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ON BUCHWALTER AND SCHMETS' THEOREMS

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Abstract

In this paper new characterizations of real-compact spaces are obtained and new proofs of Buchwalter and Schmets' theorems are given.

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1. Introduction

Throughout this paper X will stand for any Hausdorff completely regular topological space, βX for the Stone-Cech compactification of $X, \nu X$ for the Hewitt real-compactification of $X, C_c(X)$ for the space of the continuous real valued functions on $X, C(X)$, endowed with the compact-open topology and $C_s(X)$ when endowed with the pointwise convergence topology. A subset B of X is called a bounding if $f(B)$ is bounded for each $f \in C(X)$. X is called a μ-space if each bounding subset of X is relatively compact and X is called real-compact if $X = \nu X$.

The theorems of Nachbin [4] and Shirota [7] gave rise to mixed studies of General Topology and Functional Analysis. They showed that X is a μ-space if and only if $C_c(X)$ is barrelled and that X is real-compact if and only if $C_c(X)$ is bornological. Later on, De Wilde and Schmets, [2], showed the latter to be true if and only if $C_c(X)$ is ultrabornological. Buchwalter and Schmets, [1], studied the relations between the topological properties of X and $C_s(X)$ showing that $C_s(X)$ is barrelled if and only if each bounding subset of X is finite, that $C_s(X)$ is bornological if and only if X is real-compact and that $C_s(X)$ is ultrabornological if and only if X is real-compact and each compact subset of X is finite.

The aim of this paper is to obtain new characterizations of real-compact spaces and give new proofs of Buchwalter and Schmets' theorems.

2. Some properties of $C_s(X)$

Given $f \in C(X), f^* : \beta X \to \mathbf{R}^*$ will denote its continuous extension on βX, where $\mathbf{R}^*$ is the Alexandroff compactification of $\mathbf{R}$ and $supp f^*$ will denote the support of f, i.e. the closure in βX of $\{x \in \beta X : f^*(x) \neq 0\}$. For each non-void absolutely convex subset L of $C(X)$ there exists a minimum compact in βX, $supp\ L$, such that if $f \in C(X)$ and $supp\ L \cap supp f^* = \emptyset$ then $f \in L$, [6, II. 1. 3.]

Lemma 1. *Let L be an absolutely convex subset of $C_s(X)$. If L is a neighbourhood of the origin then $supp\ L$ is a finite subset of X.*

Proof. If L is an absolutely convex neighbourhood of the origin in $C_s(X)$, there exists a finite subset K of X and a positive real number ε such that $\varepsilon\Phi_K \subset L$, $\Phi_K := \{f \in C(X) : |f(x)| \leq 1\ \forall x \in K\}$. Hence $supp\ L \subset supp\ \varepsilon\Phi_K \subset K$. □

Lemma 2. *If $A \subset X$ then $\Phi_A := \{f \in C(X) : |f(x)| \leq 1 \forall x \in A\}$ is a neighbourhood of the origin in $C_s(X)$ if and only if $supp\ \Phi_A$ is a finite subset of X.*

Proof. If $supp\Phi_A$ is a finite subset of X, $\Phi_{supp\ \Phi_A} = \{f \in C(X) : \sup|f(x)| \leq 1,\ x \in supp\Phi_A\} = \{f \in C(X) : \sup|f(x)| \leq 1,\ x \in A^{-\beta X}\} = \Phi_A$ is a neighbourhood of the origin in $C_s(X)$. □

We shall say a sequence $\{f_n : n \in \mathbf{N}\}$ in $C_s(X)$ is complete if $\sum_{n=1}^{\infty} \alpha_n f_n$ is convergent for each $(\alpha_n)_{n=1}^{\infty} \in \omega = \mathbf{R}^{\mathbf{N}}$. Clearly, if this is the case,

$\{f_n : n \in \mathbf{N}\}$ is a null sequence in $C_s(X)$. Besides, it is easy to check that if $\{suppf_n : n \in \mathbf{N}\}$ is locally finite then $\{f_n : n \in \mathbf{N}\}$ is complete.

Lemma 3. *Let $A \subset X$ and $\Phi_A := \{f \in C(X) : \sup|f(x)| \leq 1,\ x \in A\}$. The following are equivalent:*

(i) A is bounding.

(ii) Φ_A is neighbourhood of the origin in the strong topology of $C_s(X)$.

(iii) Φ_A contains eventually every complete sequence $\{f_n : n \in \mathbf{N}\}$ in $C_s(X)$.

(iv) Φ_A contains eventually every sequence $\{f_n : n \in \mathbf{N}\}$ in $C(X)$ such that $\{suppf_n : n \in \mathbf{N}\}$ is locally finite.

Proof. (i) ⇒ (ii) and (iii) ⇒ (iv) are immediate.

(ii) ⇒ (iii). Assume there is a complete sequence $\{f_n : n \in \mathbf{N}\}$ in $C_s(X)$ which is not eventually contained in Φ_A. Then there exist $a_1 \in A$ and $n_1 \in \mathbf{N}$ such that $|f_{n_1}(a_1)| > 1$. And for each $p \in \mathbf{N}, \{a_1, a_2, \ldots, a_p\} \in A$ there exist $a_{p+1} \in A$ y $n_{p+1} \in \mathbf{N}$ such that $f_{n_{p+1}}(a_i) = 0$ for $i < p+1$, $|f_{n_{p+1}}(a_{p+1})| > 1$. Now $g := \sum_{p=1}^{\infty} \alpha_p f_{n_p} \in C(X)$ for each $(\alpha_n)_{n=1}^{\infty} \in \omega$. Choosing the α_p such that $g(a_i) > i$, we obtain that Φ_A is not absorbing, which is not possible.

(iv) ⇒ (i). If A is not bounding, there exist $g \in C(X)$ and a sequence $\{a_n : n \in \mathbf{N}\}$ in A such that $|g(a_n)| > |g(a_{n-1})| + 1$. Now for each $n \in \mathbf{N}$ the set $V_n := \{x \in X : |g(a_n) - g(x)| < 1/2\}$ is an open neighbourhood of a_n, so there is an $f_n \in C(X)$ such that $f_n(a_n) = 2$ and $f_n(y) = 0$ for every $y \in X \backslash V_n$. Therefore $suppf_n \subset \overline{V_n}$ and $\{suppf_n : n \in \mathbf{N}\}$ is locally finite although no f_n belongs to Φ_A. □

3. Buchwalter and Schmets' Theorems

Theorem 1. *$C_s(X)$ is barrelled if and only if each bounding subset of X is finite.*

Proof. If $C_s(X)$ is barrelled and A is bounding, Φ_A is a neighbourhood of the origin in $C_s(X)$ by Lemma 3 and, by Lemma 1, $supp\Phi_A$ is a finite subset of X, i.e. $A^{-\beta X}$ is a finite subset of X and, consequently, so is A. □

Next we shall obtain some characterizations of real-compact spaces.

Proposition 1. *Let X be a Hausdorff completely regular topological space. Then:*

(i) X is real-compact.

(ii) Each absolutely convex subset of $C_s(X)$ which absorbs every locally null sequence is a neighbourhood of the origin.

(iii) Each absolutely convex subset of $C_s(X)$ which absorbs every null sequence is a neighbourhood of the origin.

(iv) $C_s(X)$ is the inductive limit of the subspaces linear span of its null sequences.

(v) $C_s(X)$ is the inductive limit of its countable dimensional subspaces.

(vi) Each sequentially continuous linear mapping of $C_s(X)$ into any locally convex space is continuous.

(vii) Each sequentially continuous linear form on $C_s(X)$ is continuous.

Proof.

(i) $\Rightarrow$ (ii). Let L be an absolutely convex subset of $C_s(X)$ which absorbs every locally null sequence. Then, by [6, II. 3. 2], *supp* L is a finite subset of $\nu X = X$ and, by [6, II. 4. 3], there is a positive real number ε such that $\varepsilon\Phi_{suppL} \subset L$. Hence L is a neighbourhood of the origin in $C_s(X)$.

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) and (iii) $\Rightarrow$ (vi) $\Rightarrow$ (vii) are immediate.

Lastly, let us show (v) $\Rightarrow$ (i) ((vii) $\Rightarrow$ (ii)). Assume $\exists a \in \nu X \setminus X$. Let $\hat{a}$ be the linear form on $C_s(X)$ such that $\hat{a}(f) = f^*(a)$ for each $f \in C(X)$. If we set $L := \{f \in C(X) : |\hat{a}(f)| \leq 1\}$ then *supp* $L = \{a\} \not\subset X$ and, by Lemma 1, $\hat{a}$ is not continuous. On the other hand, given a countable dimensional vector subspace H of $C_s(X)$ (null sequence $\{f_n : n \in \mathbf{N}\}$ of $C_s(X)$, by [5, 10. 1. 10], $\exists b \in X$ such that $\hat{a}(f) = f(b) \forall f \in H$ ($\hat{a}(f_n) = f_n(b) \forall n \in \mathbf{N}$). Clearly, $\hat{a}$ is continuous on $H(\lim_{n\to\infty} \hat{a}(f_n) = \lim_{n\to\infty} f_n(b) = 0)$ and, consequently, $\hat{a}$ is continuous on $C_s(X)$. Contradiciton. □

(ii) of this Proposition and [3, §28.3(2)] give us:

Theorem 2. *$C_s(X)$ is bornological if and only if X is real-compact.*

Theorem 3. *$C_s(X)$ is ultrabornological if and only if X is real-compact and each compact subset of X is finite.*

Proof. ($\Rightarrow$) is clear from Theorems 1 and 2.
($\Leftarrow$) If each compact subset of X is finite $C_c(X) = C_s(X)$ and, by De Wilde and Schmets' Theorem, $C_s(X)$ is ultrabornological. □

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REZIME

O TEOREMAMA BUCHWALTER-A I SCHMETS-A

U ovom radu je data nova karakterizacija realnih kompaktnih prostora i novi dokazi teorema Buchwalter-a i Schmets-a.

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