Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 22, 2 (1992), 177-185

Review of Research
Faculty of Science
Mathematics Series

AN OUTER APPROXIMATION OF THE CONVEX HULL FOR FINITE GRID POINT SETS

Joviša Žunić
University of Novi Sad Faculty of Engineering
Institute of Applied Basic Disciplines
Trg Dositeja Obradovića 6, 21000 Novi Sad, Yugoslavia

Abstract

In this paper a new approximate convex hull algorithm is given. For the presented algorithm the outer approximation can be equal to the true convex hull for each finite grid point set. The proposed algorithm is an adaptation of the Jarvis algorithm.

AMS Mathematics Subject Classification (1991): 68E99
Key words and phrases: grid point set, convex hull, computational geometry, convex hull approximation.

1. Introduction

The determination of the convex hull of a finite set of points is important in such areas as computer graphics, robotics and pattern recognition.
Several algorithms have been presented for computing the convex hull of n planer points in $\mathcal{O}(n \cdot logn)$ worst case time [5,6]. On the other hand, it has been shown that $\Omega(n \cdot logn)$ is the lower bound for planar convex hull computation [6,10].
However, it is still possible that for some special application certain approximations of the convex hull should be prefered with respect to reducing

the computation time, without any loss of efficiency. Bentley, Faust and Preparata [1] presented a linear time algorithm for computing an approximate convex hull in a two dimensional space. That is, for given n points in the plane, a convex polygon with vertices in this points set is determined in time $O(n)$, such that this polygon is arbitrarily close to the true convex hull of the point set.

A grid point set is a finite set of points with integer coordinates. In [4] Klette proposed an algorithm for determining the outer and inner approximation for finite grid point sets. In Section 2 we shall, briefly, present Klette's algorithm.

In Section 3 we shall give a new algorithm for determining the inner and outer approximation of the convex hull of the finite grid point set. The proposed algorithm does not have some of the deficiency which Klette algorithm has.

Section 4 discusses the new algorithm.

2. Klette's algorithm

A grid point set is a set of points $P = (x_1, x_2)$ in the regular orthogonal grid having integer coordinates: x_1, x_2.

For a direction $\alpha \in [0, 2\pi)$ and a grid point $P(x_1, x_2)$ let $g(\alpha, P)$ be the stright line passing through point P in direction $\alpha + \frac{\pi}{2}$. Points $Y = (y_1, y_2)$ of $g(\alpha, P)$ satisfy the equation:

$$(y_2 - x_2)\sin\alpha + (y_1 - x_1)\cos\alpha = 0.$$

Let $Z_\alpha(Y)$ denote the function: $(y_2 - x_2)\sin\alpha + (y_1 - x_1)\cos\alpha$. A point $P = (p_1, p_2) \in G$ is called an extreme point in direction α of the grid point set G (write $P \in Ex_\alpha(G)$) iff $Z_\alpha(P) = 0$ and $Z_\alpha(Y) \leq 0$ for all $Y \in G$. (Note that then all the points of G are contained in one of the two closed half-planes defined by the stright line $g(\alpha, P)$.) If P is the extreme point in direction α, then by $hp(\alpha, G)$ we define the closed half-plane defined by $g(\alpha, G)$ and containing the set G.

Let $dir(n)$ denote the set $\{0, \frac{1}{n}2\pi, \frac{2}{n}2\pi, ..., \frac{n-1}{n}2\pi\}$ of n directions.

Let $H_n(G) = \bigcap_{\alpha \in dir(n)} hp(\alpha, G)$ be the outer approximation of $CH(G)$ called the n-hull of G, $(CH(G) \subseteq H_n G)$.

Let $A_n(G) = CH(\bigcup_{\alpha \in dir(n)} Ex_\alpha(G))$ be the inner approximation of $CH(G)$,

called the n- approximation ($A_n(G) \subseteq CH(G)$).
In [1] it is shown:
if $n \geq \frac{2\pi}{\arctan \frac{1}{2[\frac{m-1}{2}]^2}}$ then $A_n(G) = CH(G)$,
for all the grid point sets G with $\text{diam}(G) \leq m$.
($\text{diam}(G) = \max\{\max\{|x_1 - y_1|, |x_2 - y_2|\}, (x_1, x_2) \in G \text{ and } (y_1, y_2) \in G)\}$.
That means: for each set G there exists an integer n such that $A_n(G) = CH(G)$. Klette mentioned an open problem: whether there is an integer n such that $H_n(G) = CH(G)$ for the given grid point set G?
In [2] it is shown: $H_n(G) = CH(G)$ for some integer n iff $H_8(G) = CH(G)$. That means outer approximation $H_n(G)$ is equal to exact convex hull iff angles between edges of $CH(G)$ and x-axis are in the form $k \cdot \frac{\pi}{4}$. From the above considerations it follows that:

1) There are sets G for wich $H_n(G) \neq CH(G)$ for all $n \in N$.

Some of vertices of $H_n(G)$ can be points which are not grid points but:

2) The number of these points is unknown.

It is clear that some of vertices of $H_n(G)$ are points which are not convex hull points but:

3) The number of these points is unknown.

In [8] Stojmenović and Kim gave a new approximate algorithm for which outer approximation can be equal to the true convex hull for each finite grid point set G. They observed $A_n(G)$ and $H_n(G)$ with respect to the new set of directions

$$dir_1(n) = \{arctg\frac{i}{n}, arctg\frac{n}{i} \mid -n \leq i \leq n\}.$$

In both of these algorithms time complexity is $O(n \cdot N)$, where n is the number of directions and N is the number of points.
In the next section we shall give a new algorithm which has none of the defficiencies 1, 2 and 3.

3. New algorithm

The algorithm proposed in this section is an adaptation of the exact Jarvis convex hull algorithm (see fig. 1).

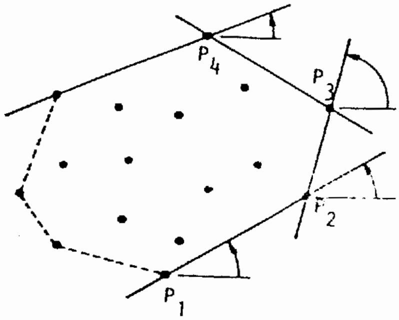

Fig. 1. The Jarvis march for constructing the convex hull. The algorithm of Jarvis finds successive hull vertices by repeatedly turning angles.

The new algorithm is iterative and in each iteration gives the outer approximation of the convex hull for finite grid point set.
We denote the outer approximation of the convex hull obtained in n-th iteration by $P_n(G)$ and call it n-approximation. For the representation of $P_n(G)$ we shall use the list of vertices of $P_n(G)$ ordered by counterclockwise. ($P_n(G)$ is always a convex set).
Let $T_{0,1}, T_{0,2}, T_{0,3}$ and $T_{0,4}$ be the vertices of the smallest rectangle with horizontal and vertical sides which contains set G.
Let us find points with the maximal (minimal) x-coordinate. From those points we choose one with maximal y-coordinate and denote it with $E'_{0,1}(E''_{0,2})$ and a point with minimal y- coordinate and denote it with $E''_{0,4}(E'_{0,3})$. Analogously, we determine the points $E''_{0,1}, E'_{0,2}, E''_{0,3}, E'_{0,4}$. Points are denoted as shown in Fig. 2. Note that among points $T_{0,1}, E'_{0,i}, E''_{0,i}$ for $i = 1,2,3,4$

there can be points which are equal.

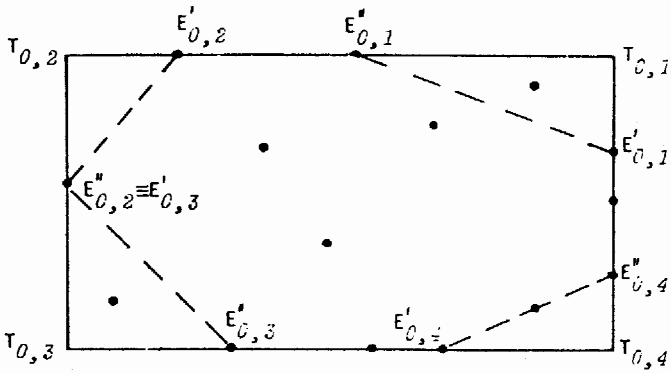

Fig. 2. Initial denotations.

Now, we shall define the 0-th approximation as:

$$E'_{0,1}T_{0,1}E''_{0,1}E'_{0,2}T_{0,2}E''_{0,2}E'_{0,3}T_{0,3}E''_{0,3}E'_{0,4}T_{0,4}E''_{0,4}.$$

Each following approximation we shall define recursively, using only the previous approxomation.
Let $P_n(G)$ be the n-th approximation, then the next approximation, $P_{n+1}(G)$, we can define as follows:
In the triangle $E'_{n,i}T_{n,i}E''_{n,i}$ (for $i = 1,2,3,4$) determine the point $E'_{n+1,i}$, such that the angle
$\measuredangle(E''_{n,i}E'_{n,i}E'_{n+1,i})$ is the greatest possible. If there is more than one point with this property, then we choose $E'_{n+1,i}$ which is the furthest from $E''_{n,i}$.
Also, in the same triangle determine point $E''_{n+1,i}$, such that angle $\measuredangle\,(E'_{n,i}E"_{n,i}E''_{n+1,i})$ is the greatest. If there are more such points, then for $E"_{n+1,i}$ choose the furthest from $E'_{n,i}$.
Let $l'_{n+1,i}$ denote the line determined by points $E'_{n+1,i}$ and $E'_{n,i}$, also let $l"_{n+1,i}$ be the line determined by $E"_{n+1,i}$ and $E"_{n,i}$. Now, let $T_{n+1,i}$ denote the intersection of lines $l'_{n+1,i}$ and $l"_{n+1,i}$ (as shown in Fig. 3.).

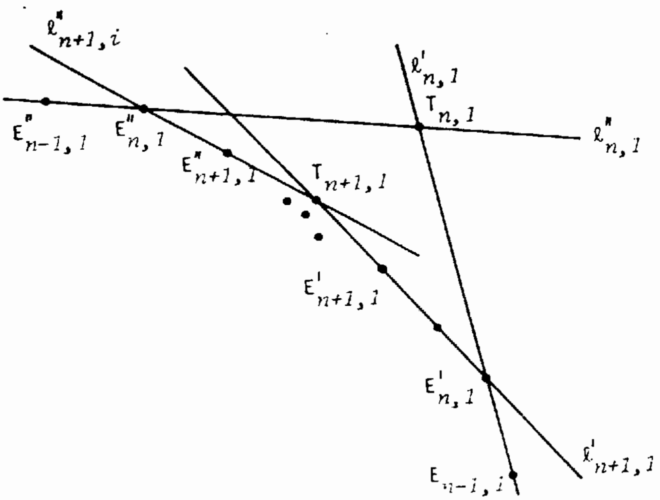

Fig. 3.

The $(n+1)$-th approximation will be obtained in the following way: In n-th approximation - $P_n(G)$, which is represented by the list of vertices of $P_n(G), T_{n,i}$ substitutes with $E'_{n+1,i}T_{n+1,i}E"_{n+1,i}$. This means that $P_{n+1}(G)$ is represented as the next list:

$$E'_{0,1} \ldots E'_{n+1,1}\ T_{n+1,1}\ E"_{n+1,1} \ldots E'_{0,1}$$

$$E'_{0,2} \ldots E'_{n+1,2}\ T_{n+1,2}\ E"_{n+1,2} \ldots E"_{0,2}$$

$$E'_{0,3} \ldots E'_{n+1,3}\ T_{n+1,3}\ E"_{n+1,3} \ldots E"_{0,3}$$

$$E'_{0,4} \ldots E'_{n+1,4}\ T_{n+1,4}\ E"_{n+1,4} \ldots E"_{0,4}$$

thus, $P_{n+1}(G) = CH(\{E'_{j,i}, E"_{j,i}T_{j,i} \mid \text{for } 1 \leq i \leq 4,\ 0 \leq j \leq n+1\})$. At the end of the $(n+1)$-th iteration, we discard the points in the interior of regions $E'_{n,i}E'_{n+1,i}E"_{n+1,i}E"_{n,i}$ for $i = 1, 2, 3, 4$. It is clear that in cases:

1) if there exist j $(j < n)$ and i $(1 \leq i \leq 4)$ so that $T_{j,i} = E'_{j,i} = E"_{j,i}$, then one can disregard triangles $E'_{k,i}T_{k,i}E"_{k,i}$ for $k > j$ (they degenerate in the convex hul point $E_{j,i}$) and

2) if for some j ($j < n$) and all i from 1,2,3,4, $T_{j,i} = E'_{j,i} = E"_{j,i}$ is valid, then $P_{j+k}(G) = P_j(G) = CH(G)$ for every integer k.

4. Discussion of the new algorithm

Lemma 1. *If $P_n(G)$ is determined then:*

1) *No more than four points from vertices of $P_n(G)$ are not grid points.*

2) *No more than four points from vertices of $P_n(G)$ are not convex hull points.*

Proof. It is obvious that points $E'_{j,i}, E"_{j,i}$ ($0 \leq j \leq n,\ 1 \leq i \leq 4$), are convex hull points and also, they are grid points. Points: $T_{n,1}, T_{n,2}, T_{n,3}$ and $T_{n,4}$ are not convex hull points (except those which satisfy $T_{n,i} = E'_{n,i}$). Points $T_{n,1}, T_{n,2}, T_{n,3}$ and $T_{n,4}$ can be grid points, but not necessarily.

Theorem 1. *For each given finite grid point set G there is n such that $P_n(G) = CH(G)$.*

Proof. Let $h(G)$ denote the number of vertices of $CH(G)$ for the finite grid point set G. For every integer n it is obviously satisfied:

1) if $h(P_{n+1}(G)) = h(P_n(G))$, then $P_n(G) = CH(G)$,

2) $h(P_{n+1}(G)) \geq h(P_n(G))$ and

3) $h(G) \geq h(P_n(G)) - 4$.

If integer n does not exist such that $P_n(G) = CH(G)$, then from 1) and 2) follows:

$$h(P_1(G)) < h(P_2(G)) < \dots < h(P_i(G)) < h(P_{i+1}(G)) < \dots$$

but it is in contradiction with $k = h(G) > h(P_n(G)) - 4$.

The performance of this algorithm is easy to analyze. Let m be the number of points of the finite grid point set G. The space that it takes is proportional to m. Finding the points $E'_{n,i}, E"_{n,i}, T_{n,i}$ requires $O(m)$ time. The running time of the proposed algorithm is therefore $O(m \cdot n)$ (in the worst case), where m is the number of points and n is the number of iterations.

5. Concluding remarks

The proposed algorithm can be used for determining the inner approximation of the convex hull for finite grid point sets. Note that if: $I_n(G) := CH(P_n(G) - \{T_{n,i} | T_{n,i} \neq E'_{n,i},\ 1 \leq i \leq 4\})$, then $I_n(G)$ is the inner approximation of the convex hull for finite grid point set G.
$I_n(G)$ can be represented by the list of vertices of $P_n(G)$ with the exception of the points from $T_{n,1}, T_{n,2}, T_{n,3}$ and $T_{n,4}$ which satisfy $T_{n,i} \neq E'_{n,i}$ for $i = 1, 2, 3, 4$.
So, the algorithm proposed here can be used for finding the inner approximation. Obviously $I_n(G) = CH(G)$ iff $P_n(G) = CH(G)$.

References

[1] Bentley J.L., Faust M.G., Preparata F.P.: Approxomation algorithms for convex hulls, Com. ACM 25, 1, 64- 68, 1982.

[2] Doroslovački, R., Žunić, J.: A characterization of finite grid point sets with exact outer approximation, Proceedings of the Eight Yugoslav Seminar on Graph Theory, 82-89, 1987.

[3] Jarvis, R.A.: On the identification of the convex hull of a finite set of points in the plane, Inf. Process. Lett., 2, 18-21, 1983.

[4] Klette, R.: On the approximation of convex hull of finite grid point sets, Patt. Rec. Lett., 2, 19-22, 1983.

[5] Preparata, F.P., Hong S.J.: Convex hulls of finite sets of points in two and three dimensions, Comm. ACM 20(2), 87-93, 1977.

[6] Shamos, M.I.: Computational geometry, Ph.D. Dissertation, Yale University, New Haven, Connecticut, 1978.

[7] Soisalon-Soininen, E.: On computing approximate convex hulls, Inf. Process. Lett. 16, 121-126, 1983.

[8] Stojmenović, I., Chul E. Kim: On approximate convex hull, Washington State University, Pullman, Washington 99164- 1210, CS-87-175, October, 1987.

[9] Stojmenović I., Soisalon-Soininen E.: A note on approximate convex hulls, Inf. Process. Lett. 22, 55-56, 1986.

[10] Yao A.C.: A lower bound to finding convex hulls, J.ACM, 28-4, 780-787, 1981.

[11] Žunić, J.: Approximate convex hull algorithm - efficiency evaluations, J. Inf. Process. Cyber. EIK 26, 3, 137- 148, 1990.

REZIME

SPOLJNA APROKSIMACIJA KONVEKSNOG OMOTAČA ZA SKUP TAČAKA SA CELOBROJNIM KOORDINATAMA

U ovom radu je dat jedan algoritam za odredjivanje aproksimacije konveksnog omotača. Predloženi algoritam daje spoljnu aproksimaciju koja može biti jednaka konveksnom omotaču za svaki skup tačaka za celobrojnim koordinatama. Ovaj algoritam je adaptacija Jarvis-ovog algoritma.

Received by the editors May 16, 1989