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PROPERTIES OF THE LAPLACE TRANSFORMATION IN COLOMBEAU'S GENERALIZED FUNCTION SPACES

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Abstract

We define and study the Laplace transformation in the spaces $\mathcal{G}_{\mathbf{a}}$, $\mathbf{a} \in \mathbf{A}$ and prove the anaogous properties to classical ones for such transformation.

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1. Introduction

We define and study the Laplace transformation in the spaces $\mathcal{G}_{\mathbf{a}}$, $\mathbf{a} \in \mathbf{A}$ via the $\mathbf{a}$-Fourier transformation developed in [3].

The definition is analogous to the distributional definition from [7]. All the properties of the Laplace transformation of distributions with suitable adaptations also hold for the Laplace transformation defined in this paper.

The definition are different from the ones used in [4], where the Paley-Wiener type theorems for compactly supported and tempered generalized functions are obtained.

2. Basic Notions

We use a slightly changed notation from [3]. By $\mathcal{A}_q$, $q \in \mathbf{N}_0$, we denote subsets of $\mathcal{D}(\mathbf{R}^n)$ with the following properties: $\phi \in \mathcal{A}_q$ if

$$\operatorname{diam}(\operatorname{supp}(\phi)) = 1, \int \phi(x)dx = 1, \int x^\alpha \phi(x)dx = 0, \ \alpha \in \mathbf{N}_0^n, \ 1 \le |\alpha| \le q.$$

Obviously, $\mathcal{A}_0 \supset \mathcal{A}_1 \supset \ldots$. By $\phi_\varepsilon(\cdot)$ is denoted by $\varepsilon^{-n}\phi(\cdot/\varepsilon)$.

$\mathcal{E}$ is defined as a set of all functions $(\phi, \varepsilon, x) \to F_{\phi,\varepsilon}(x)$, $(\phi, \varepsilon, x) \in \mathcal{A}_0 \times (0,1) \times \mathbf{R}^n$ which are smooth on $\mathbf{R}^n$ for every fixed (ϕ, ε). We denote such a function by $F_{\phi,\varepsilon}$.

$\mathbf{C}_M$ is the set of all $A_{\phi,\varepsilon} : \mathcal{A}_0 \times (0,1) \to \mathbf{C}$ such that there exist $N \in \mathbf{N}_0$ such that for every $\phi \in \mathcal{A}_N$ there exist $C > 0$ and $\eta > 0$ such that $|A_{\phi,\varepsilon}| \le C\varepsilon^{-N}$, $\varepsilon < \eta$.

$\mathcal{E}_M$ is the set of all $G_{\phi,\varepsilon} \in \mathcal{E}$ such that for every compact set K and every $\beta \in \mathbf{N}_0^n$ there exists $N \in \mathbf{N}_0$ such that for every $\phi \in \mathcal{A}_N$ there exist $C > 0$ and $\eta > 0$ such that $|\partial^\beta G_{\phi,\varepsilon}(x)| \le C\varepsilon^{-N}$, $\varepsilon < \eta$, $x \in K$.

Denote by Γ the family of all increasing sequences which tend to infinity.

$\mathbf{C}_0$ is the set of all $A \in \mathbf{C}_M$ such that there exist $g \in \Gamma$ and $N \in \mathbf{N}_0$ such that for every $\phi \in \mathcal{A}_q$, $q \ge N$, there exist $C > 0$ and $\eta > 0$ such that $|A_{\phi,\varepsilon}| \le C\varepsilon^{g(q)-N}$, $\varepsilon < \eta$.

$\mathcal{N}$ is the set of all $G \in \mathcal{E}_M$ such that for every $\beta \in \mathbf{N}_0^n$ and every compact set K there exist $N \in \mathbf{N}_0$ and $g \in \Gamma$ such that for every $\phi \in \mathcal{A}_q$, $q \ge N$, there exist $C > 0$ and $\eta > 0$ such that $|\partial^\beta G_{\phi,\varepsilon}(x)| \le C\varepsilon^{g(q)-N}$, $\varepsilon < \eta$, $x \in K$.

The spaces of Colombeau's generalized complex numbers and generalized functions are defined by $\overline{\mathbf{C}} = \mathbf{C}_M/\mathbf{C}_0$ and $\mathcal{G} = \mathcal{E}_M/\mathcal{N}$ respectively. We will denote by G or $[G_{\phi,\varepsilon}]$ the class of equivalence for $G_{\phi,\varepsilon}$.

If $g \in \mathcal{D}'$, then by

$$G_{\phi,\varepsilon}(x) = < g(\xi), \varepsilon^{-n}\phi((\xi - x)/\varepsilon) >, \ x \in \mathbf{R}^n,$$

is denoted the representative of the corresponding element in $\mathcal{G}$. Its class is called Colombeau's regularization of g is denoted by $\operatorname{Cd}(g)$. In this sense, the inclusions $\mathcal{E} \subset \mathcal{D}' \subset \mathcal{G}$ are valid.

Denote by $\mathbf{a}$ a function defined on an interval $[l_{\mathbf{a}}, \infty)$, $l_{bfa} > 0$ such that it is continuous, non-decreasing and

$$\lim_{x\to\infty} \mathbf{a}(x) = \infty \text{ and } \mathbf{a}(x) = \mathcal{O}(x),\ x \to \infty.$$

The set of such functions is denoted by $\mathbf{A}$. Note, if $\mathbf{a} \in \mathbf{A}$, then $\ln(\mathbf{a}) \in \mathbf{A}$, as well.

Let $\mathbf{a} \in \mathbf{A}$ be fixed. Then $\Theta_{\mathbf{a}}$ as the set of all functions θ defined on the interval $[0, \infty)$, with the following properties θ continuous, positive, increasing, and for every $p \geq 0$ there exists $\gamma > 0$ such that $\theta(p + \mathbf{a}(x)) = \mathcal{O}(x^\gamma)$, $x \to \infty$.

$\mathcal{E}_{\mathbf{a}}$ as the set of all elements $G_{\phi,\varepsilon} \in \mathcal{E}$ with the following property: For every $\beta \in \mathbf{N}_0^n$ there exist $N \in \mathbf{N}_0$ and $\theta \in \Theta_{\mathbf{a}}$ such that for every $\phi \in \mathcal{A}_N$ there exist $C > 0$ and $\eta > 0$ such that $|\partial^\beta G_{\phi,\varepsilon}(x)| \leq C\theta(x)\varepsilon^{-N}$, $\varepsilon < \eta$, $x \in \mathbf{R}^n$.

$\mathcal{N}_{\mathbf{a}}$ is the set of elements $G_{\phi,\varepsilon} \in \mathcal{E}_{\mathbf{t}}$ with the following property: For every $\beta \in \mathbf{N}_0^n$ there exist $\theta \in \Theta_{\mathbf{a}}$, $N \in \mathbf{N}_0$ and $g \in \Gamma$ such that for every $\phi \in \mathcal{A}_q$, $q \geq N$, there exist $C > 0$ and $\eta > 0$ such that $|\partial^\beta G_{\phi,\varepsilon}(x)| \leq C\theta(x)\varepsilon^{g(q)-N}$, $\varepsilon < \eta$, $x \in \mathbf{R}^n$.

It is an ideal of $\mathcal{E}_{\mathbf{a}}$. Colombeau's space of $\mathbf{a}$-generalized functions is defined by $\mathcal{G}_{\mathbf{a}} = \mathcal{E}_{\mathbf{a}}/\mathcal{N}_{\mathbf{a}}$.

For $\mathbf{a}_1 \geq \mathbf{a}$, $\mathcal{E}_{\mathbf{a}_1} \subset \mathcal{E}_{\mathbf{a}_1}$ but $\mathcal{G}_{\mathbf{a}_1} \not\subset \mathcal{G}_{\mathbf{a}}$, because the map $G_{\phi,\varepsilon} + \mathcal{N}_{\mathbf{a}_1} \in \mathcal{G}_{\mathbf{a}_1}$ $\to G_{\phi,\varepsilon} + \mathcal{N}_{\mathbf{a}} \in \mathcal{G}_{\mathbf{a}}$ is not injective. Because of that we shall define space of "pseudo"-$\mathbf{a}_1$ generalized functions $\mathcal{G}^{\psi}_{\mathbf{a}_1,\mathbf{a}}$ as a subspace of $\mathcal{G}_{\mathbf{a}}$.

A net of functions $\mu^{\mathbf{a}}_{0,\varepsilon}$, $\varepsilon > 0$, from $\mathcal{D}(\mathbf{R})$ is called a one dimensional unit net related to $\mathbf{a}$ if it satisfies the following properties:

1. $0 \leq \mu^{\mathbf{a}}_{0,\varepsilon}(x) \leq 1$, $x \in \mathbf{R}$, $\varepsilon > 0$.

2. For some $b > 0$ and $r > 0$,

$$\mu^{\mathbf{a}}_{0,\varepsilon}(x) = 1, \mathbf{a}(x) < b/\varepsilon,\ \mu^{\mathbf{a}}_{0,\varepsilon}(x) = 0, \mathbf{a}(x) > b/\varepsilon + r,\ \varepsilon > 0.$$

3. For every $l \in \mathbf{N}_0^n$ there exists $c_l > 0$ such that $|\partial^l \mu^{\mathbf{a}}_{0,\varepsilon}(x)| \leq c_l$, $x \in \mathbf{R}$, $\varepsilon > 0$.

We define the unit net μ_ε related to $\mathbf{a}$ (in n-dimensional case) by

$$\mu_\varepsilon^{\mathbf{a}}(x) = \mu_{0,\varepsilon}^{\mathbf{a}}(x_1)\cdot \ldots \cdot \mu_{0,\varepsilon}^{\mathbf{a}}(x_n), \text{ where } x = (x_1, \ldots, x_n) \in \mathbf{R}^n.$$

Let $\mu_\varepsilon^{\mathbf{a}}$ be a unit net related to $\mathbf{a}$, B a measurable subset of $\mathbf{R}^n$, and $G \in \mathcal{G}_{\mathbf{a}}$. Then we define

$$\int_B^{\mu^{\mathbf{a}}} G(x)dx \in \overline{\mathbf{C}} \text{ by its representative } \int_B G_{\phi,\varepsilon}(x)\mu_\varepsilon^{\mathbf{a}}(x)dx \in \mathbf{C}_M.$$

If $B = \mathbf{R}^n$ then the symbol $\int^{\mu^{\mathbf{a}}}$ is used. One can easily prove that $G_{\phi,\varepsilon} \in \mathcal{N}_{\mathbf{a}}$ implies $\int_B G_{\phi,\varepsilon}(x)\mu_\varepsilon^{\mathbf{a}}(x)dx \in \mathbf{C}_0$. Thus the definition of the integral in the $\mathcal{G}_{\mathbf{a}}$ make sense.

It is said that $G \in \mathcal{G}$ is equal to $H \in \mathcal{G}$ in generalized distribution sense, $G = H(g.d.)$, if for every $\psi \in \mathcal{D}$, $< G - H, \psi >= 0$ in $\overline{\mathbf{C}}$.

$A \in \overline{\mathbf{C}}$ is associated to $c \in \mathbf{C}$, $A \approx c$, if there exists $N \in \mathbf{N}_0$ such that $\lim_{\varepsilon\to 0} A_{\phi,\varepsilon} = c$ for every $\phi \in \mathcal{A}_N$.

$G \in \mathcal{G}$ is associated to $H \in \mathcal{G}$, $G \approx H$, if there exists $N \in \mathbf{N}_0$ such that for every $\psi \in \mathcal{D}$ and every $\phi \in \mathcal{A}_N$ $\lim_{\varepsilon\to 0} < G_{\phi,\varepsilon} - H_{\phi,\varepsilon}, \psi >= 0$.

Let $\mu^{\mathbf{a}}$ be a unit net related to $\mathbf{a}$. Then the $\mathbf{a},\mu^{\mathbf{a}}$-Fourier transformation $\mathcal{F}_{\mathbf{a},\mu^{\mathbf{a}}}$ on $\mathcal{G}_{\mathbf{a}}$ is defined by

$$\mathcal{F}_{\mathbf{a},\mu^{\mathbf{a}}}(G)(x) = \int^{\mu^{\mathbf{a}}} G(y)e^{ixy}dy,\ x \in \mathbf{R}^n.$$

It can be considered as an element of $\mathcal{G}_{\mathbf{a}_1}$, for any $\mathbf{a}_1 \in \mathbf{A}$ because

$$|\mathcal{F}_{\mathbf{a},\mu^{\mathbf{a}}}(G_{\phi,\varepsilon}(t))(x)| \leq C\varepsilon^{-N}, x \in \mathbf{R}^n.$$

The inverse $\mathbf{a},\mu^{\mathbf{a}}$-Fourier transformation is defined by

$$\mathcal{F}^{-1}_{\mathbf{a},\mu^{\mathbf{a}}}(G) = (2\pi)^{-n/2}\int^{\mu^{\mathbf{a}}} G(y)e^{-ixy}dy, x \in \mathbf{R}^n.$$

One can prove that both definitions make sense.

Now, we define a convolution in $\mathcal{G}_{\mathbf{a}}$. Let $G_1, G_2 \in \mathcal{G}_{\mathbf{a}}$, and let $\mu_\varepsilon^{\mathbf{a}}$ be a unit net related to $\mathbf{a}$. Then we define $G_1 \overset{\mathbf{a},\mu}{\star} G_2$ as an element of $\mathcal{G}$ by

$$G_1 \overset{\mathbf{a},\mu}{\star} G_2(x) = \int^{\mu^{\mathbf{a}}} G_1(x-y)G_2(y)dy, x \in \mathbf{R}^n.$$

The correctness of this definition is proved in [3]

3. Laplace transformation

Let Ω be an open set in $\mathbf{C}^n$. In [2], Colombeau defined $\mathcal{G}(\Omega)$ in the standard way by considering Ω as an open subset in $\mathbf{R}^{2n}$.

A $G \in \mathcal{G}(\Omega)$ is called holomorphic generalized function if for every $z = x + iy \in \Omega$

$$(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j})G(z) = 0,\ j = 1, \ldots, n.$$

We say that $G \in \mathcal{G}_{\mathbf{a}}$ is pseudo $\mathbf{a}_1$-bounded at infinity, $\mathbf{a}_1(x) \geq \mathbf{a}(x)$, denoted by $G \in \mathcal{G}^{\psi}_{\mathbf{a}_1,\mathbf{a}}$, if it has a representative $G^s_{\phi,\varepsilon}$ such that there exist $\theta_1 \in \Theta_{\mathbf{a}_1}$ and $N \in \mathbf{N}$ such that for every $\phi \in \mathcal{A}_N$ there exist $C > 0$ and $\eta > 0$ such that

$$|G^s_{\phi,\varepsilon}(x)| \leq C\theta_1(|x|)\varepsilon^{-N},\ x \in \mathbf{R}^n,\ \varepsilon < \eta.$$

If some operation on $\mathcal{G}_{\mathbf{a}}$ is well defined (that is, it does not depend on the representatives), then all representatives of G have the property obtained by making this operation with the "special" representative of $G \in \mathcal{G}^{\psi}_{\mathbf{a}_1,\mathbf{a}}$, $G^s_{\phi,\varepsilon}$.

Let Γ be a closed convex acute cone in $\mathbf{R}^n$ with the vertex at 0. Denote by $\Gamma^* = \{\xi|\ \xi x \geq 0,\ x \in \Gamma\}$ its conjugate cone. Put $C = \mathrm{int}\Gamma^*$ (C is an open and convex cone) and $T^C = \mathbf{R}^n + iC$.

It is said that $G \in \mathcal{G}_{\mathbf{a}}$ is bounded on the side of the cone Γ if $\mathrm{supp}G \subset \Gamma + K$, where K is a compact subset of $\mathbf{R}^n$. The space of all such generalized functions is denoted by $\mathcal{G}_{\mathbf{a}}(\Gamma+)$.

Let $\mathbf{a} \in \mathbf{A}$, $\mathbf{a}(x) \leq \ln|x|$. This means if $\theta \in \Theta_{\mathbf{a}}$, then there exists $a > 0$ such that $\theta(x) \leq e^{a|x|}$ in infinity. Let $\mu^{\mathbf{a}}$ be a unit net related to $\mathbf{a}$. Then $L_{\mu}\mathbf{a}$ is defined for $G \in \mathcal{G}_{\mathbf{a}}(\Gamma+)$ by

$$L_{\mu}\mathbf{a}(G)(z) = F_{\mathbf{a},\mu}\mathbf{a}(G(\xi)e^{-y\xi})(x)$$

$$= \int^{\mu^{\mathbf{a}}} G(\xi)e^{-y\xi}e^{ix\xi}d\xi,\ x + iy \in T^C.$$

For every $\mathbf{a}_1 \in \mathbf{A}$ and every $y \in C$, one can prove that $L_{\mu}\mathbf{a}(G)(\cdot + iy) \in \mathcal{G}_{\mathbf{a}_1}$ is well defined. In this paper we consider the case $\mathbf{a}_1 = \mathbf{t}$. Since $e^{-y\xi} \in \Theta_{\mathbf{a}}$, $\xi \in K + \gamma$, for $\mathbf{a}(x) \leq \ln|x|$, the definition makes sense. In fact, one can easily see that the definition makes sense for every $y \in \mathbf{R}^n$ when $G \in \mathcal{G}_{\mathbf{a}}(\mathbf{R}^n)$: Let $G_{\phi,\varepsilon} \in \mathcal{N}_{\mathbf{a}}$. Then (because $\mathbf{a}(x) \leq \ln x$)

$$|\partial_z^\alpha \int G_{\phi,\varepsilon}(\xi)e^{iz\xi}\mu_\varepsilon^{\mathbf{a}}(\xi)d\xi| \leq \int_{|\xi|\leq \mathbf{a}(1/\varepsilon)+r} |G_{\phi,\varepsilon}(\xi)||\xi|^{|\alpha|}e^{-y\xi}d\xi$$

$$\leq C_1(\mathbf{a}(1/\varepsilon)+r)^n \sup_{|\xi|\leq \mathbf{a}(1/\varepsilon)+r} \theta_G(|\xi|)e^{|y|\xi}\varepsilon^{q-N_G}$$

$$\leq C_1(y)\theta_1(\mathbf{a}(1/\varepsilon)+r)\varepsilon^{q-N_G} \leq C(y)\varepsilon^{q-N_G-N_1},$$

for ε small enough, where $\theta_1 \in \Theta_{\mathbf{a}}$, too. In the same manner we can see that $L_{\mu^{\mathbf{a}}}(G_{\phi,\varepsilon}) \in \mathcal{E}_{\mathbf{t}}$ for fixed y.

If G is bounded on the side of cone Γ and $y \in C$, then the constant $C(y)$ is equal to $e^{|y|C_G}$. Because of its properties, which are similar to the classical ones, we shall use only definition for such G.

$L_{\mu^{\mathbf{a}}}(G)(z)$ is a holomorphic generalized function in T^C because

$$\left(\frac{\partial}{\partial x_j}+i\frac{\partial}{\partial y_j}\right)L_{\mu^{\mathbf{a}}}(G)(z) = \left(\frac{\partial}{\partial x_j}+i\frac{\partial}{\partial y_j}\right)\int^{\mu^{\mathbf{a}}} G(\xi)e^{-y\xi}e^{ix\xi}d\xi$$

$$= \int^{\mu^{\mathbf{a}}} G(\xi)e^{-y\xi}i\xi_j e^{ix\xi}d\xi + i\int^{\mu^{\mathbf{a}}} G(\xi)e^{-y\xi}(-\xi_j)e^{ix\xi}d\xi = 0.$$

The definition of the generalized Laplace transformation coincide with classical one ([7]) in the associated sense:

Proposition 1. *Let $g \in S'$ be bounded on the side of the cone Γ. Let* $\mathrm{Cd} : S' \to \mathcal{G}_{\mathbf{a}}$*, for some* $\mathbf{a} \in \mathbf{A}$*. Then for every $y \in C = \mathrm{int}\Gamma^*$ and for every unit net $\mu^{\mathbf{a}}$ related to* $\mathbf{a}$

$$L(g)(\cdot + iy) \approx L_{\mu^{\mathbf{a}}}(\mathrm{Cd}(g)).$$

Proof. Let $\psi \in \mathcal{S}$. Let $\gamma \in C^\infty$ be equal to one on the support of g and bounded on the side of cone Γ. Then, since $\mathcal{F}(\psi)(\xi)e^{-y\xi}\gamma(\xi) \in \mathcal{S}$, we have

$$< L(g)(x+iy), \psi(x) >$$

$$=< \mathcal{F}(g(\xi)e^{-y\xi})(x), \psi(x) >=< g(\xi), \gamma(\xi)e^{-y\xi}\mathcal{F}(\psi)(\xi) >$$

and

$$< L_{\mu}\mathbf{a}(\mathrm{Cd}(g))_{\phi,\varepsilon}(x+iy), \psi(x) >=< \mathcal{F}(g*\phi_\varepsilon(\xi)e^{-y\xi}\mu_\varepsilon^{\mathbf{a}}(\xi))(x), \psi(x) >$$

$$=< g*\phi_\varepsilon(\xi)\mu_\varepsilon^{\mathbf{a}}(\xi), e^{-y\xi}\mathcal{F}(\psi)(\xi) >,$$

where L and $\mathcal{F}$ are classical Laplace and Fourier transformation respectively. Since

$$< g*\phi_\varepsilon(\xi)\mu_\varepsilon^{\mathbf{a}}(\xi), \gamma(\xi)e^{-y\xi}\mathcal{F}(\psi)(\xi) > - < g*\phi_\varepsilon(\xi), \gamma(\xi)e^{-y\xi}\mathcal{F}(\psi)(\xi) >$$

$$= (\mu_\varepsilon^{\mathbf{a}}(\xi)-1) < g*\phi_\varepsilon(\xi), \gamma(\xi)e^{-y\xi}\mathcal{F}(\psi)(\xi) >,$$

and $\mu_\varepsilon^{\mathbf{a}}(\xi)-1 \to 0$, as $\varepsilon \to 0$,

$$< g*\phi_\varepsilon(\xi), \gamma(\xi)e^{-y\xi}\mathcal{F}(\psi)(\xi) > \to < g(\xi), \gamma(\xi)e^{-y\xi}\mathcal{F}(\psi)(\xi) > \in \mathbf{C},$$

and

$$< g*\phi_\varepsilon(\xi), \gamma(\xi)e^{-y\xi}\mathcal{F}(\psi)(\xi) > - < g(\xi), \gamma(\xi)e^{-y\xi}\mathcal{F}(\psi)(\xi) > \to 0,$$

because $g \to g*\phi_\varepsilon$ in $\mathcal{S}'$, when $\phi \in \mathcal{A}_N$, N large enough, we have proved the assertion. □

The properties of the Laplace transformation, which are given in the next proposition, are analogous to the classical ones (see [7]).

Proposition 2. *Let G, G_1, G_2 be in $\mathcal{G}_{\mathbf{a}}$. Then:*

1.

$$\partial_{z_j}^{\alpha} L_{\mu}\mathbf{a}(G)(z) = L_{\mathbf{a},\mu}\mathbf{a}((i\xi_j)^{\alpha}G(\xi))(z),\ z \in T^C.$$

2.

$$L_{\mu}\mathbf{a}(\partial_j G(\xi))(z) = z_j L_{\mathbf{a},\mu}\mathbf{a}(G(\xi))(z) + M_{G,j,\mu}\mathbf{a}(z),\ z \in T^C,$$

where $M_{G,j,\mu}\mathbf{a} = [\int G_{\phi,\varepsilon}(\xi)e^{iz\xi}\partial_j\mu_\varepsilon^{\mathbf{a}}(\xi)d\xi]$

3.

$$L_{\mu}\mathbf{a}(G(\xi)e^{ia\xi})(z) = L_{\mu}\mathbf{a}(G(\xi))(z+a),\ z \in T^C,$$

for any $a \in T^C$.

4.

$$L_{\mu}\mathbf{a}(G(\xi-\xi_0))(z) = e^{iz\xi_0}(L_{\mathbf{a},\mu}\mathbf{a}(G(\xi))(z) + M_{G,\xi_0,\mu}\mathbf{a}(z)),\ z \in T^C,$$

where $M_{G,\xi_0,\mu}\mathbf{a}(z)) = [e^{iz\xi_0}\int G_{\phi,\varepsilon}(\xi)e^{iz\xi}(\mu_\varepsilon^{\mathbf{a}}(\xi+\xi_0) - \mu_\varepsilon^{\mathbf{a}}(\xi))d\xi]$, *for any* $\xi_0 \in \mathbf{R}^n$.

5.

$$L_{\mu}\mathbf{a}(G_1 \times G_2)(z_1, z_2) = L_{\mu}\mathbf{a}(G_1)(z_1)L_{\mu}\mathbf{a}(G_2)(z_2),\ z \in T^C.$$

6.

$$L_{\mu}\mathbf{a}(G_1 \overset{\mathbf{a},\mu}{\star} G_2)(z)$$
$$= L_{\mu}\mathbf{a}(G_1)(z)L_{\mu}\mathbf{a}(G_2)(z) + M_{G_1,G_2,\mu}\mathbf{a}(z),\ z \in T^C,$$

where

$$M_{G_1,G_2,\mu}\mathbf{a}(z) = [\int\int G_{1,\phi,\varepsilon}(\zeta)G_{2,\phi,\varepsilon}(\xi)$$
$$\cdot(\mu_\varepsilon^{\mathbf{a}}(\zeta+\xi) - \mu_\varepsilon^{\mathbf{a}}(\zeta))\mu_\varepsilon^{\mathbf{a}}(\xi)e^{iz(\zeta+\xi)}d\xi d\zeta].$$

Proof.

1.

$$\partial_{z_j}^{\alpha} L_{\mu}\mathbf{a}(G)(z) = \partial_{z_j}^{\alpha}\int^{\mu\mathbf{a}} G(\xi)e^{-y\xi}e^{ix\xi}d\xi =$$
$$\partial_{z_j}^{\alpha}\int^{\mu\mathbf{a}} G(\xi)e^{iz\xi}d\xi = \int^{\mu\mathbf{a}} (i\xi_j)^{\alpha}G(\xi)e^{ix\xi}d\xi =$$
$$L_{\mu}\mathbf{a}((i\xi_j)^{\alpha}G(\xi))(z).$$

2.

$$L_{\mu}\mathbf{a}(\partial_j G(\xi))(z) = \int^{\mu\mathbf{a}} \partial_j G(\xi)e^{iz\xi}d\xi =$$
$$[\int \partial_j G_{\phi,\varepsilon}(\xi)e^{iz\xi}\mu_\varepsilon^{\mathbf{a}}(\xi)d\xi] =$$
$$[\int G_{\phi,\varepsilon}(\xi)(iz_j)e^{iz\xi}\mu_\varepsilon^{\mathbf{a}}(\xi)d\xi] + [\int G_{\phi,\varepsilon}(\xi)e^{iz\xi}\partial_j\mu_\varepsilon^{\mathbf{a}}(\xi)d\xi] =$$
$$z_j L_{\mu}\mathbf{a}(G(\xi))(z) + [\int_{\mathbf{a}(b/\varepsilon)\le|\xi|\le\mathbf{a}(b/\varepsilon)+r} G_{\phi,\varepsilon}(\xi)e^{iz\xi}\partial\mu_\varepsilon^{\mathbf{a}}(\xi)d\xi] =$$
$$z_j L_{\mu}\mathbf{a}(G(\xi))(z) + M_{G,j,\mu}(z).$$

3.

$$L_{\mu}\mathbf{a}(G(\xi)e^{ia\xi})(z) = \int^{\mu^{\mathbf{a}}} G(\xi)e^{i(z+a)\xi}d\xi =$$

$$L_{\mu}\mathbf{a}(G(\xi))(z+a).$$

4.

$$L_{\mu}\mathbf{a}(G(\xi-\xi_0))(z) = \int^{\mu^{\mathbf{a}}} G(\xi-\xi_0)e^{iz\xi}d\xi =$$

$$[\int G_{\phi,\varepsilon}(\xi)e^{iz(\xi+\xi_0)}\mu_{\varepsilon}^{\mathbf{a}}(\xi+\xi_0)d\xi] =$$

$$[e^{iz\xi_0}\int G_{\phi,\varepsilon}(\xi)e^{iz\xi}\mu_{\varepsilon}^{\mathbf{a}}(\xi)d\xi] +$$

$$[e^{iz\xi_0}\int_{\substack{\mathbf{a}(b/\varepsilon)\le|\xi|\le\mathbf{a}(b/\varepsilon)+r \\ \mathbf{a}(b/\varepsilon)\le|\xi+\xi_0|\le\mathbf{a}(b/\varepsilon)+r}} G_{\phi,\varepsilon}(\xi)e^{iz\xi}(\mu_{\varepsilon}^{\mathbf{a}}(\xi+\xi_0)-\mu_{\varepsilon}^{\mathbf{a}}(\xi))d\xi] =$$

$$e^{iz\xi_0}(L_{\mu}\mathbf{a}(G(\xi))(z) + M_{G,\xi_0,\mu}(z)).$$

5.

$$L_{\mu}\mathbf{a}(G_1\times G_2)(z_1,z_2) =$$

$$\int^{\mu^{\mathbf{a}}}\int^{\mu^{\mathbf{a}}} G_1(\xi_1)G_2(\xi_2)e^{i(z_1\xi_1+z_2\xi_2)}d\xi_1 d\xi_2 =$$

$$L_{\mu}\mathbf{a}(G_1)(z_1)L_{\mu}\mathbf{a}(G_2)(z_2).$$

6.

$$L_{\mu}\mathbf{a}(G_1 \overset{\mathbf{a},\mu}{\star} G_2)(z) = \int^{\mu^{\mathbf{a}}}\int^{\mu^{\mathbf{a}}} G_1(\zeta-\xi)G_2(\xi)e^{iz\zeta}d\xi d\zeta =$$

$$[\int\int G_{1,\phi,\varepsilon}(\zeta-\xi)G_{2,\phi,\varepsilon}(\xi)\mu_{\varepsilon}^{\mathbf{a}}(\zeta)\mu_{\varepsilon}^{\mathbf{a}}(\xi)e^{iz\zeta}d\xi d\zeta] =$$

$$[\int\int G_{1,\phi,\varepsilon}(\zeta)G_{2,\phi,\varepsilon}(\xi)\mu_{\varepsilon}^{\mathbf{a}}(\zeta+\xi)\mu_{\varepsilon}^{\mathbf{a}}(\xi)e^{iz(\zeta+\xi)}d\xi d\zeta] =$$

$$[\int\int G_{1,\phi,\varepsilon}(\zeta)G_{2,\phi,\varepsilon}(\xi)\mu_{\varepsilon}^{\mathbf{a}}(\zeta)\mu_{\varepsilon}^{\mathbf{a}}(\xi)e^{iz(\zeta+\xi)}d\xi d\zeta] +$$

$$[\int\int G_{1,\phi,\varepsilon}(\zeta)G_{2,\phi,\varepsilon}(\xi)(\mu_{\varepsilon}^{\mathbf{a}}(\zeta+\xi)-\mu_{\varepsilon}^{\mathbf{a}}(\zeta))\mu_{\varepsilon}^{\mathbf{a}}(\xi)e^{iz(\zeta+\xi)}d\xi d\zeta] =$$

$$L_{\mu}\mathbf{a}(G_1)(z)L_{\mu}\mathbf{a}(G_2)(z) + M_{G_1,G_2,\mu}(z).\square$$

Corollary 1.

1. *If* G, G_1, G_2 *have compact supports, then*

$$M_{G,j\mu}\mathbf{a} = M_{G,\xi_0,\mu}\mathbf{a} = M_{G_1,G_2,\mu}\mathbf{a} = 0.$$

   That means that all six listed properties are the same as in classical case.

2. *If* $\mathbf{a}(x) = \ln(x)$ *and* G, G_1, G_2 *are in* $\mathcal{G}_{\mathbf{t}}^{\psi}$, *then* $M_{G,j\mu}\mathbf{a} \approx M_{G,\xi_0,\mu}\mathbf{a} \approx M_{G_1,G_2,\mu}\mathbf{a} \approx 0$ *for fixed large enough* $|y|$.

Proof. The proof of the first statement is obvious and the second follows from the fact that for each y and every $G_{\phi,\varepsilon} \in \mathcal{E}_{\mathbf{t}}^{\psi}$ and $\psi \in \mathcal{D}(\mathbf{R}^n)$,

$$\begin{aligned} &\int \Big(\int G_{\phi,\varepsilon}(\xi) e^{iz\xi} \mu_\varepsilon^{\mathbf{a}}(\xi) d\xi - \int G_{\phi,\varepsilon}(\xi) e^{iz\xi} d\xi | \Big) \psi(x) dx \\ &\leq C_\psi \int_{|\xi| > \mathbf{a}(1/\varepsilon)} |G_{\phi,\varepsilon}(\xi)| e^{-y\xi} d\xi \leq C_1 (\mathbf{a}(1/\varepsilon))^n \\ &\qquad \sup_{|\xi|>\mathbf{a}(1/\varepsilon), \xi = \xi_K + \xi_\Gamma, \xi_K \in K, \xi_\Gamma \in \Gamma} |G_{\phi,\varepsilon}(\xi)| e^{-y\xi} \\ &\leq C_2 (\mathbf{a}(1/\varepsilon))^n e^{|y| C_K} (1 + \mathbf{a}(1/\varepsilon))^{\gamma} \varepsilon^{-N} e^{-|y| \mathbf{a}(1/\varepsilon)} \\ &\leq C (\mathbf{a}(1/\varepsilon))^{\gamma+n} e^{|y| C_K} \varepsilon^{|y|-N} \end{aligned}$$

and the last term converge to zero as $\varepsilon \to 0$ when $|y| > N$ (N depends only on generalized function G.) □

4. Appendix. Tempered Laplace transformation

In the previous section the condition $\mathbf{a}(x) \leq \ln(x)$ is used, that means that there are no definition of generalized Laplace transformation for the space of tempered generalized functions. In this section we introduce such definition, as we said, different from the one used in [4].

Let us denote by $\mathcal{G}_{\mathbf{a}}(\Gamma)$ the set of all functions from $\mathcal{G}_{\mathbf{a}}$ which has a support inside a cone Γ.

Let $\mu^{\mathbf{t}}$ be a unit net related to $\mathbf{t}$. Let $l \in C^\infty$ be a fixed function less or equal to 1, equal to 1 on the cone Γ and there exist a compact set K such that supp$l \subset K + \Gamma$. Then the tempered Laplace transformation, $L_{\mu^{\mathbf{t}}}$, is defined for $G \in \mathcal{G}_{\mathbf{t}}(\Gamma)$ by

$$L_{\mu^{\mathbf{t}}}(G)(z) = F_{\mathbf{t},\mu^{\mathbf{t}}}(G(\xi)l(\xi)e^{-y\xi})(x)$$

$$= \int^{\mathbf{t},\mu^{\mathbf{t}}} G(\xi)l(\xi)e^{-y\xi}e^{ix\xi}d\xi \in (\mathcal{G}_{\mathrm{t}})_x \times (\mathcal{G}_{\mathrm{ln}})_y,\ (\in \mathcal{G})\ x + iy \in T^C.$$

The definition makes sense, that is, if $G_{\phi,\varepsilon} \in \mathcal{N}_{\mathbf{t}}$, then $L_{\mu^{\mathbf{t}}}(G_{\phi,\varepsilon}) \in \mathcal{N}$, or more precisely, $L_{\mu^{\mathbf{t}}}(G_{\phi,\varepsilon}) \in (\mathcal{N}_t)_x \times (\mathcal{N}_{\mathrm{ln}})_y$.

For $G_{\phi,\varepsilon} \in \mathcal{N}_{\mathbf{t}}$ we have that for every $\alpha \in \mathbf{N}_0^n$ there exist $\gamma > 0$ and $N_1 \in \mathbf{N}$ such that for every $\phi \in \mathcal{A}_q$, $q > N$ there exist $C_1 > 0$ and $\eta > 0$

$$|\partial^\alpha G_{\phi,\varepsilon}(\xi)| \le C_1(1+|\xi|)^\gamma \varepsilon^{-N_1},\ \phi \in \mathcal{A}_q,\ \varepsilon < \eta.$$

With the same notation, for $\alpha, \beta \in \mathbf{N}_0^n$,

$$|\partial_x^\alpha \partial_y^\beta \int\int G_{\phi,\varepsilon}(\xi)l(\xi)e^{(ix-y)\xi}\mu_\varepsilon^{\mathbf{t}}(\xi)d\xi|$$

$$|\int\int G_{\phi,\varepsilon}(\xi)l(\xi)(i\xi)^\alpha(-y)^\beta e^{-y\xi}e^{ix\xi}\mu_\varepsilon^{\mathbf{t}}(\xi)d\xi|$$

$$\le \sup_{|\xi|\le 1/\varepsilon+r,\xi=\xi_K+\xi_\Gamma,\xi_K\in K,\xi_\Gamma\in\Gamma} C_1(1+|\xi|)^\gamma|\xi|^{|\alpha|}|y|^{|\beta|}e^{-y\xi_K}C_2(1/\varepsilon+r)^n\varepsilon^{q-N_1}$$

$$\le Ce^{|y|C_l}\varepsilon^{q-\gamma-|\alpha|-n-N_1} \le Ce^{|y|(C_l+\delta)}\varepsilon^{q-N},\ \phi \in \mathcal{A}_q,\ \varepsilon < \eta, \text{ for every } \delta > 0.$$

Here $N > \gamma + |\alpha| + n + N_1$ and C is some suitable constant depend on ϕ. This proves that $L_{\mu^{\mathbf{t}}}(G)_{\phi,\varepsilon} \in \mathcal{N}_{\mathbf{t}} \times \mathcal{N}_{\mathbf{ln}}$ (or in $\mathcal{N}_{\mathbf{t}}$ for every $y \in C$). In the same way one can prove that $L_{\mu^{\mathbf{t}}}(G)_{\phi,\varepsilon} \in \mathcal{E}_{\mathbf{t}} \times \mathcal{E}_{\mathbf{ln}}$ (or in $\mathcal{E}_{\mathbf{t}}$ for every $y \in C$), when $G_{\phi,\varepsilon} \in \mathcal{E}_{\mathbf{t}}$.

The definition does not depend on the function l. Let l_1 and l_2 be two function which satisfy the conditions of the definition. Then $\int^{\mu^{\mathbf{t}}}(l_1(\xi) - l_2(\xi))G(\xi)e^{iz\xi}d\xi$ is given by the oridinary integral with bounds where the representative of G have the bound of the form $C(1+|\xi|)^\gamma\varepsilon^{g-N}$, so by the above procedure one can obtain that $\int^{\mu^{\mathbf{t}}}(l_1(\xi) - l_2(\xi))G(\xi)e^{iz\xi}d\xi = 0$.

In the same way as it was done in Section 3 one can prove that $L_{\mu^{\mathbf{t}}}(G)(z)$ is holomorphic function in T^C if it was taken $L_{\mu^{\mathbf{t}}}(G)(z) \in \mathcal{G}(\mathbf{R}^n)$.

By the property of the tempered Fourier transformation, one can see that, for fixed y, the tempered Laplace transformation does not depend on the unit net in (g.t.d.) sense in respect to variable x.

Remark 1.

1. *Proposition 1 holds in the case of tempered generalized functions, too.*
2. *In the Pposition 2 we have the equality in (g.t.d.) sense in all six points. For example, we have*

$$< M_{G,j,\mu}, \psi(x) >= \int\int G_{\phi,\varepsilon}(\xi)e^{(ix-y)\xi}\partial_j\mu_\varepsilon(\xi)d\xi\psi(x)dx$$

$$= \int G_{\phi,\varepsilon}(\xi)e^{-y\xi}\mathcal{F}(\psi)(\xi)\partial_j\mu_\varepsilon(\xi)d\xi \in \mathbf{C}_0,$$

because $\mathcal{F}(\psi)$ rapidly decrease to zero in infinity and the j-th projection of $supp\partial_j\mu_\varepsilon$ is a subset of $B(0,1/\varepsilon+r)\setminus B(0,1/\varepsilon)$.

The similar arguments holds for all other cases in the proposition.

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REZIME

OSOBINE LAPLASOVE TRANSFORMACIJE U KOLOMBOVIM PROSTORIMA UOPŠTENIH FUNKCIJA

U radu definišemo i proučavamo osobine Laplasove transformacije u prostorima $\mathcal{G}_{\mathbf{a}}$, $\mathbf{a} \in \mathbf{A}$. Dokazali smo analogne osobine klasičnim za ovu transformaciju.

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