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CHARACTERIZATIONS OF $\alpha-$ CONTINUOUS MULTIFUNCTIONS

Valeriu Popa
Department of Mathematics, Higher Education Institute,
5500 Bacău, Romania

Takashi Noiri
Department of Mathematics, Yatsushiro College of Technology
Yatsushiro, Kumamoto 866 Japan

Abstract

The purpose of the present paper is to introduce the notion of $\alpha-$ continuous multifunctions and to obtain several characterizations of such functions.

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1. Introduction

In 1965, Njåstad [11] introduced a weak form of open sets called $\alpha-$ sets. In 1982, the second author [13] of the present paper defined a function of a topological space into a topological space to be strongly semi - continuous if the inverse image of each open set is an $\alpha-$ set. Mashhour et al. [9] called strongly semi - continuous functions $\alpha-$ continuous and obtained several properties of such functions. In 1986, Neubrunn [10] extended these functions to multifunctions and introduced the notion of $\alpha-$ upper ($\alpha-$ lower) continuous multifunctions. The purpose of the present paper is to introduce the notion of $\alpha-$ continuous multifunctions and to obtain several characterizations of $\alpha-$ continuous multifunctions.

2. Preliminaries

Let X be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $Cl(A)$ and $Int(A)$, respectively. A subset A is said to be $\alpha-$ open [11] (resp. semi - open [5], preopen [8]) if $A \subset Int(Cl(Int(A)))$ (resp. $A \subset Cl(Int(A))$, $A \subset Int(Cl(A))$). The family of all $\alpha-$ open (resp. semi - open, preopen) sets in X is denoted by $\alpha(X)$ (resp. $SO(X), PO(X)$). For these three families, it is shown in [14, Lemma 3.1] that $\alpha(X) = SO(X) \cap PO(X)$. Since $\alpha(X)$ is a topology for X [11, Proposition 2], by $\alpha Cl(A)$ we shall denote the closure of A with respect to $\alpha(X)$. The complement of a semi - open (resp. $\alpha-$ open) set is said to be semi - closed (resp. $\alpha-$ closed). The intersection of all semi - closed sets of X containing A is called the semi - closure [3] and is denoted by $sCl(A)$. The union of all semi - open sets of X contained in A is called the semi - interior of A and is denoted by $sInt(A)$. A subset A is said to be feebly open [7] if thre exists an open set U such that $U \subset A \subset sCl(U)$. The complement of a feebly open set is called feebly - closed. Since $sCl(U) = Int(Cl(U))$ for any open set U [4, Lemma 2.1], it follows from [14, Lemma 4.12] that the notion of feebly open sets is equivalent to that of $\alpha-$ open sets. Maheshwari and Jain [6] defined a function to be feebly continuous if the inverse image of every open set is feebly open. However, we realize that feeble continuity is equivalent to $\alpha-$ continuity, that is, strong semi - continuity.

Throughout this paper, (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces and $F : X \to Y$ (resp. $f : X \to Y$) presents a multivalued (resp. single valued) function. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a set G of Y by $F^+(G)$ and $F^-(G)$, respectively, that is

$$F^+(G) = \{x \in X \mid F(x) \subset G\} \quad \text{and} \quad F^-(G) = \{x \in X \mid F(x) \cap G \neq \emptyset\}.$$

3. $\alpha-$continuous multifunctions

Definition 1. *A multifunction $F : X \to Y$ is said to be $\alpha-$ continuous at a point x of X if for any open sets G_1 and G_2 of Y such that $F(x) \subset G_1$ and $F(x) \cap G_2 \neq \emptyset$, there exists $U \in \alpha(X)$ containing x such that $F(U) \subset G_1$ and $F(u) \cap G_2 \neq \emptyset$ for every $u \in U$.*

A multifunction $F : X \to Y$ is said to be $\alpha-$ continuous if it is $\alpha-$

continuous at every point of X.

Theorem 1. *The following are equivalent for a multifunction $F : X \to Y$:*

(1) F is $\alpha-$ continuous at $x \in X$.

(2) For any open sets G_1, G_2 of Y such that $F(x) \subset G_1$ and $F(x) \cap G_2 \neq \emptyset$,
$x \in sC1(Int(F^+(G_1) \cap F^-(G_2)))$.

(3) For any open sets G_1, G_2 of Y such that $F(x) \subset G_1$ and $F(x) \cap G_2 \neq \emptyset$ and any $U \in SO(X)$ containing x, there exists a nonempty open set G_U of X such that $G_U \subset U$, $F(G_U) \subset G_1$ and $F(g) \cap G_2 \neq \emptyset$ for every $g \in G_U$.

Proof. (1)⇒(2): Let G_1, G_2 be any open sets of Y such that $F(x) \subset G_1$ and $F(x) \cap G_2 \neq \emptyset$. There exists $U \in \alpha(X)$ containing x such that $F(U) \subset G_1$ and $F(u) \cap G_2 \neq \emptyset$ for every $u \in U$. Therefore, we have $x \in U \subset F^+(G_1) \cap F^-(G_2)$. Since $U \in \alpha(X)$, by Theorem 1 of [15] we obtain

$$x \in U \subset sC1(Int(U)) \subset sC1(Int(F^+(G_1) \cap F^-(G_2))).$$

(2)⇒(3): Let G_1, G_2 be any open sets of Y such that $F(x) \subset G_1$ and $F(x) \cap G_2 \neq \emptyset$. Then $x \in sC1(Int(F^+(G_1) \cap F^-(G_2)))$. Let U be any semi - open set of X containing x. By Lemma 3 of [12] and Theorem 1.9 of [3], we have $\emptyset \neq U \cap Int(F^+(G_1) \cap F^-(G_2)) \in SO(X)$. Put $G_U = Int(U \cap Int(F^+(G_1) \cap F^-(G_2)))$, then G_U is a nonempty open set of X such that $G_U \subset U$, $G_U \subset F^+(G_1)$ and $G_U \subset F^-(G_2)$. Therefore, we obtain $F(G_U) \subset G_1$ and $F(g) \cap G_2 \neq \emptyset$ for every $g \in G_U$.

(3) ⇒(1): Let $SO(X, x)$ be the family of all semi - open sets of X containing $x \in X$. Let G_1, G_2 be any open sets of Y such that $F(x) \subset G_1$ and $F(x) \cap G_2 \neq \emptyset$. For each $U \in SO(X, x)$, there exists a non - empty open set G_U such that $G_U \subset U$, $F(G_U) \subset G_1$ and $F(g) \cap G_2 \neq \emptyset$ for every $g \in G_U$. Let $W = \bigcup\{G_U \mid U \in SO(X, x)\}$. Then W is open in X, $x \in sC1(W), F(W) \subset G_1$ and $F(w) \cap G_2 \neq \emptyset$ for every $w \in W$.

Put $S = W \bigcup \{x\}$, then $W \subset S \subset sC1(W) = Int(C1(W))$. Therefore, S is an $\alpha-$ open set containing x [14, Lemma 4.12], $F(S) \subset G_1$ and $F(s) \cap G_2 \neq \emptyset$ for every $s \in S$. □

A function $f : X \to Y$ is said to be $\alpha-$ continuous, [9], if $f^{-1}(V)$ is $\alpha-$ open in X for every open set V of Y. Since $\alpha-$ open sets and feebly open sets are equivalent, $\alpha-$ continuity is equivalent to feeble continuity due to Maheshwari and Jain [6]. The following corollary is an immediate consequence of Theorem 1.

Corollary 1. *(Popa [15]). The following are equivalents for a function* $f : X \to Y$:

(1) f *is feebly continuous at* $x \in X$.

(2) For any open set G *containing* $f(x)$, $x \in sC1(Int(f^{-1}(G)))$.

(3) For any open set G *of* Y *containing* $f(x)$ *and any* $U \in SO(X)$ *containing* x, *there exists a nonempty open set* G_U *such that* $G_U \subset U$ *and* $f(G_U) \subset G$.

Let A be a subset of a topological space X. A subset V of X is said to be an $\alpha-$ neighbourhood of A if there exists $U \in \alpha(X)$ such that $A \subset U \subset V$. A subset V is called a neighbourhood which intersects A if there exists an open set G of X such that $G \subset V$ and $G \cap A \neq \emptyset$.

Theorem 2. *The following are equivalent for a multifunction* $F : X \to Y$:

(1) F *is* $\alpha-$ *continuous.*

(2) $F^+(G_1) \cap F^-(G_2) \in \alpha(X)$ *for any open sets* G_1, G_2 *of* Y.

(3) $F^-(V_1) \cup F^+(V_2)$ *is* $\alpha-$ *closed in* X *for any closed sets* V_1, V_2 *of* Y.

(4) $sInt(C1(F^-(B_1) \cup F^+(B_2))) \subset F^-(C1(B_1)) \cup F^+(C1(B_2))$ *for any subsets* B_1, B_2 *of* Y.

(5) $\alpha C1(F^-(B_1) \cup F^+(B_2)) \subset F^-(C1(B_1)) \cup F^+(C1(B_2))$ *for any subsets* B_1, B_2 *of* Y.

(6) For each point x *of* X, *for each neighbourhood* V_1 *of* $F(x)$ *and for each neighbourhood* V_2 *which intersects* $F(x)$, $F^+(V_1) \cap F^-(V_2)$ *is an* $\alpha-$ *neighbourhood of* x.

(7) Foe each point x *of* X, *for each neighbourhood* V_1 *of* $F(x)$ *and for each neighbourhood* V_2 *which intersects* $F(x)$, *there exists an* $\alpha-$ *neighbourhood* U *of* x *such that* $F(U) \subset V_1$ *and* $F(u) \cap V_2 \neq \emptyset$ *for every* $u \in U$.

Proof. (1) $\Rightarrow$ (2): Let G_1, G_2 be any open sets of Y and $x \in F^+(G_1) \bigcap F^-(G_2)$. Then $F(x) \subset G_1$ and $F(x) \cap G_2 \neq \emptyset$ and by Theorem 1 $x \in sC1(Int(F^+(G_1) \cap$

$F^-(G_2)))$. Therefore, we obtain

$$F^+(G_1) \cap F^-(G_2) \subset sCl(Int(F^+(G_1) \cap F^-(G_2))).$$

It follows from [15, Theorem 1] that $F^+(G_1) \cap F^-(G_2) \in \alpha(X)$.

(2) ⇒ (3): This follows from the relations: $F^-(Y - B) = X - F^+(B)$ and $F^+(Y - B) = X - F^-(B)$ for any subset B of Y.

(3) ⇒ (4): Let B_1, B_2 be any subsets of Y, then $F^-(Cl(B_1)) \cup F^+(Cl(B_2))$ is an $\alpha-$ closed set in Y. By [15, Theorem 2], we have

$$sInt(Cl(F^-(Cl(B_1)) \cup F^+(Cl(B_2)))) \subset F^-(Cl(B_1)) \cup F^+(Cl(B_2)).$$

Since $F^-(B_1) \subset F^-(Cl(B_1))$ and $F^+(B_2) \subset F^+(Cl(B_2))$, we have

$$sInt(Cl(F^-(B_1) \cup F^+(B_2))) \subset F^-(Cl(B_1)) \cup F^+(Cl(B_2)).$$

(4) ⇒ (5): By Lemma 2.1 of [4], we have $Int(Cl(Int(A))) = sCl(Int(A))$ and hence $sInt(Cl(A)) = Cl(Int(Cl(A)))$ for any subset A of X. Therefore, it follows from [1, Corollary 2.4] that $\alpha Cl(A) = A \cup Cl(Int(Cl(A))) = A \cup sInt(Cl(A))$. This completes the proof of the implication.

(5) ⇒ (3): Let B_1, B_2 be any closed sets of Y. Then we have

$$\alpha Cl(F^-(B_1) \cup F^+(B_2)) \subset F^-(Cl(B_1)) \cup F^+(Cl(B_2)) = F^-(B_1) \cup F^+(B_2).$$

This shows that $F^-(B_1) \cup F^+(B_2)$ is $\alpha-$ closed in X.

(2) ⇒ (6): Let $x \in X$, V_1 be a neighbourhood of $F(x)$ and V_2 a neighbourhood which intersects $F(x)$. There exist two open sets U_1 and U_2 such that $F(x) \subset U_1 \subset V_1$, $U_2 \subset V_2$ and $F(x) \cap U_2 \neq \emptyset$. Therefore, we obtain $x \in F^+(U_1) \cap F^-(U_2) \subset F^+(V_1) \cap F^-(V_2)$. Since $F^+(U_1) \cap F^-(U_2) \in \alpha(X)$, it follows that $F^+(V_1) \cap F^-(V_2)$ is an $\alpha-$ neighbourhood of x.

(6) ⇒ (7): Let $x \in X$, V_1 be a neighbourhood of $F(x)$ and V_2 a neighbourhood which intersects $F(x)$. Put $U = F^+(V_1) \cap F^-(V_2)$, then U is an $\alpha-$ neighbourhood of x, $F(U) \subset V_1$ and $F(u) \cap V_2 \neq \emptyset$ for every $u \in U$.

(7) ⇒ (1): Let $x \in X$ and G_1, G_2 be any open sets of Y such that $F(x) \subset G_1$ and $F(x) \cap G_2 \neq \emptyset$. Then G_1 is a neighbourhood of $F(x)$ and G_2 is a neighbourhood which intersects $F(x)$. There exists an $\alpha-$ neighbourhood U of x such that $F(U) \subset G_1$ and $F(u) \cap G_2 \neq \emptyset$ for every $u \in U$. Therefore,

there exists $A \in \alpha(X)$ such that $x \in A \subset U$; hence $F(A) \subset G_1$ and $F(a) \cap G_2 \neq \emptyset$ for every $a \in A$.

A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be $\alpha-$ upper (resp. $\alpha-$ lower) continuous [10] if $F : (X, \alpha(X, \tau)) \to (Y, \sigma)$ is upper (resp. lower) continuous.

Remark 1. Since $\alpha(X)$ is a topology, the intersection of two $\alpha-$ open sets is $\alpha-$ open. Therefore, a multifunction $F : X \to Y$ is $\alpha-$ continuous if and only if it is $\alpha-$ upper continuous and $\alpha-$ lower continuous.

Corollary 2. *(Popa [15]) and Mashhour et al. [9]). The following are equivalent for a function $f : X \to Y$:*

(1) f is feebly continuous.

(2) f is $\alpha-$ continuous.

(3) $f^{-1}(V)$ is $\alpha-$ closed in X for every closed set V of Y.

(4) $sInt(C1(f^{-1}(B))) \subset f^{-1}(C1(B))$ for any subset B of Y.

(5) $\alpha C1(f^{-1}(B)) \subset f^{-1}(C1(B))$ for any subset B of Y.

(6) For each $x \in X$ and each neighbourhood V of $f(x)$, $f^{-1}(V)$ is an $\alpha-$ neighbourhood of x.

(7) For each $x \in X$ and each neighbourhood V of $f(x)$, there exists and $\alpha-$ neighbourhood U of x such that $f(U) \subset V$.

The following lemma was shown by Mashhour et al. [9] and Reilly and Vamanamurthy [18].

Lemma 1. *Let A and B be subsets of a topological space X.*

(1) If $A \in SO(X) \cup PO(X)$ and $B \in \alpha(X)$, then $A \cap B \in \alpha(A)$.

(2) If $A \subset B \subset X$, $A \in \alpha(B)$ and $B \in \alpha(X)$, then $A \in \alpha(X)$.

Theorem 3. *If a multifunction $F : X \to Y$ is $\alpha-$ continuous and $X_0 \in PO(X) \cup SO(X)$, then the restriction $F \downarrow X_0 : X_0 \to Y$ is $\alpha-$ continuous.*

Proof. Let $x \in X$ and G_1, G_2 be any open sets of Y such that $(F \mid X_0)(x) \subset G_1$ and $(F \mid X_0)(x) \cap G_2 \neq \emptyset$. Since F is $\alpha-$ continuous and $(F \mid X_0)(x) =$

$F(x)$, there exists $U \in \alpha(X)$ containing x such that $F(U) \subset G_1$ and $F(u) \cap G_2 \neq \emptyset$ for every $u \in U$. Set $U_0 = U \cap X_0$, then by Lemma 1 we have $x \in U_0 \in \alpha(X_0)$, $(F \mid X_0)(U_0) \subset G_1$ and $(F \mid X_0)(u) \cap G_2 \neq \emptyset$ for every $u \in U_0$. This shows that $F \mid X_0$ is $\alpha-$ continuous.

Theorem 4. *A multifunction $F : X \to Y$ is $\alpha-$ continuous if for each $x \in X$ there exists $X_0 \in \alpha(X)$ containing x such that the restriction $F \mid X_0 : X_0 \to Y$ is $\alpha-$ continuous.*

Proof. Let $x \in X$ and G_1, G_2 be any open set of Y such that $F(x) \subset G_1$ and $F(x) \cap G_2 \neq \emptyset$. There exists $X_0 \in \alpha(X)$ containing x such that $F \mid X_0$ is $\alpha-$ continuouss. Therefore, there exists $U_0 \in \alpha(X_0)$ containing x such that $(F \mid X_0)(U_0) \subset G_1$ and $(F \mid X_0)(u) \cap G_2 \neq \emptyset$ for every $u \in U_0$. By Lemma 1, $U_0 \in \alpha(X)$ and $F(u) = (F \mid X_0)(u)$ for every $u \in U_0$. This shows that $F : X \to Y$ is $\alpha-$ continuous.

Corollary 3. *Let $\{U_\alpha \mid \alpha \in \nabla\}$ be an $\alpha-$ open cover of X. A multifunction $F : X \to Y$ is $\alpha-$ continuous if and only if the restriction $F \mid U_\alpha : U_\alpha \to Y$ is $\alpha-$ continuous for each $\alpha \in \nabla$.*

Proof. This is an immediate consequence of Theorem 3 and 4.

Definition 2. *A multifunction $F : X \to Y$ is said to be precontinuous if for each $x \in X$ and any open sets G_1, G_2 of Y such that $F(x) \subset G_1$ and $F(x) \cap G_2 \neq \emptyset$, there exists $U \in PO(X)$ containing x such that $F(U) \subset G_1$ and $F(u) \cap G_2 \neq \emptyset$ for every $u \in U$.*

Remark 2. If a multifunction is precontinuous, then it is upper and lower precontinuous by Theorems 2.3 and 2.4 of [17].

Lemma 2. *A multifunction $F : X \to Y$ is precontinuous if and only if for any open sets G_1, G_2 of Y, $F^+(G_1) \cap F^-(G_2) \in PO(X)$.*

Proof. Necessity. Let F be precontinuous and G_1, G_2 any open sets of Y. Let $x \in F^+(G_1) \cap F^-(G_2)$. Then $F(x) \subset G_1$ and $F(x) \cap G_2 \neq \emptyset$ and hence there exists $U \in PO(X)$ containing x such that $F(U) \subset G_1$ and $F(u) \cap G_2 \neq \emptyset$ for every $u \in U$. Therefore, we have $U \subset F^+(G_1) \cap F^-(G_2)$ and hence

$$x \in U \subset Int(Cl(U)) \subset Int(Cl(F^+(G_1) \cap F^-(G_2))).$$

This shows that $F^+(G_1) \cap F^-(G_2) \in PO(X)$.

Sufficiency. Let $x \in X$ and G_1, G_2 be any open sets of Y such that $F(x) \subset G_1$ and $F(x) \cap G_2 \neq \emptyset$. Put $U = F^+(G_1) \cap F^-(G_2)$, then $x \in U \in PO(X), F(U) \subset G_1$ and $F(u) \cap G_2 \neq \emptyset$ for every $u \in U$. Therefore, F is precontinuous.

Definition 3. *A multifunction $F : X \to Y$ is said to be quasicontinuous [2] if for each $x \in X$, any neighbourhood U of x and any open sets G_1, G_2 of Y such that $F(x) \subset G_1$ and $F(x) \cap G_2 \neq \emptyset$, there exists a nonempty open set $G_U \subset U$ such that $F(G_U) \subset G_1$ and $F(g) \cap G_2 \neq \emptyset$ for every $g \in G$.*

Lemma 3. *(Banzaru [2]). A multifunction $F : X \to Y$ is quasi continuous if and only if for any open sets G_1, G_2 of Y, $F^+(G_1) \cap F^-(G_2) \in SO(X)$.*

Remark 3. If a multifunction is quasi continuous, then it is upper and lower quasi continuous by Theorems 2.3 and 2.4 of [6].

Theorem 5. *A multifunction is $\alpha-$ continuous if and only if it is precontinuous and quasi continuous.*

Proof. This follows from Theorem 1, Lemmas 2 and 3 and [14, Lemma 3.1].

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REZIME

KARAKTERIZACIJE α - NEPREKIDNIH MULTIFUNKCIJA

U radu je uvedena notacija $\alpha-$ neprekidnih multifunkcija i date su neke njihove karakteristike.

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