_{j=0}^m \frac{A_j}{S - \omega_j} = X_1.$$

Similarly as in the previous case, we can decompose the other terms in relation (10), and obtain the following form of equation (8):

$$(14) \quad X = \sum_{j=1}^m \left( \frac{A_j + e^{-s\tau} C_j}{(S - \omega_j)} + \sum_{k=1}^{\infty} \sum_{i=0}^{k-1} \left( \frac{G_{i,j}}{(S - \omega_j)^i} e^{-sk\tau} \right) + \right. \\ \left. + F \sum_{k=0}^{\infty} \sum_{i=0}^{k-1} \left( \frac{D_{i,j}}{(S - \omega_j)^i} e^{-sk\tau} \right) \right).$$

The coefficients  $C_j$ ,  $G_{i,j}$  and  $D_{i,j}$  can be found in a similar way as  $A_j$ ,

$$A_j = \frac{\sum_{\mu=0}^{m-1} q_{\mu} \omega_j^{\mu}}{P'(\omega_j)}, \quad P'(\omega_j) = \sum_{\mu=1}^{m-1} \mu a_{\mu}(s) \omega_j^{\mu-1},$$

and we shall omit these somewhat cumbersome expressions.

It can be remarked that  $A_j \in \mathcal{K}$ ,  $j = 1, \dots, m$ , and, also, all coefficients  $C_j$ ,  $G_{i,j}$  and  $D_{i,j}$  belong to  $\mathcal{K}$ .

From relation (14) it follows that the solution of equation (2) with conditions (5) in the space  $\mathcal{F}$  can be written in the form

$$(15) \quad x(\lambda) = \sum_{j=1}^m e^{\lambda \omega_j} \left( A_j + e^{-s\tau} C_j + \sum_{k=1}^{\infty} \sum_{i=0}^{k-1} \frac{G_{i,j}}{i!} \lambda^i e^{-sk\tau} \right) +$$

$$+ \int_0^\lambda f(\lambda - u) \left( \sum_{k=0}^\infty \left( \sum_{i=0}^{k-1} \frac{D_{i,j}}{i!} u^i \right) e^{-sk\tau} e^{u\omega_j} \right) du.$$

**Lemma 4.** *If in equation (2) (or (1))  $m_1 < m$ , then the equation is logarithmic ([9]) when the equation (its first part)*

$$(16) \quad \sum_{\mu=0}^m \sum_{\nu=0}^n \alpha_{\mu,\nu} \frac{\partial^{\mu+\nu} x(\lambda, t)}{\partial \lambda^\mu \partial t^\nu} = 0$$

is logarithmic.

*Proof.* From the form of the exact solution given by relation (15) it follows that there appear only Mikusiński operator functions  $e^{u\omega_j}$  for  $0 \leq u \leq \lambda$  and  $\omega_j$  are, in fact, solutions of the characteristic equation of the equation

$$(17) \quad \sum_{\mu=0}^m a_\mu x^\mu(\lambda) = 0,$$

( see [3] page 18). Equation (17), in the field  $\mathcal{F}$ , corresponds to equation (16).□

## 2. The form of the approximate solution

Let us consider the approximate solution of the characteristic equation of equation (17) in the space  $\mathcal{F}$

$$(18) \quad \omega_{j,n} = \sum_{i=0}^n c_{i,j} l^{i\alpha_j - \beta_j},$$

where  $l \in \mathcal{F}$  and  $\alpha_j, \beta_j, j = 1, \dots, m$ , satisfy the same conditions as in (12).

The approximation of the first part of the solution of equation (8) in the space  $Q(\mathcal{A})$  can be written as

$$(19) \quad X_{1,n} = \sum_{j=1}^m \frac{A_{j,n}}{(S - \omega_{j,n})} = \sum_{j=1}^m \frac{\sum_{\mu=0}^{m-1} q_\mu \omega_{j,n}^\mu}{P'(\omega_{j,n})(S - \omega_{j,n})}.$$

Similarly, by replacing  $\omega_j$  with  $\omega_{j,n}$ , we obtain the approximate solution of equation (8) in the space  $Q(\mathcal{A})$ :

$$(20) \quad X_n = \sum_{j=1}^m \left( \frac{A_{j,n} + e^{-s\tau} C_{j,n}}{(S - \omega_{j,n})} + \sum_{k=1}^{\infty} \sum_{i=0}^{k+1} \left( \frac{G_{i,j,n}}{(S - \omega_{j,n})^i} e^{-sk\tau} \right) + \right. \\ \left. + F \sum_{k=0}^{\infty} \sum_{i=0}^{k+1} \left( \frac{D_{i,j,n}}{(S - \omega_{j,n})^i} e^{-sk\tau} \right) \right).$$

The coefficients  $C_{j,n}$ ,  $G_{i,j,n}$ ,  $D_{i,j,n}$  are also, obtained as an approximations  $C_j$ ,  $G_{i,j}$  and  $D_{i,j}$  by replacing  $\omega_j$  with  $\omega_{j,n}$ .

In the space of Mikusiński operators the approximate solution of equation (2) with conditions (5) has the form

$$(21) \quad x_n(\lambda) = \sum_{j=1}^m (e^{\lambda\omega_{j,n}} (A_{j,n} + e^{-s\tau} C_{j,n}) + \sum_{k=1}^{\infty} \sum_{i=0}^{k+1} \frac{G_{i,j,n}}{i!} \lambda^i e^{-sk\tau}) + \\ + \int_0^\lambda f(\lambda - u) \sum_{k=0}^{\infty} \sum_{i=0}^{k+1} \frac{D_{i,j,n}}{i!} u^i e^{-sk\tau} e^{u\omega_{j,n}} du.$$

As usual, we denote by  $\mathcal{L}$  the space of locally integrable functions on  $[0, \infty)$ , and by  $\mathcal{L}_0$  the subspace of  $\mathcal{L}$  which consists of all functions  $f$  such that

$$\|f\|_T = \int_0^T |f(t)| dt > 0,$$

for any  $T > 0$ . By  $\mathcal{F}_0$  we denote the subspace of  $\mathcal{F}$ , of the elements of the form  $f/g$ , where  $f \in \mathcal{L}$  and  $g \in \mathcal{F}_0$ .

The operator  $g_k = \frac{k!}{i+k}$  for  $k \in \mathbb{N}$  represents a continuous function from  $\mathcal{L}_0$ . Since, in the most interesting cases of equation (2), it holds that if  $g_k x(\lambda) \in \mathcal{L}$  and  $g_k x_n(\lambda) \in \mathcal{L}$  (for fixed  $\lambda \in [0, \lambda_0]$ ), hence  $x(\lambda) \in \mathcal{F}_0$  and  $x_n(\lambda) \in \mathcal{F}_0$ , we shall observe only the equations with such properties.

Therefore, we can estimate the error of approximation in the space  $\mathcal{F}_0$ , and in the space  $\mathcal{L}$  if the exact and the approximate solutions belong to  $\mathcal{L}$ .



### 3. The measure of approximation

Let us introduce the functional ([5]):

$$(22) \quad A(x) = \sum_{i=0}^{\infty} \frac{B_{i,1/i}(x)}{e^{ie^{i^2}}(1 + B_{i,1/i}(x))}, \quad x \in \mathcal{F}_0,$$

where the functional  $B_{T,\epsilon}(x)$  introduced by J. Burzyk in [1], has the form

$$(23) \quad B_{T,\epsilon}(x) = \inf\{\|f\|_T : x = \frac{f}{g}, \|g\|_T < 1, \|l - lg\|_T < \epsilon\}, \quad x \in \mathcal{F}_0,$$

As in [5], we have

**Definition 1.** The operator  $y \in \mathcal{F}_0$  is the approximation of the operator  $x \in \mathcal{F}_0$  according to the functional  $A$  (given by (22)) with the measure of approximation  $\delta > 0$ , if  $A(x - y) < \delta$ .

The measure of approximation for the approximate solution given by (21) can be obtained by the estimation of the expression  $A(x(\lambda) - x_n(\lambda))$ , where  $x(\lambda)$  is given by 15 and  $x_n(\lambda)$  by (21). Therefore, first we have to estimate the functional

$$B_{T,1/k}(x(\lambda) - x_n(\lambda)), \quad k \in \mathbb{N}.$$

The operator  $g_k = \frac{1k}{T+k}$  satisfies the inequalities  $\|g_k\|_T < 1$  and  $\|l - lg_k\| < \frac{1}{k}$ , for every  $T > 0$ .

Using (23) we can write

$$(24) \quad B_{T,1/k}(x(\lambda) - x_n(\lambda)) \leq \| \epsilon g_k(\lambda, t) \|_T,$$

where

$$\frac{g_k(x(\lambda) - x_n(\lambda))}{g_k} =: \left\{ \frac{\epsilon g_k(\lambda, t)}{g_k} \right\}.$$

**Lemma 5.** If  $x(\lambda)$  and  $x_n(\lambda)$  are given by relations (15) and (21) respectively, then for fixed  $\lambda = \lambda_1 \in [0, \lambda_0]$  we have

$$(25) \quad B_{T,1/k}(x(\lambda) - x_n(\lambda)) = \sum_{j=1}^{\infty} \frac{1}{\Gamma\left(\frac{(n+1)\alpha_j - \beta_j}{2} + 1\right)} k_{1,j} e^{k_{2,j} e^{k_{3,j}}},$$

where  $\alpha_j$  and  $\beta_j$  appear in (12) and  $k_{1,j}, k_{2,j}, k_{3,j}$  do not depend on  $T$ .

*Proof.* Since for the translation operator from  $\mathcal{F}$  it holds that

$$e^{-s\tau} = sH_\tau, \quad H_\tau = \begin{cases} 0, & t \leq \tau, \\ 1, & t > \tau, \end{cases}$$

and  $le^{s\tau} = H_\tau$ , the infinite sums in relations (15) and (21) become finite on every interval  $[0, T]$  for  $T > 0$  (see [5]). Observe, that  $\phi(\lambda, t)$  is a bounded function for  $t \in [0, T]$  and  $\lambda \in [0, \lambda_0]$ .

So, similarly as in papers [5] and [7], we can find  $k_{1,j}, k_{2,j}, k_{3,j}$  satisfying relation (25).  $\square$

From relation (22) and Lemma 5, one can find  $Q_j$  (depending only on  $\lambda = \lambda_1 \in [0, \lambda]$ ) such that

$$(26) \quad A(x(\lambda) - x_n(\lambda)) \leq \sum_{j=1}^{\infty} \frac{1}{\Gamma\left(\frac{(n+1)\alpha_j - \beta_j}{2} + 1\right)} Q_j.$$

The convergence defined by the functional  $A(x)$  from relation (22) is equivalent to the convergence type  $I'$ . Relation (26) implies

**Proposition 1.** *The sequence of the approximate solutions  $\{x_n(\lambda)\}$  of equation (1), given by (21) converges in  $\mathcal{F}_0$ , when  $n \rightarrow \infty$  to the exact solution given by (15) in the convergence type  $I'$ .*

Using definition (1) we can say that the measure of approximation in  $\mathcal{F}_0$  for the approximate solution given by (21) can be expressed by relation (26).

#### 4. The measure of approximation in $\mathcal{L}$

If the exact and approximate solution represent functions from  $\mathcal{L}$ , for fixed  $\lambda = \lambda_1 \in [0, \lambda_0]$ , then using the functional

$$(27) \quad F(f) = \sum_{i=0}^{\infty} \frac{\|f\|_i}{e^{ie^{i^2}}(1 + \|f\|_i)}, \quad f \in \mathcal{L},$$

we can find  $Q_{1,j}$ , depending on  $\lambda = \lambda_1 \in [0, \lambda_0]$ , such that

$$(28) \quad F(x(\lambda) - x_n(\lambda)) \leq \sum_{j=1}^{\infty} \frac{1}{\Gamma\left(\frac{(n+1)\alpha_j - \beta_j}{2} + 1\right)} Q_{1,j},$$

where  $\alpha_j$  and  $\beta_j$  appeared in relation (12).

The convergence defined by the functional  $F(f)$  from (27) is equivalent to the convergence in  $\mathcal{L}$ . So, from relation (26) follows

**Proposition 2.** *The sequence of the approximate solutions  $\{x_n(\lambda)\}$  of equation (1), given by (21), converges in  $\mathcal{L}$  when  $n \rightarrow \infty$ , to the exact solution given by (15).*

The measure of approximation in  $\mathcal{L}$  can be expressed by relation (25).

## 5. Example

Let us consider the differential-difference equation

$$(29) \quad \frac{\partial x(\lambda, t)}{\partial \lambda \partial t} - \frac{\partial x(\lambda, t)}{\partial \lambda} - x(\lambda, t) + x(\lambda, t-1) = 0$$

with conditions

$$(30) \quad \frac{\partial x(\lambda, 0)}{\partial \lambda} = 0, \quad x(\lambda, t) = 0, \quad t \leq 0, \\ x(0, t) = e^t.$$

In the field of Mikusiński operators, this equation corresponds to the equation

$$(31) \quad (s-1)x'(\lambda) - x(\lambda) + e^{-s}x(\lambda) = 0,$$

with condition

$$(32) \quad x(0) = \frac{1}{s-1}.$$

In the space  $Q(\mathcal{A})$ , the corresponding equation is

$$(33) \quad (s-1)SX - X - e^{-s}X = 1$$

The solution of equation (33) can be written as

$$X = \frac{1}{(s-1)S-1+e^{-s}} = \frac{1}{s-1} \cdot \frac{1}{S-\frac{1}{s-1}} + \\ + \sum_{k=1}^{\infty} \frac{(-1)^k e^{-sk}}{(s-1)^{k+1} (S-\frac{1}{s-1})^{k+1}}.$$

The approximate solution in the space  $\Omega(A)$  is

$$X_n = \left( \sum_{i=0}^n l^{i+1} \right) \frac{1}{S - \left( \sum_{i=0}^n l^{i+1} \right)} + \\ + \sum_{k=1}^{\infty} \left( \sum_{i=0}^n l^{i+1} \right)^{k+1} \cdot \frac{(-1)^k}{\left( S - \sum_{i=0}^n l^{i+1} \right)^{k+1}} e^{-sk}.$$

In the space  $\mathcal{F}$  the corresponding solution can be written as

$$x_n(\lambda) = \left( \sum_{i=0}^n l^{i+1} \right) \exp\left( \lambda \sum_{i=0}^n l^{i+1} \right) + \\ + \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \left( \sum_{i=0}^n l^{i+1} \right)^{k+1} \left( \exp(\lambda) \sum_{i=0}^n l^{i+1} \right) e^{-sk}.$$

The error of approximation can be estimated as

$$A(x(\lambda) - x_n(\lambda)) \leq \frac{Q}{\Gamma\left(\frac{n+2}{2}\right)},$$

where

$$Q \geq 2e^{e^4} + \frac{1}{e^4 - 1}.$$

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## REZIME

### PRIBLIŽNO REŠENJE NEHOMOGENE PARCIJALNE DIFERENCIJNO-DIFERENTNE JEDNAČINE

U radu se konstruiše približno rešenje parcijalne diferencijalno-diferentne jednačine pomoću dvodimenzionalnog računa koji je uveo T. Ogata [4] kao i pomoću rezultata J. Wloke [8] za obične diferencijalno-diferentne jednačine. Takodje se daje i ocena greške.

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