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WEAK CONTRACTOR DIRECTIONS AND WEAK DIRECTIONAL CONTRACTIONS FOR A SET VALUED OPERATOR WITH A CLOSED RANGE

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Abstract

The weak contractor directions and weak directional contractions for nonlinear set - valued operators are defined and used for obtaining very general solvability theorem for a class of nonlinear set - valued operator equations. The given theorems improve and generalize some impotrant results in [1 – 5].

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1. Introduction

The theory of Altman's contractor directions [1] and [2] provides extremely useful and important tools for obtaining solvability theorems of nonlinear operators with closed ranges. As a further development of the general theory of contractor directions, Altman [3] introduced two new concepts: weak contractor directions and weak directional contractions and obtained a general solvability theorem which generalizes some of the known in literature.

In [5], we introduced the concepts of weak contractor directions and weak directional contractions for nonlinear set-valued operators with closed ranges and obtained general solvability theorems for nonlinear set-valued operator equations which generalize the corresponding results in [1], [2] and [3].

In this paper, we shall define weak contractor directions and weak directional contractions for nonlinear set-valued operators and allow the contractor directions to have more general nonlinear majorant functions. Using the concepts, we obtain very general solvability theorem for nonlinear set-valued operator equations. As a consequence, a solvability theorem can be obtained for a class of nonlinear set-valued operators which are called weak directional contractions. Our theorems improve and generalize some important results in [1], [2], [3], [4] and [5].

2. Preliminary definitions and lemmas

Let Y be a Banach space. $CB(Y)$ denotes the family of all the non-empty bounded closed subsets of Y. For $y \in Y$, $A \subset Y$ $D(y, A) = \inf\{d(y, a) : a \in A\}$ and $H(\cdot, \cdot)$ denote the Hausdorff metric on $CB(Y)$ deduced by the norm in Y.

Let $\mathbf{R}_+$ be the set of all the positive real numbers. We denote by F the set of all the functions $q : \mathbf{R}_+ \to \mathbf{R}_+$ such that $q(t) < t$ for each $t > 0$ and $\limsup_{s \to t^+} q(s) < t$ for every $t > 0$. Moreover, we denote by F^* the set of all the functions $f : \mathbf{R}_+ \to \mathbf{R}_+$ which are upper semicontinuous and such that $f(t) < t$ for each $t > 0$. Obviously, the inclusion $F^* \subset F$ holds and the inverse is not true (see [6]).

Lemma 2.1. *Let the function $q \in F$. Then, for any $a, b \in \mathbf{R}_+$ with $0 < a \leq b < \infty$, the number*

$$c = \inf\{1 - \frac{q(t)}{t} : a \leq t \leq b\}$$

satisfies that $0 < c < 1$.

Proof. Clearly,

$$\sup\{\frac{q(t)}{t} : a \leq t \leq b\} > 0.$$

Now, we shall show that

$$\sup\{\frac{q(t)}{t} : a \leq t \leq b\} < 1$$

for any $a, b \in \mathbf{R}_+$, $0 < a \leq b < \infty$. If it is not true, we may assume that there exist $a, b \in \mathbf{R}_+$ with $0 < a \leq b < \infty$ such that

$$\sup\{\frac{q(t)}{t} : a \leq t \leq b\} = 1.$$

Thus, we easily choose a decreasing sequence $\{t_n\} \subset [a, b]$ such that $q(t_n)/t_n > 1 - 1/(n+1)$, $n = 1, 2, \ldots$. Let $t_n \to t^*$. Then we have $\limsup_{s \to t^{*+}} q(s) \geq t^*$. This is in contradiction with $\limsup_{s \to t^{*+}} q(s) < t^*$. Hence, we have that

$$0 < \sup\{\frac{q(t)}{t} : a \leq t \leq b\} < 1$$

for all $a, b \in \mathbf{R}_+$ with $0 < a \leq b < \infty$, and so

$$\begin{aligned} 0 < c &= \inf\{1 - \frac{q(t)}{t} : 0 < a \leq t \leq b\} \\ &= 1 - \sup\{\frac{q(t)}{t} : 0 < a \leq t \leq b\} < 1. \square \end{aligned}$$

In this paper, we assume throughout that the nonlinear majorant functions of weak contractor directions and weak directional contractions belong to F.

Definition 2.1. *Let X be an abstract set and $P : X \to CB(Y)$ a set-valued operator of X into $CB(Y)$. Given a function $q \in F$. We define sets $\Gamma_X^*(P) \subset Y$ of weak contractor directions for P at x, as follows. An element $y \in Y$ is a weak contractor direction, i. e., $y \in \Gamma_X^*(P)$ if there exists a positive $\varepsilon = \varepsilon(x, y) \leq 1$ and an element $\bar{x} \in X$ such that*

$$H(P\bar{x}, Px + \varepsilon y) \leq \varepsilon q(\max\{\|y\|, D(0, Px), D(0, Px)\}), \tag{1}$$

where 0 is the zero element of Banach space Y. If $q(t) = kt$, with $0 < k < 1$, then $\Gamma_X^*(P)$ is a set of contractor directions for P at x denoted by $\Gamma_X(P)$, which is also a generalization of the corresponding concepts in [1] and [2].

Lemma 2.2. *Let $A, B \in CB(Y)$ and $a \in A$. Then for each real number $r > 1$, there exists an element $b \in B$ such that*

$$\|a - b\| \leq rH(A, B).$$

Lemma 2.3. *([2]) Let α be an ordinal number of the first or second class and let $\{t_\gamma\}_{0 \leq \gamma \leq \alpha}$ be a well-ordered sequence of real numbers provided, for ordinal numbers β of the second kind (= limit number), we have*

$$t_\beta = \lim_{\gamma \to \beta} t_\gamma.$$

Then, the following equality holds,

$$t_\alpha = t_0 + \sum_{0 \leq \gamma < \alpha} (t_{\gamma+1} - t_\gamma).$$

Lemma 2.4. *([1]) Let alpha be an ordinal number of the first or second class and let $\{x_\gamma\}_{0 \leq \gamma \leq \alpha}$ be a well-ordered sequence of elements of metric space X provided*

$$x_\beta = \lim_{\gamma \to \beta} x_\gamma.$$

Then,

$$d(x_\alpha, x_0) \leq \sum_{0 \leq \gamma < \alpha} d(x_{\gamma+1}, x_\gamma).$$

Lemma 2.5. *Let $A, B \in CB(Y)$ and $y \in Y$. Then,*

$$H(A, B) \leq H(A, B - y) + \|y\|.$$

Proof. This lemma easily follows rom the definition of a Hausdorff metric.
□

Lemma 2.6. *Let α be an ordinal number of the second class and let $\{A_\gamma\}_{0 \leq \gamma < \alpha}$ be a well ordered sequence of elements of $CB(Y)$ such that*

$$\lim_{\gamma \to \alpha} H(A_\gamma, A_\alpha) = 0$$

where $A_\alpha \in CB(Y)$. If $y_\gamma \in A_\gamma$, $\forall 0 \leq \gamma < \alpha$ and $\lim_{\gamma \to \alpha} y_\gamma = y_\alpha$, then $y_\alpha \in A_\alpha$.

Proof. For all $0 \leq \gamma < \alpha$, we have that

$$D(y_{\alpha}, A_{\alpha}) \leq \|y_{\gamma} - y_{\alpha}\| + H(A_{\gamma}, A_{\alpha}).$$

Letting $\gamma \to \alpha$, we obtain $D(y_{\alpha}, A_{\alpha}) = 0$. Since A_{α} is closed, therefore $y_{\alpha} \in A_{\alpha}$. □

3. Weak contractor directions

In order to prove a solvability theorem for nonlinear set-value operator P with weak contractor directions, we shall begin with the following:

Theorem 3.1. *Let X be an abstract set, Y a Banach space, $P : X \to CB(Y)$ a nonlinear set-valued operator such that $P(X) = \bigcup_{x \in X} Px$ is closed in Y and $\{Px :\ x \in X\}$ is closed in $(CB(Y), H)$. If $-Px \subset \Gamma^{*}_{X}(P)$ for all $x \in X$, that the set-valued operator equation*

$$0 \in Px$$

has a solution in X.

Proof. We can assume that there exists a constant $k > 0$ such that for all $x \in X$

$$D(0, Px) > k. \tag{2}$$

For, if this is not a case, then there exists a sequence $\{x_n\} \subset X$ such that $D(0, Px_n) \to 0$ as $n \to \infty$. It follows that there exists $y_n \in Px_n \subset P(X)$, $n = 1, 2, \ldots$, such that $y_n \to 0$ as $n \to \infty$. Since $P(X)$ is closed in Y, we have $0 \in Px$. Hence, there exists an element $x^* \in X$ such that $0 \in Px^*$, i. e., the theorem holds. Now, we denote by Λ the set of all countable ordinal numbers. That is Λ is the set f all the ordinal numbers less than Ω, the first uncountable ordinal. Now we can construct well ordered sequences of nonnegative real numbers t_{γ}, $\gamma \in \Lambda$, elements $y(t_{\gamma}) \in Y$ and sets $P(t_{\gamma}) \in \{Px :\ x \in X\}$ as follows. Let $t_0 = 0$ and let x_0 be an arbitrary element of X, and put $P(t_0) = Px_0$. Let $y(t_0)$ be an arbitrary element of $P(t_0)$ Suppose that t_{γ}, $P(t_{\gamma})$ and $y(t_{\gamma}) \in P(t_{\gamma})$ have been constructed for all $\gamma < \alpha$, and satisfy:

$$\text{(3)} \qquad \text{(i)} \qquad y(t_{\gamma}) \in P(t_{\gamma}),$$

(ii) for any ordinal numbers $\gamma < \alpha$, we have that

$$\|y(t_\gamma)\| \leq \exp(-ct_\gamma)\|y(t_0)\|, \tag{4}$$

where

$$c = \inf\{1 - \frac{q(s)}{s} : \ k \leq s \leq \|y(t_0)\|\}, \tag{5}$$

with k determinated by (2). It follows from Lemma 2.1 that $0 < c < 1$.
(iii) for the first ordinal numbers $\gamma + 1 < \alpha$ the following inequalities are satisfied:

$$H(P(t_{\gamma+1}), P(t_\gamma)) \leq 2\|y(t_0)\|(t_{\gamma+1} - t_\gamma)\exp(-ct_\gamma), \tag{6}$$

$$\|y(t_{\gamma+1}) - y(t_\gamma)\| \leq 2\|y(t_0)\|(t_{\gamma+1} - t_\gamma)\exp(-ct_\gamma), \tag{7}$$

and

$$0 < t_{\gamma+1} - t_\gamma \leq 1. \tag{8}$$

(iv) for the second ordinal numbers $\gamma < \alpha$ the following relations hold:

$$t_\alpha = \lim_{\gamma\to\alpha} t_\gamma, \ y(t_\alpha) = \lim_{\gamma\to\alpha} y(t_\gamma), \ P(t_\alpha) = \lim_{\gamma\to\alpha} P(t_\gamma). \tag{9}$$

Then, from (6), (8), (9), Lemma 2.3 and 2.4, it follows that for arbitrary $\lambda < \gamma < \alpha$ we have

$$\begin{aligned}
(10)\ H(P(t_\gamma), P(t_\lambda)) &\leq \sum_{\lambda\leq\beta<\gamma} H(P(t_{\beta+1}), P(t_\beta)) \\
&\leq 2\|y(t_0)\| \sum_{\lambda\leq\beta<\gamma} \exp(-ct_\beta)(t_{\beta+1} - t_\beta) \\
(11) \qquad &= 2\|y(t_0)\| \sum_{\lambda\leq\beta<\gamma} \exp(c(t_{\beta+1} - t_\beta)) nonumber \\
&\quad \exp(-ct_\beta)(t_{\beta+1} - t_\beta) \\
&\leq 2\|y(t_0)\|\exp(c) \sum_{\lambda\leq\beta<\gamma} \exp(-ct_{\beta+1})(t_{\beta+1} - t_\beta) \\
&\leq 2\|y(t_0)\|\exp(c) \sum_{\lambda\leq\beta<\gamma} \int_{t_\beta}^{t_{\beta+1}} \exp(-ct)dt \\
&= 2\|y(t_0)\|\exp(c) \int_{t_\lambda}^{t_\gamma} \exp(-ct)dt.
\end{aligned}$$

Similarly, from (7), (8), (9), Lemma 2.3 and 2.4, we obtain that

$$\|y(t_\gamma) - y(t_\lambda)\| \le 2\|y(t_0)\| \exp(c) \int_{t_\lambda}^{t_\beta} \exp(-ct)dt \tag{12}$$

Now, assume that α is the first kind ordinal number. If $0 \in P(t_{\alpha-1})$, then the theorem holds from $P(t_{\alpha-1}) \in \{Px : x \in X\}$. THus we may assume that $0 \notin P(t_{\alpha-1})$, and let $x \in X$ such that $P(t_{\alpha-1}) = Px$. Then, by the hypotheses of the theorem, there exist a positive $\varepsilon \le 1$ and an element $\bar{x} \in X$ such that for all $-y \in -Px \subset \Gamma_X^*(P)$, (1) holds. Put $\varepsilon_{\alpha-1} = \varepsilon \le 1$, and

$$t_\alpha = t_{\alpha-1} + \varepsilon_{\alpha-1},\ P(t_\alpha) = P\bar{x}. \tag{13}$$

Then we obtain by using (1) and $y(t_{\alpha-1}) \in P(t_{\alpha-1})$ that

$$\begin{aligned} & H(P(t_\alpha), P(t_{\alpha-1}) - \varepsilon y(t_{\alpha-1})) \\ \le\ & \varepsilon q(\max\{\|y(t_{\alpha-1})\|, D(0, P(t_{\alpha-1})), D(0, P(t_\alpha))\}) \\ \le\ & \varepsilon q(\max\{\|y(t_{\alpha-1})\|, D(0, P(t_\alpha))\}). \end{aligned} \tag{14}$$

Now, we shall show that $D(0, P(t_\alpha)) \le \|y(t_{\alpha-1})\|$. If $D(0, P(t_\alpha)) > \|y(t_{\alpha-1})\|$, it follows from (14) and Lemma 2.5 that

$$\begin{aligned} D(0, P(t_\alpha)) &\le H(P(t_\alpha), P(t_{\alpha-1}) - y(t_{\alpha-1})) \\ &= H(P(t_\alpha), P(t_{\alpha-1}) - \varepsilon y(t_{\alpha-1})) - (1-\varepsilon)\|y(t_{\alpha-1})\| \\ &\le (1-\varepsilon)\|y(t_{\alpha-1})\| \\ &+ \varepsilon q(\max\{\|y(t_{\alpha-1})\|, D(0, P(t_{\alpha-1})), D(0, P(t_\alpha))\}) \\ &\le (1-\varepsilon)D(0, P(t_\alpha)) + q(D(0, P(t_\alpha))) \\ &< (1-\varepsilon)D(0, P(t_\alpha)) + \varepsilon D(0, P(t_\alpha)) = D(0, P(t_\alpha)). \end{aligned}$$

This is contradiction. Hence, we have

$$D(0, P(t_\alpha)) \le \|y(t_{\alpha-1})\|. \tag{15}$$

From $0 < c < 1$ and $0 < \varepsilon \le 1$ it follows that

$$\min\{\frac{1}{1-c}, 1 + \frac{\varepsilon c^2}{1-c}(\frac{1}{2} - \frac{\varepsilon c}{6})\} > 1.$$

Let the real number b satisfy

$$1 < b < \min\{\frac{1}{1-c}, 1 + \frac{\varepsilon c^2}{1-c}(\frac{1}{2} - \frac{\varepsilon c}{6})\}. \tag{16}$$

By Lemma 2.2 and (15), we obtain that there exists an element $y(t_\alpha) \in P(t_\alpha)$ such that

$$
\begin{aligned}
(17) \qquad & \|y(t_\alpha) - (1-\varepsilon)y(t_{\alpha-1})\| \\
\leq\ & bH(P(t_\alpha), P(t_{\alpha-1}) - \varepsilon y(t_{\alpha-1})) \\
\leq\ & b\varepsilon q(\max\{\|y(t_{\alpha-1})\|, D(0, P(t_{\alpha-1})), D(0, P(t_\alpha))\}) \\
\leq\ & b\varepsilon q(\|y(t_{\alpha-1})\|).
\end{aligned}
$$

From (17), (16), (4) and (5), it follows that

$$
\begin{aligned}
(18) \qquad \|y(t_\alpha)\| &\leq (1-\varepsilon)\|y(t_{\alpha-1})\| + b\varepsilon q(\|y(t_{\alpha-1})\|) \\
&= (1-\varepsilon)\|y(t_{\alpha-1})\| + b\varepsilon \frac{q(\|y(t_{\alpha-1})\|)}{\|y(t_{\alpha-1})\|}\|y(t_{\alpha-1})\| \\
&\leq (1-\varepsilon)\|y(t_{\alpha-1})\| + b\varepsilon(1-c)\|y(t_{\alpha-1})\| \\
&= (1-\varepsilon(1-b(1-c)))\|y(t_{\alpha-1})\| \\
&\leq (1 - c\varepsilon + \frac{c^2\varepsilon^2}{2} - \frac{c^3\varepsilon^3}{6})\|y(t_{\alpha-1})\| \\
&< \exp(-c\varepsilon)\|y(t_{\alpha-1})\| \\
&\leq \exp(-c\varepsilon)\exp(-ct_{\alpha-1})\|y(t_0)\| \\
&= \exp(-ct_\alpha)\|y(t_0)\|.
\end{aligned}
$$

Since $y(t_\alpha) \in P(t_\alpha)$, by virtue of Lemma 2.5, (14), (15), (5), (18) and (4), we get

$$
\begin{aligned}
H(P(t_\alpha), P(t_{\alpha-1})) &\leq H(P(t_\alpha), P(t_{\alpha-1}) - \varepsilon y(t_{\alpha-1})) + \varepsilon\|y(t_{\alpha-1})\| \\
&\leq \varepsilon\|y(t_{\alpha-1})\| + \varepsilon q(\|y(t_{\alpha-1})\|) \\
&\leq (1 + (1-c))\varepsilon\|y(t_{\alpha-1})\| \\
&\leq 2\|y(t_0)\|(t_\alpha - t_{\alpha-1})\exp(-ct_{\alpha-1}).
\end{aligned}
$$

Furthermore, we obtain from (17), (16), (13) and (5)

$$
\begin{aligned}
\|y(t_\alpha) - y(t_{\alpha-1})\| &\leq \varepsilon\|y(t_{\alpha-1})\| + b\varepsilon q(\|y(t_{\alpha-1})\|) \\
&\leq \varepsilon(1 + b(1-c))\|y(t_{\alpha-1})\| \\
&\leq 2\|y(t_0)\|(t_\alpha - t_{\alpha-1})\exp(-ct_{\alpha-1}).
\end{aligned}
$$

Thus, induction assumptions (3_α), (4_α), (6_α) - (8_α) are satisfied for t_α.

Now, suppose that α is an ordinal number of the second kind and put $t_\alpha = \lim_{\gamma\to\alpha} t_\gamma$. Let $\{\gamma_n\}$ be an increasing sequence converging to α. It follows from (10) and (12) that $\{P(t_{\gamma_n})\}$ and $\{y(t_{\gamma_n})\}$ are Cauchy sequences, and so are $\{P(t_\gamma)\}$ and $\{y(t_\gamma)\}$. Let $P(t_\alpha) = \lim_{\gamma\to\alpha} P(t_\gamma)$ and $y(t_\alpha) = \lim_{\gamma\to\alpha} y(t_\gamma)$. It follows from Lemma 2.6 that $y(t_\alpha) \in P(t_\alpha)$. Since $y(t_\gamma)$ satisfy (4_γ), therefore $y(t_\alpha)$ satisfies (4_α). This process will terminate if $t = +\infty$, where α is of the second kind. In the case (4_α) yields $y(t_\alpha) = 0$ and so $0 \in P(t_\alpha) \in \{Px : x \in X\}$. The proof of theorem is completed. □

Corollary 3.1. *Let X be an abstract set, Y a Banach space, $P : X \to Y$ a nonlinear point-valued operator such that $P(X)$ is closed in Y. If $-Px \in \Gamma_X^*$ for each $x \in X$, then equation $Px = 0$ has a solution in X.*

Remark 3.1. *Note that our definitions of nonlinear majorant functions and the set of weak contractor directions are more general that those in [3], [5]. Thus Theorem 3.1 and Corollary 3.1 improve and generalize Theorem 1.1 of [3] and Theorem 3.1 and Corollary 3.1 of [5].*

Theorem 3.2. *Let X be an abstract set, Y a Banach space. The function $q \in F$ and $P : X \to CB(Y)$ is such that $P(X) = \bigcup_{x\in X} Px$ is closed in Y and $\{Px : x \in X\}$ is closed in $(CB(Y), H)$. If for all $x \in X$ and $y \in Y$, there exist positive $\varepsilon = \varepsilon(x, y) \leq 1$ and $\bar{x} \in X$ such that*

$$H(P\bar{x}, Px + \varepsilon y) \leq \varepsilon q(\max\{\|y\|, D(u, Px), D(u, P\bar{x})\}) \tag{19}$$

for any $u \in Y$, then the nonlinear set-valued operator equation $u \in Px$ has a solution in X and $P(X) = Y$.

Proof. For an arbitrary given $u \in Y$, put $\bar{P}x = Px - u$. By (19)

$$H(\bar{P}\bar{x}, \bar{P}x + \varepsilon y) \leq \varepsilon q(\max\{\|y\|, D(u, \bar{P}x), D(u, \bar{P}\bar{x})\}).$$

Since $\Gamma_X^*(\bar{P}) = \Gamma_X^*(P) = Y$, for all $x \in X$, it follows that $-Px \in \Gamma_X^*(\bar{P})$ for all $x \in X$. Clearly, $\bar{P}x = Px - u$ is closed in Y and $\{\bar{P}x : x \in X\} = \{Px - u : x \in X\}$ also is closed in $(CB(Y), H)$. Hence, by Theorem 3.1, equation $0 \in \bar{P}x$ has a solution in X, and consequently, equation $u = Px$ has a solution in X. This completes the proof. □

Remark 3.2. *Theorem 3.2 of [1] (see also Theorem 1.2 of [2]), Theorem 1.2 of [3] and Therem 3.2 of [5] are very special cases of Theorem 3.2.*

4. Weak directional set-valued contractions

Definition 4.1. *Let X ba a Banach space and a function $q \in F$. A mapping $T : D(T) \to CB(Y)$ is called a weak directional set-valued contraction if for each $x \in D(T)$ and $y \in X$ there exits a positive $\varepsilon = \varepsilon(x, y) \leq 1$ such that $x + \varepsilon y \in D(T)$ and*

$$
\begin{aligned}
(20) \qquad & H(T(x+\varepsilon y), Tx) \\
\leq \; & \varepsilon q(\max\{\|y\|, D(y, x - Tx), D(y, x + \varepsilon y - T(x + \varepsilon y))\}).
\end{aligned}
$$

The following global solvability theorem holds for weak directional set-valued contractions.

Theorem 4.1. *Let $T : D(T) \to CB(Y)$ be a weak directional set-valued contraction. If $P = I - T$ (I is identity mapping) is such that the set $P(D(T)) = \bigcup_{x \in D(T)} Px$ is closed in X and the set $\{Px : x \in D(T)\}$ is closed in $(CB(X), H)$, then for each $y \in X$, set-valued operator equation $y \in Px$ has a solution in $D(T)$. Furthermore, we have $X = P(D(T))$.*

Proof. Since $P = I - T$, by (20), we have

$$
\begin{aligned}
H(P(x+\varepsilon y), Px + \varepsilon y) &= H(x + \varepsilon y - T(x+\varepsilon y), x - Tx + \varepsilon y) \\
&= H(T(x+\varepsilon y), Tx) \\
&\leq \varepsilon q(\max\{\|y\|, D(y, Px), D(y, P(x+\varepsilon y))\}).
\end{aligned}
$$

It is easy to check that the hypotheses of Theorem 3.2 are satisfied and the proof is completed. □

Remark 4.1. *Theorem 4.1 is the improvement and generalization of Theorem 2.1 of [3].*

Corollary 4.1. *Let X and q satisfy the hypotheses in Theorem 4.1. Suppose that $T : D(T) \subset X \to CB(X)$ such that for each $x \in D(X)$ and $y \in X$, there exist a positive $\varepsilon = \varepsilon(x, y) \leq 1$ such satisfying: $x + \varepsilon y \in D(T)$ and*

$$
(21) \qquad H(T(x+\varepsilon y), Tx) \leq \varepsilon q(\|y\|).
$$

If $P = I - T$ is such that $P(D(T))$ is closed in X and $\{Px : x \in X\}$ is closed in $(CB(X), H)$. Then for $y \in X$, equation $y \in Px$ has a solution in $D(T)$ and so $P(D(T)) = X$. Specially, T has a fixed point in $D(T)$.

Remark 4.2. *Letting $\varepsilon(x,y) = 1$ or all $x \in D(T)$ and $y \in X$, Corollary 4.1 is the set-valued generalization of the theorem of Boyd and Wong [4].*

5. Weak directional set-valued contractions in a narrow sense

Definition 5.1. *Let W be a convex subset of a Banach space X and a function $q \in F$. A mapping $T : W \subset X \to CB(W)$ is called a weak directional set-valued contraction in a narrow sense if for each $x \in W$ there exits a positive $\varepsilon = \varepsilon(x) \leq 1$ such that for each $y \in Tx - x$*

$$(22) H(T(x+\varepsilon y), Tx) \leq \varepsilon q(\max\{\|y\|, D(x,Tx), D(x+\varepsilon y, T(x+\varepsilon y))\}).$$

Theorem 5.1. *Let $T : W \subset X \to CB(W)$ be a weak directional set-valued contraction in a narrow sense. If $P = I - T$ is such that $P(W) = \bigcup_{x \in W} Px$ is closed in X and $\{Px : x \in W\}$ is closed in $(CB(X), H)$, then T has a fixed point in W.*

Proof. By $P = I - T$ and (22), for each $x \in W$ and $y \in Tx - x$, we have

$$\begin{aligned} H(P(x+\varepsilon y), Px+\varepsilon y) &= H(x+\varepsilon y - T(x+\varepsilon y), x - Tx + \varepsilon y) \\ &= H(T(x+\varepsilon y), Tx) \\ &\leq \varepsilon q(\max\{\|y\|, D(0,Px), D(0, T(x+\varepsilon y))\}). \end{aligned}$$

Since $\bar{x} = x + \varepsilon y \in x + \varepsilon(Tx - x) = (1-\varepsilon)x + \varepsilon Tx \subset W$ and $Tx - x \subset \Gamma_X^*(P)$, i. e. $-Px = Tx - x \subset \Gamma_X^*(P)$ for all $x \in W$. From Theorem 3.1 it follows that there exists an element $x^* \in W$ such that $0 \in Px^* = x^* - Tx^*$ and hence $x^* \in Tx^*$. This completes the proof. □

Corollary 5.1. *Let W be a convex subset of a Banach space X and a function $q \in F$. Suppose that $T : W \subset X \to W$ satisfies that for each $x \in W$, there exists a positive $\varepsilon = \varepsilon(x) \leq 1$ such that for each $y = Tx - x$,*

$$\|T(x+\varepsilon y) - Tx\| \leq \varepsilon q(\max\{\|y\|, \|x - Tx\|, \|x + \varepsilon y - T(x+\varepsilon y)\|\}).$$

If the range $(I - T)(W)$ is closed in X, then T has a fixed point in W.

Remark 5.1. *Theorem 5.1 and Corollary 5.1 are generalizations of Theorem 3.4 of [3].*

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REZIME

SLABE DIREKCIONE KONTRAKCIJE I SLABI PRAVCI KONTRAKCIJE ZA SKUPOVNE OPERATORE

Slabi pravci kontrakcije i slabe direkcione kontrakcije za nelinearne skupovne operatore su definisani i korišćeni za dobijanje vrlo opštih teorema o rešenju za jednu klasu nelinearnih skupovnih operatorskih jednačina. Date teoreme poboljšavaju i uopštavaju neke vrlo važne rezultate u [1 – 5].

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