Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 23, 1 (1993), 65 - 86

Review of Research
Faculty of Science
Mathematics Series

(α,β)– CONVOLUTION IN SPACES WITH THE LAGUERRE EXPANSIONS AND ITS APPLICATIONS

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Abstract

We develope a theory of generalized function spaces LG'_α and its generalizations $LG'_{e\alpha}, \alpha > -1$, which elements have orthonormal expansions with respect to the Laguerre orthonormal systems $l_{n,\alpha}$, $n \in \mathbf{N}_0$, $\alpha > -1$. We define the (α, β) - convolution product and find conditions of solvability of a convolution equations in these spaces. Finally, we give some applications of it in solving integral equations.

AMS Mathematics Subject Classification (1991): 46F12, 34A25, 45D05
Key words and phrases: Laguerre expansions, spaces of generalized functions, convolution equations, integral equations.

1. Introduction

Zemanian introduced in ([10], Ch. 9) the $\mathcal{A}'$ - types spaces of generalized functions which elements have orthonormal series expansions with respect to various orthonormal systems. The generalization of these Zemanian results was given in [3] as $(\exp\mathcal{A}')$– types spaces, which contain $\mathcal{A}'$– type spaces as proper subspaces.

The expansion of generalized functions with respect to the Laguerre orthonormal system has been studied by Zayed [9], Duran [1] and Pilipović [4], [5].

In [6] we have studied the expansion of elements of generalized function spaces LG'_0 and LG'_{e0} with respect to the Laguerre orthonormal system and we present a numerical method for solving convolution equations. Using this numerical approach we found a new sufficient conditions for the existence of solutions of convolution equations which are supplement to Vladimirov's results ([8], Ch. 2, § 13).

In [7] we introduced the so - called (α,β) - convolution for solving convolution equations of the form

$$f \overset{(\alpha,\beta)}{*} g = (x^{\alpha/2}f) * (x^{\beta/2}g) = x^{(\alpha+\beta+1)/2}h,$$

$f \in LG'_\alpha$, $g \in LG'_\beta$, $h \in LG'_{\alpha+\beta+1}$, $*$ is the ordinar convolution of tempered distributions, based on the generalized Laguerre polynomials, $\alpha, \beta > -1$. We gave examples of expansions of elements from LG'_α, $\alpha > -1$, into Laguerre series and characterized the coefficients which appear in these expansions.

In this paper we give conditions for the solvability of $(\alpha,\beta)-$ convolution equations, $\alpha, \beta > -1$, in LG'_α and $LG'_{e\alpha}$ spaces. The spaces $LG'_{e\alpha}$ are the types of exponentional generalized function spaces $(\exp \mathcal{A}')$. First, we give basic facts concerning the fundamental spaces which will be examined. We deduce that the spaces $LG_{e\alpha}$ are equal to the spaces $\exp LG_\alpha$. Secondly, the convolution product is defined in $LG'_{e\alpha}$. The Section 5 is concerned on the mapping between LG'_α spaces. In Section 6. we prove in a simple way that $(\alpha,\beta)-$ convolution equation is solvable in $LG'_{e\alpha}$ iff the first coefficient in the Laguerre expansion of f is different from zero. In Section 7. we give some comments on solvability of $(\alpha,\beta)-$ convolution equations over the spaces LG'_β. We consider $(\alpha,\beta)-$ fundamental solution of $(\alpha,\beta)-$ convolution equation Section 8. In Section 9. we give a remark concerning the approximate error estimate of $(\alpha,\beta)-$ convolution equation. We give finally some applications of these results, on solving Volterra's integral equation of the first kind, and we include some numerical method for solving integral equations. In [6] is proposed an explicit method for finding solution through the Laguerre polynomials, concerning the ordinar convolution equation:

$$f * g = h, \quad f, h \in LG'_0 \quad g \text{ is unknown.}$$

This equation can be treated analitically by using the Laplace transform but this method is impractical from numerical point of view. We propose an explicit method for finding solution trought the Laguerre polynomials. The usefulness of this approach is selfevident in solving equations which can not be performed analitically. So, in Section 10. we present a new numerical method by using $(\alpha,\beta)-$ convolution form of integral equations. For this we employ a system of algebraic equations as in [6].

2. Fundamental spaces

Throughout the paper we shall assume that $\alpha > -1$. Also, we shall use the following notation: $\mathbf{R}_+ = (0,\infty)$, $\overline{\mathbf{R}}_+ = [0,\infty)$, $\mathbf{Z}_+ = \mathbf{R} + i\mathbf{R}_+$, $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$, $\mathbf{N}$ is the set of naturals.

Consider the Laguerre orthonormal system $l_{n,\alpha}$ $n \in \mathbf{N}_0$, in $L^2(\mathbf{R}_+)$:

$$l_{n,\alpha}(x) = \tau_{n,\alpha} x^{\alpha/2} L_n^\alpha(x) e^{-x/2}, \quad x \in \mathbf{R}_+,$$

where

$$\tau_n = (\Gamma(n+1)/\Gamma(n+\alpha+1))^{1/2}$$

and

$$L_n^\alpha(x) = \sum_{m=0}^{n} \binom{n+\alpha}{n-m} \frac{(-x)^m}{m!}, \quad n \in \mathbf{N}_0,$$

are generalized Laguerre polynomials. The $l_{n,\alpha}$ are eigenfunctions for the operator

$$\mathcal{R}_\alpha = x^{\alpha/2} e^{x/2} D x^{\alpha+1} e^{-x} D x^{-\alpha/2} e^{x/2}, \quad \text{i.e.} \quad \mathcal{R}_\alpha(l_{n,\alpha}) = -n l_{n,\alpha}, \quad n \in \mathbf{N}_0.$$

LG_α is the space of smooth functions $\Phi \in C^\infty(\mathbf{R}_+)$ for which all the norms $\|\Phi\|_k$ are finite:

$$\|\Phi\|_k = \|\mathcal{R}_\alpha^k \Phi\|_0 = (\int_0^\infty |\mathcal{R}_\alpha^k \Phi(x)|^2 dx)^{1/2}, \quad k \in \mathbf{N}_0,$$

and for the operator $\mathcal{R}_\alpha$ the following holds:

$$(\mathcal{R}_\alpha^k \Phi, l_{n,\alpha}) = (\Phi, \mathcal{R}_\alpha^k l_{n,\alpha}), \quad k, n \in \mathbf{N}_0;$$

$\mathcal{R}_\alpha^0$ is the identity operator and $\mathcal{R}_\alpha^{k+1} = \mathcal{R}_\alpha(\mathcal{R}_\alpha^k)$. The $L^2(\mathbf{R}_+)-$ inner product is denoted by

$$(\Phi, \psi) = \int_0^\infty \Phi(t)\overline{\psi}(t)dt = <\Phi, \overline{\psi}>, \quad \Phi, \psi \in L^2(\mathbf{R}_+).$$

It is proved in [4] that the space LG_0' is in fact S_+', the space of tempered distributions supported by $\overline{\mathbf{R}}_+$. As well in [4] is given the connection between LG_α' and LG_0' which follows from:

$$\Phi \in LG_\alpha \quad \text{iff} \quad \Phi = x^{\alpha/2}\psi \quad \text{for some} \quad \psi \in LG_0,$$

$$\Phi_n \to 0 \quad \text{in} \quad LG_\alpha,\ n \to \infty, \quad \text{iff} \quad \Phi = x^{\alpha/2}\psi_n,\ \psi_n \in LG_0,\ \ n \in \mathbf{N} \quad \text{and}$$

$$\Phi_n \to 0 \quad \text{in} \quad LG_0,\ \ n \to \infty.$$

Thus, we have $LG_\alpha' = x^{-\alpha/2}LG_0'$ where the dual pairing between $f \in LG_\alpha'$ and $\varphi \in LG_\alpha$ is given by $< f, \varphi > = < x^{\alpha/2}f, x^{-\alpha/2}\varphi >$.

Since we are interested in series expansions of elements on LG_α'– spaces we shall give the equivalent definition of them.

The space $L_{k,\alpha}$, $k \geq 0$, is defined as follows.

$$L_{k,\alpha} = \left\{ \ \Phi \overset{2}{=} \sum_{n=0}^{\infty} a_{n,\alpha} l_{n,\alpha} \in L^2(\mathbf{R}_+);\ |\ \|\Phi\|\ |_{k,\alpha} < \infty \ \right\}.$$

where

$$\|\ |\ \Phi\ |\ \|_{k,\alpha} = \left(\ |a_{0,\alpha}|^2 + \sum_{n=1}^{\infty} |a_{n,\alpha}|^2 n^{2k} \ \right)^{1/2}.$$

$$LG_\alpha = \text{proj} \lim_{k\to\infty} L_{k,\alpha} \qquad ([10]).$$

The dual

$$L_{k,\alpha}' = \left\{ \ \psi = \sum_{n=0}^{\infty} b_{n,\alpha} l_{n,\alpha};\ \|\ |\ \psi\ |\ \|_{k,\alpha}' < \infty \ \right\},\ k \geq 0.$$

$$\|\ |\ \psi\ |\ \|_{k,\alpha} = \left(\ |b_{0,\alpha}|^2 + \sum_{n=1}^{\infty} |b_{n,\alpha}|^2 n^{-2k} \ \right)^{1/2}.$$

$$LG_\alpha' = \text{ind} \lim_{k\to\infty} L_{k,\alpha}'.$$

The strong and the weak convergence in LG_α' are equivalent.

The criteria for the convergence of sequences is similar to the corresponding one in LG_0' ([6]).

Denote by $L_{ek,\alpha}$, $k \geq 0$, the space

$$L_{ek,\alpha} = \left\{ \Phi \overset{2}{=} \sum_{n=0}^{\infty} a_{n,\alpha} l_{n,\alpha} \in L^2(\mathbf{R}_+);\ \| \mid \Phi \mid \|_{ek,\alpha} < \infty \right\},$$

where

$$\| \mid \Phi \mid \|_{ek,\alpha} = \left(|a_{0,k}|^2 + \sum_{n=1}^{\infty} |a_{n,\alpha}|^2 k^{2n} \right)^{1/2} < \infty.$$

When $\alpha = 0$ this is the space L_{ek} ([6]).

These spaces have the properties:

a) $L_{ek,\alpha}$, $k \in \mathbf{R}_+$ are Banach - spaces;

b) the inclusion mapping $L_{ek,\alpha} \to L_{el,\alpha}$, $k > l$, is compact;

c) $LG_{e\alpha} = \text{proj}\lim_{k\to\infty} L_{ek,\alpha}$,

$$LG'_{e\alpha} = \text{ind} \lim_{k\to\infty} L'_{ek,\alpha}.$$

Here, $LG'_{ek,\alpha}$ are duals of $LG_{ek,\alpha}$ supplied by the dual norms.

$$(d) L'_{ek,\alpha} = \left\{ \sum_{n=0}^{\infty} b_{n,\alpha} l_{n,\alpha};\ \left(|b_{0,\alpha}|^2 + \sum_{n=1}^{\infty} |b_{n,\alpha}|^2 k^{-2n} \right)^{1/2} < \infty \right\} = L_{e1/k,\alpha},$$

$k > 0$; (for $k = 0$, $L'_{e0,\alpha} = L^2(\mathbf{R}_+)$).

When $\alpha = 0$,

$$LG_e = \text{proj} \lim_{k\to\infty} L_{ek}, \quad \text{and} \quad LG'_e = \text{ind} \lim_{k\to\infty} L'_{ek} \quad \text{and}$$

$$L'_{ek} = \left\{ \sum_{n=0}^{\infty} b_{n,0} l_{n,0};\ \left(|b_{0,0}|^2 + \sum_{n=1}^{\infty} |b_{n,0}|^2 k^{-2n} \right)^{1/2} < \infty \right\} = L_{e1/k}.$$

The weak and the strong convergence in $LG'_{e\alpha}(LG'_e)$ are equivalent and the sequence

$$f_n = \sum_{m=0}^{\infty} b_{m,n} l_{m,\alpha} \quad \text{converges to} \quad f = \sum_{m=0}^{\infty} b_m l_{m,\alpha}$$

iff for some $k > 0$,

$$\sum_{m=0}^{\infty} |b_{m,n} - b_m|^2 k^{-2m} \to 0, \quad n \to \infty.$$

3. Some isomorphism

The generalization of $\mathcal{A}'-$ type spaces, spaces of $\exp\mathcal{A}'-$ type, were introduced in [3]. We shall consider the special case of these spaces with the generalized Laguerre orthonormal systems. We take $\mathcal{A} = LG_\alpha$, and we shall find the connection between them and $LG_{e\alpha}$.

Define the space $\exp LG_\alpha$ as

$$\exp LG_\alpha = \left\{ \Phi = \sum_{n=0}^{\infty} a_{n,\alpha} l_{n,\alpha}; \quad \text{iff for every } \ k > 0, \ \sum_{n=0}^{\infty} |a_{n,\alpha}|^2 k^{2n} < \infty \right\}.$$

It means, for every $k > 0$ there exists γ_k such that $|a_{n,\alpha}|, \gamma_k k^{-n}$, $n \in \mathbf{N}_0$.

Similarly,

$$\exp LG'_\alpha = \left\{ f = \sum_{n=0}^{\infty} b_{n,\alpha} l_{n,\alpha}; \quad \text{iff for some } \ k > 0, \ \sum_{n=0}^{\infty} |b_{n,\alpha}|^2 k^{-2n} < \infty \right\},$$

or there exists $k > 0$, and γ_k such that $|b_{n,\alpha}| < \gamma_k k^n$, $n \in \mathbf{N}_0$.

By the representation theorem from [3], we have

If $f \in \exp LG'_\alpha$ then there exists a sequence $\{f_n\}_{n=0}^{\infty}$ from $L^2(\mathbf{R}_+)$ and $k \geq 0$ such that

$$(i) \qquad f = \sum_{n=0}^{\infty} \frac{k^n}{n!} \mathcal{R}^n f_n, \qquad\qquad (ii) \qquad \sup_{n\in\mathbf{N}_0} \|f_n\|_2 < \infty.$$

Conversely, if a sequence $\{f_n\}$ from $L^2(\mathbf{R}_+)$ satisfies (ii), with the series (i), a unique element from $\exp LG'_\alpha$ is defined.

We have

Proposition 1. $LG_{e\alpha} = \exp LG_\alpha$.

This gives, particularly, $LG_e = \exp LG_0$.

4. The convolution in $LG'_{e\alpha}$

The definition and the basic properties of $(\alpha, \beta)-$ convolution for the spaces LG'_α were given in [7]. Here, we shall give its $LG'_{e\alpha}-$ form.

Recall, for $f \in LG'_\alpha$, $g \in LG'_\beta$,

$$f \overset{(\alpha,\beta)}{*} g = x^{\alpha/2} f * x^{\beta/2} g, \tag{1}$$

and for $f, g \in LG'_e$ we have:

$$f \overset{(0,0)}{*} g = \lim_{N\to\infty} \left(\sum_{n=0}^{N} b_{n,0} l_{n,0} * \sum_{n=0}^{N} c_{n,0} l_{n,0} \right),$$

where (in the limit) expansions of f and g appear.

Let for $\alpha, \beta > -1$, $\beta - \alpha > -1$,

$$f = \sum_{n=0}^{\infty} b_{n,\alpha} l_{n,\alpha} \in LG'_{e\alpha}, \quad g = \sum_{n=0}^{\infty} x_{n,\beta} l_{n,\beta} \in LG'_{e\beta}, \tag{2}$$

$$h = \sum_{n=0}^{\infty} c_{n,\alpha+\beta+1} l_{n,\alpha+\beta+1} \in LG'_{e\alpha+\beta+1}.$$

Then, in the same way as for LG'_α and LG'_0 ([4]) one can prove:

Proposition 2. *The mapping from $LG'_{e\alpha}$ into LG'_{e0} defined by $f \to x^{\alpha/2} f$ is continuous.*

The convolution $f \overset{(\alpha,\beta)}{*} g$ is defined by

$$f \overset{(\alpha,\beta)}{*} = \lim_{N\to\infty} \left(x^{\alpha/2} \sum_{n=0}^{N} b_{n,\alpha} l_{n,\alpha} \right) * \left(x^{\beta/2} \sum_{n=0}^{N} x_{n,\beta} l_{n,\beta} \right).$$

Proposition 3.

(i) With the $(\alpha, \beta)-$ convolution the space $LG'_{e\alpha}$ is an algebra;

(ii) The explicit form of $(\alpha, \beta)-$ convolution product of f, g is given by

$$f \overset{(\alpha,\beta)}{*} g = x^{(\alpha+\beta+1)/2} h \tag{3}$$

where f, g are from (2) and

$$h = \sum_{n=0}^{\infty} n!/(\Gamma(\alpha+\beta+n+2))(\sum_{p+q=n} (\Gamma(\alpha+p+1)\Gamma(\beta+q+1)/(p!q!))^{1/2}.$$

$$\cdot b_{p,\alpha} x_{q,\beta}) l_{n,\alpha+\beta+1}.$$

(iii) $f \overset{(\alpha,\beta)}{*} g = g \overset{(\beta,\alpha)}{*} f$. *(follows from (ii))* .

We shall give the algebraic representation for $(1,0)-$ convolution product and $(0,\beta)$ - one for later use. With the notation as in (2) we have that h in (3) is equal to:

$$(4) \qquad \sum_{n=0}^{\infty} 1/((n+1)(n+2))(\sum_{p+q=n} (p+1)^{1/2} b_{p,1} x_{q,0}) l_{n,2} = \sum_{n=0}^{\infty} c_{n,2} l_{n,2}$$

$$(5) \qquad \sum_{n=0}^{\infty} n!/\Gamma(\beta+n+2)(\sum_{p+q=n} (\Gamma(\beta+q+1)/q!)^{1/2} b_{p,0} x_{q,\beta}) l_{n,\beta+1} =$$

$$= \sum_{n=0}^{\infty} c_{n,\beta+1} l_{n,\beta+1}.$$

5. The mapping between LG'_α spaces

Proposition 4. *If $f \in LG'_\alpha$, then the mapping: $f \to xf$, from LG'_α onto $LG'_{\alpha-2}$, $\alpha > 1$ is the bijection.*

Proof. The formula

$$(6) \qquad x^2 L_n^\alpha = (n+\alpha)(n+\alpha+1) L_n^{\alpha-2} - 2(n+\alpha)(n+1)$$

$$L_{n+1}^{\alpha-2} + (n+1)(n+2) L_{n+2}^{\alpha-2},$$

$n \in \mathrm{N}_0$, follows from $xL_n^\alpha = (n+\alpha)L_n^{\alpha-1} - (n+1)L_{n+1}^{\alpha-1}$ ([2], (23), p. 190).

If $g = xf$, where

$$f = \sum_{n=0}^{\infty} b_{n,\alpha} l_{n,\alpha}, \text{ than } g = \sum_{n=0}^{\infty} b_{n,\alpha} x l_{n,\alpha}.$$

By using (6) we have

$$g = \sum_{n=0}^{\infty} b_{n,\alpha} \{ \ x^{(\alpha-2)/2} e^{-x/2} (x^2 L_n^\alpha) \ \} = \sum_{n=0}^{\infty} b_{n,\alpha} \{ \ x^{(\alpha-2)/2} e^{-x/2} [(n+\alpha) \cdot$$

$$\cdot(n+\alpha+1)L_n^{\alpha-2} - 2(n+\alpha)(n+1)L_{n+1}^{\alpha-2} + (n+1)(n+2)L_{n+2}^{\alpha-2}]\} ,$$

and thus,

$$(7) \quad g = \sum_{n=0}^{\infty} b_{n,\alpha}\{(n+\alpha)(n+\alpha+1)l_{n,\alpha-1} - 2(n+\alpha)(n+1)l_{n+1,\alpha-2} +$$

$$+(n+1)(n+2)l_{n+2,\alpha-2}\} =$$

$$= \alpha(\alpha+1)b_{0,\alpha}l_{0,\alpha} + ((\alpha+1)(\alpha+2)b_{1,\alpha-2} - 2\alpha b_{0,\alpha-2})l_{1,\alpha} + \sum_{n=2}^{\infty}[b_{n,\alpha}(n+\alpha)\cdot$$

$$\cdot(n+\alpha+1) - 2n(n-1+\alpha)b_{n-1,\alpha} + n(n-1)b_{n-2,\alpha}]l_{n,\alpha-2}.$$

Since $g \in LG'_{\alpha-2}$, it has the expansion $g = \sum_{n=0}^{\infty} x_{n,\alpha-2}l_{n,\alpha-2}$. From (7) we have:

$$x_{0,\alpha-2} = b_{0,\alpha}\alpha(\alpha+1),$$

$$x_{1,\alpha-2} = -2\alpha b_{0,\alpha} + b_{1,\alpha}(1+\alpha)(2+\alpha),$$

$$x_{2,\alpha-2} = 2b_{0,\alpha} - 4(\alpha+1)b_{1,\alpha} + (2+\alpha)(3+\alpha)b_{2,\alpha},$$

$$\vdots$$

$$x_{n,\alpha-2} = b_{n-2,\alpha}(n-1)n - 2b_{n-1,\alpha}(n-1+\alpha)n + b_{n,\alpha}(n+\alpha)(n+\alpha+1).$$

If $g = 0$ then $x_{0,\alpha-2} = x_{1,\alpha-2} = ... = x_{n,\alpha-2} = ... = 0$ and since $\alpha > 1$, we get $f = 0$. It means that the mapping is the injection.

The surjection follows from the solvability of the above system of equations on $b_{n,\alpha}$, $n \in \mathbf{N}_0$ bacause $\alpha > 1$.

For given g the coefficients of f are the following:

$$b_{0,\alpha} = x_{0,\alpha-2}/(\alpha(\alpha+1)),$$

$$b_{1,\alpha} = 1/((\alpha+1)(\alpha+2))(x_{1,\alpha-2} + 2x_{0,\alpha-2}/(\alpha+1)),$$

$$b_{2,\alpha} = 1/((\alpha+2)(\alpha+3))\{x_{2,\alpha-2} + 2/(\alpha+2)x_{1,\alpha-2} + 4/(\alpha+2)x_{0,\alpha-2}$$

$$-2x_{0,\alpha-2}/(\alpha(\alpha+1))\}$$

....

So we have proved that the quoted mapping is the bijection. □

Procceding it by induction on k, we obtain:

Proposition 5. *If $f \in LG'_\alpha$, than the mapping: $f \to x^k f$, $k \in \mathbf{R}_+$, $\alpha - 2k > -1$, from LG'_α onto $LG'_{\alpha-2k}$ is the bijection.*

Examples. The following two examples give another proof, based on the $(1,0)$-convolution, that the mapping $f \to xf$ is a bijection of LG'_2 on LG'_0.

1^0 Using $(1,0)$– convolution equation $x^{1/2}(x^{-1/2}\delta) * g = xh$ where $x^{-1/2}\delta = \sum_{n=0}^{\infty} l_{n,1}$, ([7]), and g, h are from (2) we obtain

$$\sum_{n=0}^{\infty} l_{n,1} \overset{(1,0)}{*} \sum_{n=0}^{\infty} x_{n,0} l_{n,0} = xh,$$

where $h = \sum_{n=0}^{\infty} c_{n,2} l_{n,2}$ and thus,

$$c_{n,2} = 1/((n+2)(n+1)) \sum_{p+q=n} (p+1)^{1/2} x_{q,0}.$$

The solution of this system is:

$$x_{0,0} = 2c_{0,2},\ x_{1,0} = 6c_{1,2} - 2\sqrt{2}c_{0,2},\ x_{2,0} = (12 - 6\sqrt{2})c_{1,2} + (4 - 2\sqrt{3})c_{0,2}, \ldots$$

and

$$xh = \sum_{n=0}^{\infty} x_{n,0} l_{n,0} = (2c_{0,2}) l_{0,0} + (6c_{1,2} - 2\sqrt{2}c_{0,2}) l_{1,0} +$$

$$+ [(12 - 6\sqrt{2})c_{1,2} + (4 - 2\sqrt{3})c_{0,2}] l_{2,0} + \ldots.$$

2^0 Let $g = (1/x)h$, and

$$h = \sum_{n=0}^{\infty} c_{n,0} l_{n,0} \quad g = \sum_{n=0}^{\infty} x_{n,2} l_{n,2}.$$

Then $h = xg$ and from the system from example 1^0, we have

$$x_{n,2} = 1/((n+2)(n+1)) \sum_{p+q=n} (p+1)^{1/2} c_{q,0}.$$

6. The solvability of (α, β) - convolution equations in $LG'_{e\alpha}$

In [7] we have shown trought the algebraic representation of (3) that the necessary condition for the solvability of the convolution equation

$$f \overset{(\alpha,\beta)}{*} g = (x^{\alpha/2} f) * (x^{\beta/2} g) = x^{(\alpha+\beta+1)/2} h, \tag{8}$$

where $f \in LG'_{\alpha}$, $g \in LG'_{\beta}$, $h \in LG'_{\alpha+\beta+1}$, is that the first coefficient in Laguerre expansion of f is different from zero.

The $(\alpha, \beta)-$ convolution form of (8) can be expressed as

$$F * G = H, \qquad F, G, H \in LG'_0,$$

where $F = x^{\alpha/2} f$, $G = x^{\beta/2} g$, $H = x^{(\alpha+\beta+1)/2} h$.

In [6] we have proved that if $F = \sum_{n=0}^{\infty} d_{n,0} l_{n,0} \in LG'_{e0}$ and $d_{0,0} \neq 0$, then for any $H \in LG'_{e0}$ there exists a solution of equation $F * G = H$ which belongs to LG'_{e0}. The function $F = x^{\alpha/2} f$ can be treated in two different ways:

If

$$f = \sum_{n=0}^{\infty} b_{n,\alpha} l_{n,\alpha} \in LG'_{e\alpha}$$

than

$$x^{\alpha/2} f = \sum_{n=0}^{\infty} b_{n,\alpha} x^{\alpha/2} l_{n,\alpha} = F = \sum_{n=0}^{\infty} d_{n,0} l_{n,0} \in LG'_{e0}.$$

The first coefficient is $d_{0,0} :< F, l_{0,0} >=< F, e^{-x/2} >= d_{0,0}$.

Since $l_{0,0} = e^{-x/2}$, and $l_{0,\alpha} = \tau_{0,\alpha} x^{\alpha/2} e^{-\frac{x}{2}} = \tau_{0,\alpha} x^{\alpha/2} l_{0,0}$, the following holds:

$$< x^{\alpha/2} f, l_{0,0} >=< f, x^{\alpha/2} l_{0,0} >=< \sum_{n=0}^{\infty} b_{n,\alpha} l_{n,\alpha}, x^{\alpha/2} l_{0,0} >=$$

$$=< \sum_{n=0}^{\infty} b_{n,\alpha} l_{n,\alpha}, l_{0,\alpha}/\tau_{0,\alpha} >= b_{0,\alpha}/\tau_{0,\alpha}.$$

So we obtain $d_{0,0} = b_{0,\alpha}/\tau_{0,\alpha}$. It gives

Proposition 6. *The convolution equation*

$$f \overset{(\alpha,\beta)}{*} g = x^{(\alpha+\beta+1)/2}h$$

is solvable in $LG'_{e\beta}$ for any $h \in LG'_{e\alpha+\beta+1}$ iff in the Laguerre expansion of $f = \sum_{n=0}^{\infty} b_{n,\alpha} l_{n,\alpha}$ the first coefficient $b_{0,\alpha} \neq 0$.

7. The solvability of $(\alpha, \beta)-$ convolution equations in LG'_β

In this section we shall give the conditions for the solvability of $(\alpha,\beta)-$ convolution equations which are analogous to the conditions of Vladimirov ([8], Ch. 2 § 13), given for the space of tempered distributions.

Consider LG'_0 - from of $(\alpha,\beta)-$ convolution equation

$$f \overset{(\alpha,\beta)}{*} g = (x^{\alpha/2}f) * (x^{\beta/2}g) = x^{(\alpha+\beta+1)/2}h, \tag{9}$$

where $f \in LG'_\alpha$, $g \in LG'_\beta$, $h \in LG'_{\alpha+\beta+1}$, and $x^{\alpha/2}f, x^{\beta/2}g, x^{(\alpha+\beta+1)/2}h \in LG'_0$, $*$ is the ordinar (tempered) convolution.

Proposition 7. *If $f = x^{-\alpha/2}\sum_{n=0}^{k} a_n\delta^{(n)}$, then for any $h \in LG'_{\alpha+\beta+1}$ there exists a solution of $(\alpha,\beta)-$ convolution equation in LG'_β iff $P(-iz) =$*
$\sum_{n=0}^{k} a_n$
$(iz)^n \neq 0$ in $\mathbf{Z}_+$.

Proof. When $f = x^{-\alpha/2}\sum_{n=0}^{k} a_n\delta^{(n)}$ then (9) becomes

$$\sum_{n=0}^{k} a_n\delta^{(n)} * (x^{\beta/2}g) = x^{(\alpha+\beta+1)/2}h$$

where $*$ is ordinar convolution in LG'_0 and $hx^{(\alpha+\beta+1)/2} \in LG'_0$. From ([8], Ch 2. § 13) follows that there exists the solution of this equation in LG'_0. From Proposition 5 we have that there exists $g \in LG'_\beta$ which satisfies (9).
□

Proposition 8. *Let $f \in LG'_0$. If $f(\bar{z}) = \mathcal{L}(f(t))(z)$, $z \in \mathbf{Z}_+, \mathcal{L}$ is the Laplace transform, has nonnegative imaginary part, then for all $h \in LG'_{\alpha+\beta+1}$ there exists a solution of*

$$(10) \qquad (x^{-\alpha/2} f) \overset{(\alpha,\beta)}{*} g = x^{(\alpha+\beta+1)/2} h, \quad \text{in } LG'_\beta.$$

Proof. The ordinar convolution form of (10) is

$$f * (x^{\beta/2} g) = x^{(\alpha+\beta+1)/2} h.$$

Since $x^{(\alpha+\beta+1)/2} h \in LG'_0$, then by following Vladimirov ([8], Ch. § 13), or ([6], condition (B)), (10) is solvable in LG'_0 iff $f(z)$ has nonnegative imaginary part. Thus, $x^{\beta/2} g \in LG'_0$. From Proposition 5 $g \in LG'_\beta$. □

We shall give examples concerning these propositions.

Example 1^0.

$$f \overset{(2,2)}{*} g = x^{5/2} h, \quad h \in LG'_5.$$

Let

$$f = x^{-1}\delta = \sum_{n=0}^{\infty} [A_n/((n+2)(n+1))] l_{n,2}, \; A_n = \sum_{i=0}^{n} (i+1)^{1/2}$$

([7]), then with the notation as in (2), the corresponding system of equations $xg = x^{5/2} h$ is

$$\Gamma(3)/\Gamma(6)(1/2) x_{0,2} = c_{0,5},$$

$$1/\Gamma(7)\{(\Gamma(4)\Gamma(3))^{1/2}[(1+\sqrt{2})/6 + 1/2] x_{0,2} + (\Gamma(3)\Gamma(4))^{1/2}(1/2) x_{1,2})\} = c_{1,5},$$

$$2/\Gamma(8)\{(1/2\Gamma(3)\Gamma(5))^{1/2}[(1+\sqrt{2}+\sqrt{3})/12 + 1/6(1+\sqrt{2}) + 1/2] x_{0,2} +$$
$$+ \Gamma(4)[(1+\sqrt{2})/6 + 1/2] x_{1,2} + (1/2\Gamma(3)\Gamma(5))^{1/2}(1/2) x_{2,2}\} = c_{2,5}.$$

$$\dots$$

Example 2^0.

$$f \overset{(1,2)}{*} g = x^2 h, \quad h \in LG'_4, \; f \in LG'_1, \; g \in LG'_2.$$

Suppose $f = x^{1/2}\delta = \sum_{n=0}^{\infty} 4(-1)^n l_{n,1}$ ([7]). Then we obtain

$$4/\Gamma(5)(\Gamma(2)\Gamma(3))^{1/2} x_{0,2} = c_{0,4},$$

$$1/\Gamma(6)\{\Gamma(3)\Gamma(-4)x_{0,2}+(\Gamma(2)\Gamma(4))^{1/2}4x_{1,2}\}=c_{1,4},$$
$$2/\Gamma(7)\{(\Gamma(4)\Gamma(3))/2)^{1/2}4x_{0,2}+(\Gamma(3)\Gamma(4)))^{1/2}4x_{1,2}+$$
$$+(\Gamma(2)\Gamma(4))/1)^{1/2}4x_{2,2}\}=c_{2,4}.$$
$$\dots$$

Appropriate versions of Proposition 7, 8 hold for $(0,\beta)-$ convolution equations.

We shall give two examples concerning this case.

Example 3^0. Consider the equation $P(\delta)\overset{(0,\beta)}{*}g=x^{(\beta+1)/2}h, g, h$ are as in (2). Suppose

$$P(\delta)=\sum_{n=0}^{2}a_n\delta^{(n)}=a_0\delta+a_1\delta'+a_2\delta''.$$

From [7]

$$\delta(x)=\sum_{n=0}^{\infty}l_{n,0}(x),\delta'(x)=\sum_{n=0}^{\infty}(n+1/2)l_{n,0}(x),$$
$$\delta''(x)=\sum_{n=0}^{\infty}((n-1)/2+n+1/4)l_{n,0}(x),$$

and

$$P(\delta)=\sum_{n=0}^{\infty}\{a_0+a_1(n+1/2)+a_2((n-1)/2+n+1/2)\}l_{n,0}(x).$$

Then the algebraic system (5) gives

$$(1/\Gamma(\beta+2))(\Gamma(\beta+1))^{1/2}(a_0+a_1/2+a_2/4)x_{0,\beta}=c_{0,\beta+1}.$$
$$(1/\Gamma(\beta+3))\{(a_0+a_1/2+a_2/4)(\Gamma(\beta+2))^{1/2}x_{1,\beta}+$$
$$+(a_0+3/2a_1+5/4a_2)(\Gamma(\beta+1))^{1/2}x_{0,\beta}\}=c_{1,\beta+1},$$
$$(1/\Gamma(\beta+4))\{(a_0+a_1/2+a_2/4)(\Gamma(\beta+3)/2)^{1/2}x_{2,\beta}+(a_0+3/2a_1+$$
$$+5/4a_2+(\Gamma(\beta+2))^{1/2}x_{1,\beta}+(a_0+5/2a_1+11/4a_2)\cdot$$
$$\cdot(\Gamma(\beta+1))^{1/2}x_{0,\beta}\}=c_{1,\beta+1}.$$
$$\dots$$

Remark. If $a_0, a_2 = 0$, $a_1 = 1$ in f and $(\beta = 1)$ we obtain the solution of ordinar differential equation $(x^{1/2}g)' = xh$, where $g \in LG'_0$, $h \in LG'_2$. Convolution form of this equation is $\delta' \overset{(0,1)}{*} g = xh$, and coefficients of the solution in our notation are:

$$x_{0,1} = 8\sqrt{6}c_{0,2}, \quad x_{1,1} = 20\sqrt{6}/a_0(c_{1,2} - 3/5a_1c_{0,2}),$$

$$x_{2,1} = 48\sqrt{15}/a_1\{c_{2,2} - 1/6a_1/a_0c_{1,2} + 1/10a_1^2/a_0c_{0,2} + 1/60a_1c_{0,2}\},$$

$$\dots$$

8. On the $(\alpha,\beta)-$ fundamental solution

The $(\alpha,\beta)-$ fundamental solution of $(\alpha,\beta)-$ convolution equation

$$(11) \qquad f \overset{(\alpha,\beta)}{*} g = (x^{\alpha/2}f) * (x^{\beta/2}g) = x^{(\alpha+\beta+1)^{1/2}}h, \ \ h \in LG'_{\alpha+\beta+1},$$

denoted by G, is the solution of

$$f \overset{(\alpha,0)}{*} G = \delta.$$

By the associativity of convolution in LG'_0, (11) implies

$$(x^{\beta/2}g) = G * x^{(\alpha+\beta+1)/2}h, \quad \text{i.e. } g \overset{(\beta,0)}{*} \delta = G \overset{(0,\alpha+\beta+1)}{*} h \quad \text{where}$$

$$g = x^{-\beta/2}(G * x^{(\alpha+\beta+1)/2}h) = x^{-\beta/2}(G \overset{(0,\alpha+\beta+1)}{*} h).$$

As an example we shall construct the fundamental solution and the solution of $(1,2)-$ convolution equation. Let $f = \sum_{n=0}^{\infty} b_{n,1}l_{n,1}$, $G = \sum_{n=0}^{\infty} y_{n,0}l_{n,0}$.

We shall solve the equation

$$f \overset{(1,0)}{*} G = \delta = x(x^{-1}\delta).$$

Since $x^{-1}\delta = \sum_{n=0}^{\infty} A_n/((n+2)(n+1))l_{n,2}$ by (4) we have

$$1/2b_{0,1}y_{0,0} = 1/2$$

$$1/6\{\sqrt{2}b_{1,1}y_{0,0}+b_{0,1}y_{1,0}\}=1/2+1/6(1+\sqrt{2})$$
$$1/12\{\sqrt{3}b_{2,1}y_{0,0}+\sqrt{2}b_{1,1}y_{1,0}+b_{0,1}y_{2,0}\}$$
$$=1/2+1/6(1+\sqrt{2})+1/12(1+\sqrt{2}+\sqrt{3}).$$
$$\dots$$

Thus, we have the coefficients of G :

$$y_{0,0}=b_{0,1},\quad y_{1,0}=-\sqrt{2}b_{1,1}-3/y_{0,0}(1+1/3(1+\sqrt{2})),$$
$$y_{2,0}=1/b_{0,1}\{9+3\sqrt{2}+\sqrt{3}-\sqrt{2}b_{2,1}b_{0,1}+2b_{1,1}^2-3\sqrt{2}/y_{0,0}\cdot$$
$$(1+1/3(1+\sqrt{2}))\},\dots$$

For the solution $f\overset{(1,2)}{*}g=x^2h$ we need to find x^2h for given $h=\sum_{n=0}^{\infty}c_{n,4}l_{n,4}\in LG'_4$. First put $h_1=xh=\sum_{n=0}^{\infty}d_{n,2}l_{n,2}$. The coefficients of h_1 we seek from Example 1^0, Section 5.

$$h_1=x\sum_{n=0}^{\infty}c_{n,4}l_{n,4}=\sum_{n=0}^{\infty}d_{n,2}l_{n,2}=(2c_{0,4})l_{0,2}+(6c_{1,4}-2\sqrt{2}c_{0,4})l_{1,2}+$$
$$+[(12-6\sqrt{2})c_{1,4}+(4-2\sqrt{3})c_{0,4}]l_{2,2}+\dots$$

Applying Example 1^0 once again we obtain:

$$h_2=x^2h=x(xh)=\sum_{n=0}^{\infty}s_{n,0}l_{0,0},\quad\text{or}$$

$$h_2=x\sum_{n=0}^{\infty}d_{n,2}l_{n,2}=(2d_{0,2})l_{0,0}+(6d_{1,2}-2\sqrt{2}d_{0,2})l_{1,0}+$$
$$+[(12-6\sqrt{2})d_{1,2}+(4-2\sqrt{3})d_{0,2}]l_{2,0}+\dots$$

and

$$x^2h=(4c_{0,4})l_{0,0}+\{36c_{1,4}-16\sqrt{2}c_{0,4}\}l_{1,0}+\{36c_{1,4}(2-\sqrt{2})+$$
$$+c_{0,4}(32-24\sqrt{2}-4\sqrt{3})\}l_{2,0}+\dots$$

By using the approximation formula for ordinar convolution given in [6], we obtain

$$(xg)=G*h_2=\sum_{n=0}^{\infty}y_{n,0}l_{n,0}*\sum_{n=0}^{\infty}s_{n,0}l_{n,0}=\sum_{m=0}^{\infty}(\sum_{p+q=m}y_{p,0}s_{q,0}-$$

$$\sum_{p+q=m-1} y_{p,0}s_{q,0})l_{m,0}\cdot \quad (\text{ we take } \sum_{p+q=-1} = 0)$$

$$y_{0,0}s_{0,0} = z_{0,0}$$

$$y_{0,0}s_{1,0} + (y_{1,0} - y_{0,0})s_{0,0} = z_{1,0},$$

$$y_{0,0}s_{2,0} + (y_{1,0} - y_{0,0})s_{1,0} + (y_{2,0} - y_{1,0})s_{0,0} = z_{2,0},$$

$$\ldots$$

From Example 2^0 in Section 5. we obtain the coefficients of g from $xg = \sum_{n=0}^{\infty} z_{n,0}l_{n,0}$:

$$x_{n,2} = 1/((n+1)(n+2)) \sum_{p+q=n} (p+1)^{1/2} z_{q,0}.$$

Remark. Consider the $(0,4)$-convolution form of

$$xg = G \overset{(0,4)}{*} h.$$

Equation (5) for $\beta = 4$ gives the system of equations:

$$1/\Gamma(6)(\Gamma(5))^{1/2} y_{0,0}c_{0,4} = p_{0,5},$$

$$1/\Gamma(7)\{(\Gamma(6))^{1/2} y_{1,0}c_{0,4} + (\Gamma(5))^{1/2} y_{0,0}c_{1,4}\} = p_{1,5},$$

$$1/\Gamma(8)\{(\Gamma(7)/2)^{1/2} y_{2,0}c_{0,4} + (\Gamma(6))^{1/2} y_{1,0}c_{0,4} +$$

$$+(\Gamma(5))^{1/2} y_{0,0}c_{2,4}\} = p_{2,5}.$$

$$\ldots$$

It yields

$$xg = x^{5/2} \sum_{n=0}^{\infty} p_{n,5}l_{n,5}, \quad \text{i.e.}$$

$$g = x^{3/2} \sum_{n=0}^{\infty} p_{n,5}l_{n,5}.$$

9. Remark on the error estimate of $(\alpha,\beta)-$ convolution

Denote by subindex $(\cdot)_N$ the $N-$ th partial summ of Laguerre's expansion of some generalized function. Then, $g_N = x^{-\beta/2}(G_N * x^{(\alpha+\beta+1)/2}h_N)$, $g = x^{-\beta/2}(G * x^{(\alpha+\beta+1)/2}h)$ and if g, h and G are functions, we have

$$x^{\beta/2}|g_N - g| \leq |G * x^{(\alpha+\beta+1)/2}(h-h_N)| + |G-G_N| * |x^{(\alpha+\beta+1)/2}||h_N|,\ x>0.$$

Let $hx^{(\alpha+\beta+1)/2} \in L^p(\mathbf{R}_+)$, $G \in L^q(\mathbf{R}_+)$, where $p,q \in [1,\infty)$ and satisfy $1/p+1/q \geq 1$.

If $1/r = 1/p+1/q-1$, then $g \in L^r(\mathbf{R}_+)$ and

$$(\int_0^\infty (x^{\beta/2}|g_N(x)-g(x)|)^r dx)^{1/r} \leq (\int_0^\infty |G(x)|^q dx)^{1/q}\cdot$$

$$\cdot(\int_0^\infty (|x^{(\alpha+\beta+1)/2}||h_N-h|)^p dx)^{1/p} + (\int_0^\infty |G(x)-G_N(x)|^q dx)^{1/q}$$

$$\cdot(\int_0^\infty (|x^{\alpha+\beta+1)/2}||h_N|)^p dx)^{1/p}.$$

Let $p=q=2$. Then $r=\infty$ and

$$\sup\{x^{\beta/2}|g_N-g|,\ \ x\in\mathbf{R}_+\} \leq (\int_0^\infty |G(x)|^2 dx)^{1/2}\cdot$$

$$\cdot(\int_0^\infty (|x^{(\alpha+\beta+1)/2}||h_N-h|)^2 dx)^{1/2} \leq$$

$$\leq (\int_0^\infty |G(x)-G_N(x)|^2 dx)^{1/2}(\int_0^\infty (|x^{(\alpha+\beta+1)/2}||h_N|)^2 dx)^{1/2}.$$

10. On Volterra's type integral equations

We give an application of $(\alpha,\beta)-$ convolution equation in solving Volterra's integral equation of the first kind. We shall consider separately, $(\alpha,0)$ and $(0,\beta)$ convolution form of Volterra's integral equation, and we give two algorithms for their solving.

10.1 Solving Volterra's equation by $(\alpha, 0)-$ convolution

Volterra's integral equation of the first kind

$$(12) \qquad \lambda g(t) + \int_0^\infty \varphi(t) g(x-t)dt = x^{(\alpha+1)/2} h,$$

where $\varphi \in L^1(\mathbf{R}_+) \cap L^2(\mathbf{R}_+)$, $\lambda \neq -\int_0^\infty \varphi(t)e^{izt}dt$, $z \in \mathbf{Z}_+ \cup \dot{\mathbf{R}}_+$, where $\dot{\mathbf{R}}_+$ is the completition of the real line, $h \in LG'_{\alpha+1}$, can be solved by using the following $(\alpha, 0)-$ convolution form

$$(13) \qquad (\lambda x^{-\alpha/2}\delta + x^{-\alpha/2}\varphi) \overset{(\alpha,0)}{*} g = x^{(\alpha+\beta+1)/2} h.$$

Under the above assumptions on λ there exists a solution of (12) for any $h \in LG'_{\alpha+1}$.

Algorithm I

When $\alpha = 1$, $\beta = 0$ in (13), we have

$$x^{1/2}(\lambda x^{-1/2}\delta + x^{-1/2}\varphi) * g = xh.$$

Since

$$x^{-1/2}\delta = \sum_{n=0}^{\infty} l_{n,1} \text{ and } x^{-1/2}\varphi = \sum_{n=0}^{\infty} 1/(n+1)(\sum_{i=0}^{n} a_i) l_{n,1}, \text{ ([7]) , where}$$

$$\varphi = \sum_{n=0}^{\infty} a_{n,0} l_{n,0} \text{ and } g, h \text{ are from (2), } f = \sum_{n=0}^{\infty} [\lambda + (1/(n+1))(\sum_{i=0}^{n} a_i)] l_{n,1}.$$

Then the algebraic form (4) becomes

$$(1/2)(\lambda + a_0)x_{0,0} = c_{0,2},$$

$$(1/6)\{x_{1,0}(\lambda + a_0) + \sqrt{2}(\lambda + (1/2)(a_0 + a_1))x_{0,0}\} = c_{1,2},$$

$$(1/12)\{x_{2,0}(\lambda + a_0) + \sqrt{2}(\lambda + (1/2)(a_0 + a_1))x_{1,0} + \sqrt{3}(\lambda + (1/3)(a_0 + a_1 + a_2))x_{0,0}\} = c_{2,2}.$$

$$\dots$$

10.2 $(0,\beta)-$ convolution form of Volterra's equation

The $(0,\beta)-$ convolution form of Volterra's integral equation of the first kind

$$\lambda x^{\beta/2}g + \int_0^{\infty} \varphi(x-t)g(t)t^{\beta/2}dt = x^{(\beta+1)/2}h, \quad \beta > -1 \tag{14}$$

where

$$\lambda \neq \int_{\mathbf{R}_+} \varphi(t)e^{izt}dt,\ \varphi \in L^1(\mathbf{R}_+) \cup L^2(\mathbf{R}_+),\ z \in \mathbf{Z}_+ \cup \mathbf{R}_+,$$

$$g \in LG'_{\beta},\ h \in LG'_{\beta+1}$$

is

$$(\lambda\delta + \varphi) * (x^{\beta/2}g) = x^{(\beta+1)/2}h. \tag{15}$$

Under the given condition on λ this equation is solvable for any $h \in LG'_{\beta+1}$.

Algorithm II

If $\beta = 1$, we have

$$(\lambda\delta + \varphi) * (x^{1/2}g) = xh, \quad g \in LG'_1,\ \ h \in LG'_2, \tag{16}$$

$$\delta = \sum_{n=0}^{\infty} l_{n,0}, \quad \varphi = \sum_{n=0}^{\infty} a_{n,0}l_{n,0},$$

g, h are given in (2). Then (16) becomes system (4), i.e.

$$(1/2)(\lambda + a_{0,0})x_{0,1} = c_{0,2},$$

$$(1/6)\{\sqrt{2}(\lambda + a_{0,0})x_{1,1} + (\lambda + a_{1,0})x_{0,1}\} = c_{1,2},$$

$$(1/12)\{\sqrt{3}(\lambda + a_{0,0})x_{2,1} + \sqrt{2}(\lambda + a_{1,0})x_{1,1} + (\lambda + a_{2,0})x_{0,1}\} = c_{2,2},$$

$$\dots$$

At the end we give one more application of the $(\alpha,\beta)-$ convolution. Consider the integral equation

$$\lambda g(x) + x^{-\beta/2}\int_0^{\infty} \varphi(t)(x-t)^{\beta/2}g(x-t)dt = xh, \quad \beta > -1, \tag{17}$$

where $\varphi \in L^1(\mathbf{R}_+) \cap L^2(\mathbf{R}_+)$, $h \in L^2(\mathbf{R}_+)$. This equation can not be solved by Laplace transformation.

$(1, \beta)-$ convolution form of (17)

$$x^{1/2} f * x^{\beta/2} g = x^{1/2}(\lambda x^{-1/2}\delta + x^{-1/2}\varphi) * x^{\beta/2} g = x^{(\beta+2)/2} h,$$

where $g \in LG'_\beta$, $h \in LG'_{\beta+2}$, transforms it in solvable equation if $\lambda \neq \int_{\mathbf{R}_+} \varphi(t) e^{izt} dt$, $z \in \mathbf{Z}_+$, where

$$f = \sum_{n=0}^{\infty} [\lambda + (1/(n+1)) \sum_{i=0}^{n} a_i] l_{n,1}.$$

Putting (2) we obtain

$$\sum_{n=0}^{\infty} c_{n,\beta+2} l_{n,\beta+2} = \sum_{n=0}^{\infty} (n!/(\Gamma(\beta+n+3)))$$

$$(\sum_{p+q=n} ((\Gamma(\beta+q+1)/q!)(p+1))^{1/2} \cdot b_{p,1} x_{q,\beta}) l_{n,\beta+2},$$

where $b_{p,1} = \lambda + (1/(p+1)) \sum_{i=0}^{p} a_i$.

Its developing form is

$$(1/\Gamma(\beta+3))(\lambda + a_0) x_{0,\beta} = c_{0,\beta+2},$$

$$(1/\Gamma(\beta+4))\{(\Gamma(\beta+2))^{1/2}(\lambda + a_0) x_{1,\beta} + (2\Gamma(\beta+1))^{1/2}(\lambda +$$
$$+ 1/2(a_0 + a_1)) x_{0,\beta}\} = c_{1,\beta+2},$$

$$(2/\Gamma(\beta+5))\{((\Gamma(\beta+3))/2)^{1/2}(\lambda + a_0) x_{2,\beta} + (2\Gamma(\beta+2))^{1/2}(\lambda + 1/2(a_0 +$$
$$+ a_1)) x_{1,\beta} + (3\Gamma(\beta+1))^{1/2}(\lambda + 1/3(a_0 + a_1 + a_2)) x_{0,\beta}\} = c_{2,\beta+2}.$$

...

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REZIME

(α, β) - KONVOLUCIJA U PROSTORIMA SA LAGEROVOM EKSPANZIJOM I NJENE PRIMENE

Razvijamo teoriju prostora uopštenih funkcija LG'_α i njihove generalizacije $LG'_{e\alpha}, \alpha > -1$, čiji elementi imaju ortonormalne razvoje u odnosu na Lagerov ortonormalni sistem $\ell n, \alpha$, $n \in \mathrm{N}_0$, $\alpha > -1$.

Definišemo (α, β) - konvolucioni proizvod i nalazimo uslove rešivosti konvolucionih jednačina u tim prostorima. Konačno, dobijamo primenu toga u rešavanju integralnih jednačina.

Received by the editors September 24, 1990