Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 23, 1 (1993), 121 - 134

Review of Research
Faculty of Science
Mathematics Series

n–PARTITIONS OF TOPOLOGICAL SPACES

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Abstract

A partition of type n on a set X is a family Π of its subsets, such that any n elements of X are contained in exactly one subset and every subset contains at least n elements of X. If X is a Hausdorff space and if Π satisfies certain topological conditions (expressed by the convergence of nets) then we consider Π as a topological space.Some examples of such partitions are given here. Topological projective and Euclidean planes appear as special cases of 2-partitions.

AMS Mathematics Subject Classification (1991): 51H10, 54B20
Key words and phrases: topological spaces, generalized partitions, topological planes.

1. Preliminaries

The well-known notion of the n–partition on a set X appears in many different branches of mathematics. In combinatorics it is investigated as a block-scheme. In algebra were studied the lattices on n–partitions and the connection of n–partitions with quasigroups. Geometrist sees an n-partition as an incidence structure.

The idea to topologize an n–partition is a result of a conversation with professor J.Ušan who considered n–partitions from the aspect of n–ary relations and hyperoperations. I am grateful to him for many useful suggestions.

Finally, something about notation. If X is a set $|X|$ will be its cardinality, $P(X)$ the power set of X and $[X]^n$ the family of subsets of X with exactly n elements.By N, R and C we will denote the set of natural, real and complex numbers respectively. The set of all permutations of $\{1,2,...,n\}$ will be denoted with $\{1,2,...,n\}!$. If $(X,\mathcal{O})$ is a topological space, letters $\mathcal{P}$ and $\mathcal{B}$ will be reserved for the subbase and base for the topology $\mathcal{O}$. At last, the fact that a net $< x_\sigma >$ converges to x will be denoted by $< x_\sigma > \rightarrow x$.

2. Topological n–partitions

Let X be a set with at least n elements. By [2], a collection $\Pi \subset P(X)$ is an n–partition on X iff (i) each $p \in \Pi$ contains at least n elements; (ii) any n distinct elements of X are contained in some $p \in \Pi$; and (iii) two distinct elements of Π may have at most $n-1$ common elements.

According to the definition, n distinct elements $x^1,...,x^n \in X$ determine the unique element of Π, denoted by $p(x^1,...,x^n)$, (notation $p(\{x^1,...,x^n\})$ or $p_{\{x^1,...,x^n\}}$ would be better but more complicated). Clearly, $p(x^1,...,x^n) = p(x^{\pi_1},...,x^{\pi_n})$ for every $\pi \in \{1,2,...,n\}!$. Also, if $y^1,...,y^n \in p(x^1,...,x^n)$ are distinct, then $p(x^1,...,x^n) = p(y^1,...,y^n)$.

To simplify the following definition, if $x^1,...,x^n \in X$ are not distinct (that is if $|\{x^1,...,x^n\}| < n$) we define $p(x^1,...,x^n) = X$. Then, of course, $p(x^1,...,x^n) \notin \Pi$ (exept if $\Pi = \{X\}$).

Definition 1. *Let $(X,\mathcal{O})$ be a Hausdorff space. An n– partition Π on X is topological iff it holds: if $x^1,x^2,...,x^n,z \in X$ are arbitrary distinct points and $< x^i_\sigma \mid \sigma \in \Sigma >$ are nets, such that $< x^i_\sigma > \rightarrow x^i$, $i = 1,...,n$; where $p(x^1_\sigma,...,x^n_\sigma) = p_\sigma$ and $p(x^1,...,x^n) = p$, then*

(a) if $< z_\sigma > \rightarrow z$ and $z_\sigma \in p_\sigma$ for all $\sigma \in \Sigma$, then $z \in p$.

(b) if $z \in p$, there is a net $< z_\sigma \mid \sigma \in \Sigma >$ such that $< z_\sigma > \rightarrow z$ and $z_\sigma \in p_\sigma$ for all $\sigma \in \Sigma$.

The set of all topological n–partitions on X will be denoted by $TP_n(X)$.

Since X is a Hausdorff space and nets $< x^i_\sigma >$ converge to distinct points, there is $\sigma_0 \in \Sigma$ such that $x^1_\sigma, ..., x^n_\sigma$ are distinct for $\sigma \geq \sigma_0$. Therefore, for $\sigma \geq \sigma_0$, we have $p(x^1_\sigma, ..., x^n_\sigma) \in \Pi$.

Theorem 1. *Elements of* Π *are closed in* X.

Proof. Let $p \in \Pi, z \in \bar{p}$ and $x^1, ..., x^n \in p$ where $x^1, ..., x^n$ are distinct. Then there is a net $< z_\sigma \mid \sigma \in \Sigma >$ in p such that $< z_\sigma > \rightarrow z$. If we define $x^i_\sigma = x^i$ for $i = 1, ..., n$ and $\sigma \in \Sigma$, we have $< x^i_\sigma > \rightarrow x^i$ for $i = 1, ..., n$ and for all $\sigma \in \Sigma$, $z_\sigma \in p(x^1_\sigma, ..., x^n_\sigma) = p$. According to (a) of the previous definition, $z \in p$. □

If X is a first countable space, nets can be replaced by sequences.

Theorem 2. *Let* $(X, \mathcal{O})$ *be a first countable, Hausdorff space. An* $n-$*partition* Π *on* X *is topological iff it holds: if* $x^1, ..., x^n$, $z \in X$ *are arbitrary distinct points and* $< x^i_k \mid k \in N >$ *are sequences such that* $< x^i_k > \rightarrow x^i$ *for all* $i = 1, ..., n$; *where* $p(x^1_k, ..., x^n_k) = p_k$, $k \in N$ *and* $p(x^1, ..., x^n) = p$, *then*

(a') if $< z_k > \rightarrow z$ *and* $z_k \in p_k$ *for all* $k \in N$, *then* $z \in p$.

(b') if $z \in p$, *there is a sequence* $< z_k \mid k \in N >$ *such that* $< z_k > \rightarrow z$ *and* $z_k \in p_k$ *for all* $k \in N$.

Proof. ($\Rightarrow$) Is obvious because each sequence is a net.

($\Leftarrow$) Suppose that $x^1, ..., x^n, z \in X$ are distinct points and $< x^i_\sigma > \rightarrow x^i$ for $i = 1, ..., n$. Let $\mathcal{B}(x^i) = \{B^i_k \mid k \in N\}$ be a monotonous, countable local base at the point x^i, $i = 1, ..., n$ and $\mathcal{B}(z)$ such a base at the point z.

(a) Let $< z_\sigma > \rightarrow z$ where $z_\sigma \in p_\sigma$, $\sigma \in \Sigma$; and let $k \in N$. Because of convergence of the observed nets, there exists σ^i_k such that for $\sigma \geq \sigma^i_k$, $x^i_\sigma \in B^i_k$, $i = 1, ..., n$, and there is σ^z_k such that $z_\sigma \in B^z_k$ for $\sigma \geq \sigma^z_k$. If we choose $\sigma_k \in \Sigma$ such that $\sigma_k \geq \sigma^1_k, ..., \sigma^n_k, \sigma^z_k$, then $x^i_{\sigma_k} \in B^i_k$ for $i = 1, ..., n$ and $z_{\sigma_k} \in B^z_k$. This holds for all $k \in N$, so $< x^i_{\sigma_k} > \rightarrow x^i$, $i = 1, ..., n$ and $< z_{\sigma_k} > \rightarrow z$. Since $z_{\sigma_k} \in p_{\sigma_k}$, $k \in N$, according to (a') we have $z \in p$.

(b) Let $z \in p$. Firstly we will prove that

$$(1) \qquad \forall n \in N \;\; \exists \sigma_n \in \Sigma \;\; \forall \tau \geq \sigma_n \;\; p_\tau \cap B^z_n \neq \emptyset.$$

Suppose that (1) does not hold. Then there is $n_0 \in N$ such that $T = \{\tau \in \Sigma \mid p_\tau \cap B^z_{n_0} = \emptyset\}$ is cofinal in Σ, so $< x^i_\tau \mid \tau \in T > \rightarrow x^i$ for $i = 1, ..., n$. By the

construction given in (a) we get sequences $< x^i_{\tau_k} \mid k \in N > \to x^i$, $i = 1, ..., n$. Because of (b') there is a sequence $< z_k > \to z$ such that $z_k \in p_{\tau_k}$. Now, for $B^z_{n_0}$ there is $k_1 \in N$ such that $z_{k_1} \in B^z_{n_0}$. But $z_{k_1} \in p_{\tau_{k_1}}$
$capB^z_{n_0} = \emptyset$. A contradiction.

According to (1) there is $\sigma_1 \in \Sigma$ such that for $\sigma \geq \sigma_1$ we have $p_\sigma \cap B^z_1 \neq \emptyset$. Then $N(\sigma) = \{k \in N \mid p_\sigma \cap B^z_k \neq \emptyset\} \neq \emptyset$, since $1 \in N(\sigma)$. Since $\mathcal{B}(z)$ is a monotonous local base, $N(\sigma) = N$ or there is $m_0 = \max N(\sigma)$. For $\sigma \geq \sigma_1$ we define:

$$B_\sigma = \begin{cases} p_\sigma \cap B^z_{m_\sigma} & \text{if } m_\sigma \text{ exists} \\ \{z\} & \text{otherwise.} \end{cases}$$

If m_σ exists, then $m_\sigma \in N(\sigma)$, so $B_\sigma = p_\sigma \cap B^z_{m_\sigma} \neq \emptyset$, and $B_\sigma \subset p_\sigma$. If $N(\sigma) = N$ then for all $k \in N$, $p_\sigma \cap B^z_k \neq \emptyset$, that is $z \in \bar{p}_\sigma = p_\sigma$. (Theorem 2.1), so $B_\sigma = \{z\} \subset p_\sigma$. Therefore for all $\sigma \geq \sigma_1$, $\emptyset \neq B_\sigma \subset p_\sigma$.

For $\sigma \geq \sigma_1$ we choose $z_\sigma \in B_\sigma$. For the rest of $\sigma \in \Sigma$ we define $z_\sigma = x^1_\sigma$. Now $z_\sigma \in p_\sigma$ for all $\sigma \in \Sigma$.

Let $n \in N$. According to (1) for $\sigma \geq \sigma_n$, $p_\sigma \cap B^z_n \neq \emptyset$. If m_σ exists, then $n \leq m_\sigma$ and $B^z_{m_\sigma} \subset B^z_n$, so $z_\sigma \in B_\sigma = p_\sigma \cap B^z_{m_\sigma} \subset B^z_n$. If m_σ does not exist, then $z_\sigma = z \in B^z_n$. Hence for each $\sigma \geq \sigma_n$, $z_\sigma \in B^z_n$, so we have $< z_\sigma > \to z$. $\square$

Now, let $(X, \mathcal{O})$ be a Hausdorff space and $\Pi \in TP_n(X)$. If for $O \in \mathcal{O}$ we define $O^* = \{p \in \Pi \mid p \cap O \neq \emptyset\}$, then $\mathcal{P}_\Pi = \{O^* \mid O \in \mathcal{O}\}$ is a subbase for some topology on Π, denoted by $\mathcal{O}_\Pi$. From now on we will consider the space $(\Pi, \mathcal{O}_\Pi)$.

3. Base, local base and density

Lemma 1. *Let $p \in \Pi$ and let $x^1, ..., x^n \in p$ be distinct points. If $\mathcal{B}(x^i)$ is a local base at the point x^i, $i = 1, ..., n$, then*

(i) if $O \in \mathcal{O}$ and $p \in O^*$, then there is $(B_1, ..., B_n) \in \Pi^n_{i=1}\mathcal{B}(x^i)$ where

$$p \in \bigcap_{i=1}^{n} B^*_i \subset O^*$$

(ii) if $k \in N$ and $O_j \in \mathcal{O}$ for $j = 1, ..., k$, and $p \in \bigcap_{j=1}^{k} O^*_j$, then there is

$(B_1, ..., B_n) \in \Pi_{i=1}^n \mathcal{B}(x^i)$ *such that*

$$p \in \bigcap_{i=1}^{n} B_i^* \subset \bigcap_{j=1}^{k} O_j^*.$$

Proof. Firstly let us notice that $\mathcal{B}(x^i)$, $i = 1, ..., n$ are directed sets, so $\Pi_{i=1}^n \mathcal{B}(x^i)$ is directed too. Let $B_i^0 \in \mathcal{B}(x^i)$ be arbitrary, but fixed.

(i) Let $p \in O^*$. Suppose that for each $(B_1, ..., B_n) \in \Pi_{i=1}^n \mathcal{B}(x^i)$ there exists $q_{(B_1,...,B_n)} \in \Pi$ such that $q_{(B_1,...,B_n)} \in \bigcap_{i=1}^n B_i^*$ and $q_{(B_1,...,B_n)} \bigcap O = \emptyset$. For all $i \in \{1, ..., n\}$ and $(B_1, ..., B_n) \in \Pi_{i=1}^n \mathcal{B}(x^i)$ we choose $x^i_{(B_1,...,B_n)} \in q_{(B_1,...,B_n)} \bigcap B_i$. Then $< x^i_{(B_1,...,B_n)} >$ is a net for $i = 1, ..., n$. Let $B_1' \in \mathcal{B}(x^1)$ and $(B_1, ..., B_n) \geq (B_1', B_2^0, .., B_n^0)$. Then $B_1 \subset B_1'$ and $x^1_{(B_1,...,B_n)} \in B_1 \subset B_1'$. Therefore $< x^1_{(B_1,...,B_n)} > \rightarrow x^1$. Similarly for other sequences. If $z \in p \cap O$, then according to (b) there is a net $< z_{(B_1,...,B_n)} > \rightarrow z$ and $z_{(B_1,...,B_n)} \in q_{(B_1,...,B_n)}$. But O is a neighborhood of z, so there is $z_{(B"_1,...,B"_n)} \in O$. Now $z_{(B"_1,...,B"_n)} \in q_{(B"_1,...,B"_n)} \cap O = \emptyset$. A contradiction.

Thus there exists $(B_1, ..., B_n) \in \Pi_{i=1}^n \mathcal{B}(x^i)$ such that for each $q \in \Pi$, $q \in \bigcap_{i=1}^n B_i^*$ implies $q \in O^*$, that is $\bigcap_{i=1}^n B_i^* \subset O^*$.

(ii) Let $p \in \bigcap_{j=1}^k O_j^*$ and $j \in \{1, ..., k\}$. Then $p \in O_j$ and because of (i) there is $(B_1^j, ..., B_n^j) \in \Pi_{i=1}^n \mathcal{B}(x^i)$ satisfying $p \in \bigcap_{i=1}^n B_i^{j*} \subset O_j^*$. Now $p \in \bigcap_{j=1}^k \bigcap_{i=1}^n B_i^{j*} \subset \bigcap_{j=1}^k O_j^*$, but $\bigcap_{j=1}^k \bigcap_{i=1}^n B_i^{j*} = \bigcap_{i=1}^n B_i^*$ where $B_i = \bigcap_{j=1}^k B_i^j$ for $i = 1, ..., n$. □

Now, we are able to prove:

Theorem 3. *Let $(X, \mathcal{O})$ be a Hausdorff space and $\Pi \in TP_n(X)$. Then*

(i) if $\mathcal{B}$ is a base for the topology on X, then $\mathcal{B}'_\Pi = \{\bigcap_{i=1}^n B_i^* \mid (B_1, ..., B_n) \in \mathcal{B}^n\}$ is a base for the topology on Π.

(ii) if $w(X) \geq \aleph_0$, then $w(\Pi) \leq w(X)$.

(iii) if $p \in \Pi$ if $x^1, ..., x^n \in p$ are distinct points and $\mathcal{B}(x^i)$ is a local base at the point x^i, $i = 1, ..., n$, then $\mathcal{B}(p) = \{\bigcap_{i=1}^n B_i^* \mid (B_1, ..., B_n) \in \Pi_{i=1}^n \mathcal{B}(x^i)\}$ is a local base at the point $p \in \Pi$.

(iv) if $p \in \Pi$, then $\chi(p) \leq \min\{\Pi_{i=1}^n \chi(x^i) \mid \{x^1, ..., x^n\} \in [p]^n\}$.

(v) if D is dense in X, then $D_\Pi = \{p(d^1, ..., d^n) \mid \{d^1, ..., d^n\} \in [D]^n\}$ is dense in Π.

(vi) If $d(x) \geq \aleph_0$, then $d(\Pi) \leq d(X)$.

Proof. (i) One base for the topology on Π is $\mathcal{B}_\Pi = \{\bigcap_{j=1}^k O_j^* \mid k \in N;\ O_j \in \mathcal{O},\ j = 1, ..., k\}$. Obviously $\mathcal{B}'_\Pi \subset \mathcal{B}_\Pi \subset \mathcal{O}_\Pi$. Let $p \in \bigcap_{j=1}^k O_j^*$, $\{x^1, ..., x^n\} \in [p]^n$ and let $\mathcal{B}(x^i)$ be a local base at the point x^i, $i = 1, ..., n$.

According to Lemma 3.1, there is $(B_1, ..., B_n) \in \Pi_{i=1}^n \mathcal{B}(x^i)$ such that $p \in \bigcap_{i=1}^n B_i^* \subset \bigcap_{j=1}^k O_j^*$. Since $\mathcal{B}$ is a base, there are $B_i' \in \mathcal{B}$ where $x^i \in B_i' \subset B_i$. Now, $p \in B_p \subset \bigcap_{j=1}^k O_j^*$ where $B_p = \bigcap_{i=1}^n B_i'^* \in \mathcal{B}'_\Pi$, so $\mathcal{B}'_\Pi$ is also a base.

(ii) If $\mathcal{B}$ is a base on X satisfying $|\mathcal{B}| = w(X)$, then from (i) we have $w(\Pi) \leq |\mathcal{B}'_\Pi| \leq |\mathcal{B}|^{n-1} = |\mathcal{B}| = w(X)$.

The proofs of the other statements are also direct . □

Corollary 1. *If X is a second countable (first countable, separable) space then Π is a second countable (first countable, separable) space.*

Proof. See (ii), (iv) and (vi) of the preceeding theorem . □

Theorem 4. *If the space X is discrete, then Π is a discrete space too.*

Proof. Let $p \in \Pi$ and $\{x^1, ..., x^n\} \in [p]^n$. Since X is discrete, $\{x^i\} \in \mathcal{O}$ for $i = 1, ..., n$. Thus $\bigcap_{i=1}^n \{x^i\}^* = \{p\} \in \mathcal{O}_\Pi$, so Π is discrete. □

Topological n-partition is defined if $(X, \mathcal{O})$ is a Hausdorff space. So it holds.

Theorem 5. *Π is a Hausdorff space.*

Proof. Let $p, q \in \Pi$ and $p \neq q$. Let $\{x^1, ..., x^n\} \in [p]^n$, $x^{n+1} \in q - p$. X is a Hausdorff space, so there are local bases $\mathcal{B}(x^i)$, $i = 1, ..., n$, such that for $i \neq j$, and for $B_i \in \mathcal{B}(x^i)$, $B_j \in \mathcal{B}(x^j)$ it holds $B_i \cap B_j = \emptyset$. Let us suppose that for every $(B_1, ..., B_{n+1}) \in \Pi_{i=1}^{n+1} \mathcal{B}(x^i)$ there exists $p_{(B_1,...,B_{n+1})} \in \bigcap_{i=1}^{n+1} B_i^*$. Let $x^i_{(B_1,...,B_{n+1})} \in p_{(B_1,...,B_{n+1})} \cap B^i$, then as in Lemma 3.1 one can prove that $< x^i_{(B_1,...,B_{n+1})} > \to x^i$ for all $i = 1, ..., n+1$. Since $x^{n+1}_{(B_1,...,B_n)} \in p(x^1_{(B_1,...,B_n)}, ..., x^n_{(B_1,...,B_n)})$ and $\Pi \in TP_n(X)$, the condition (a) gives $x^{n+1} \in p(x^1, ..., x^n) = p$. A contradiction. Therefore there are $B_1, ..., B_{n+1}$ such that $(\bigcap_{i=1}^n B_i^*) \cap B_{n+1}^* = \emptyset$. These are disjoint neighborhoods of the points p and q. □

4. Topological planes as topological 2-partitions

In [4] Skornyakov investigated topological planes. In the further consideration the space of lines in a topological plane will be observed as a subset of the power set of the set of its points, and the incidence relation will be interpreted as the relation of membership ($\in$).

A pair (X, Π) where X (the set of points) and Π (the set of lines) are nonempty sets, $\Pi \subset P(X)$, and where (1) each line contains at least two points; (2) every two points are contained in some line; (3) two distinct lines can have at most one common point; (4_p) two distinct lines have at least one common point; is a projective plane.

If, instead of (4_p), holds (4_E) if the point x does not belong to the line p, there is a unique line q which contains x and which is disjoint from p; then (X, Π) is an Euclidean plane.

If X and Π are Hausdorff spaces and if (X, Π) is a projective plane such that it holds: $(\alpha)\forall x, y \in X, \quad \forall p \in \Pi[x \neq y \wedge x, y \in p \Rightarrow \forall O \in \mathcal{U}(p)\exists V \in \mathcal{U}(x)\exists W \in \mathcal{U}(y) V^* \cap W^* \subset O]$ and $(\beta)\forall p, q \in \Pi, \quad \forall x \in X[p \neq q \wedge x \in p \cap q \Rightarrow \forall O \in \mathcal{U}(x)\exists V \in \mathcal{U}(p)\exists W \in \mathcal{U}(q)(\bigcup V) \cap (\bigcup W) \subset O]$ then (X, Π) is a topological projective plane.

If (X, Π) is a Euclidean plane and, except of (α) and (β) it holds $(\gamma)\forall p, q \in \Pi \quad \forall x \in X \quad \forall U \in \mathcal{U}(p)[(p \cap q = \emptyset \vee p = q) \wedge x \in p] \Rightarrow [\exists V \in \mathcal{U}(x)\exists W \in \mathcal{U}(q) \quad \forall r(r \in V^* \wedge \exists s \in W \; r \cap s = \emptyset) \Rightarrow r \in U]$ then (X, Π) is a topological Euclidean plane.

Now we are able to prove:

Theorem 6. *If (X, Π) is a topological projective plane, then Π is a topological $2-$partition on X.*

Proof. The conditions (1), (2) and (3) are in fact the conditions (i), (ii) and (iii) in the definition of the $n-$partition, for $n = 2$. Let us prove that Π is a topological partition .

Let $x, y, z \in X$ be different points; $\mathcal{B}(x), \mathcal{B}(y)$ and $\mathcal{B}(z)$ their local bases and $< x_\sigma > \rightarrow x$, $< y_\sigma > \rightarrow y$.

(a) Let $< z_\sigma > \rightarrow z$ and $z_\sigma \in p(x_\sigma, y_\sigma)$ for $\sigma \in \Sigma$. According to $(\alpha)\mathcal{B}(p(x, y)) = \{B_x^* \cap B_y^* \mid B_x \in \mathcal{B}(x), B_y \in \mathcal{B}(y)\}$ and $\mathcal{B}(p(y, z)) =$

$\{B_y^* \cap B_z^* \mid B_y \in \mathcal{B}(y), B_z \in \mathcal{B}(z)\}$ are local bases for $p(x,y), p(y,z) \in \Pi$. If $B_x \in \mathcal{B}(x), B_y \in \mathcal{B}(y)$ and $B_z \in \mathcal{B}(z)$, from the convergence of the observed nets if follows that there are $\sigma_1, \sigma_2, \sigma_3 \in \Sigma$ such that for $\sigma \geq \sigma_1, x_\sigma \in B_x$; for $\sigma \geq \sigma_2, y_\sigma \in B_y$ and for $\sigma \geq \sigma_3, z_\sigma \in B_z$. Let $\sigma_{12} \geq \sigma_1, \sigma_2$ and $\sigma_{23} \geq \sigma_2, \sigma_3$. Now, for $\sigma \geq \sigma_{12}, p(x_\sigma, y_\sigma) \in B_x^* \cap B_y^*$ and for $\sigma \geq \sigma_{23}, p(y_\sigma, z_\sigma) \in B_y^* \cap B_z^*$. Since B_x, B_y and B_z are arbitrary we have $< p(x_\sigma, y_\sigma) > \rightarrow p(x,y)$ and $< p(y_\sigma, z_\sigma) > \rightarrow p(y,z)$. Finally $z_\sigma \in p(x_\sigma, y_\sigma)$ imply $p(x_\sigma, y_\sigma) = p(y_\sigma, z_\sigma)$ and, since Π is a Hausdorff space, it follows $p(x,y) = p(y,z)$, that is $z \in p(x,y)$.

(b) Let $z \in p(x,y)$ and $t \notin p(x,y)$. Now we define

$$z_\sigma = \begin{cases} z & \text{if } p(x_\sigma, y_\sigma) = p(t,z) \\ p(x_\sigma, y_\sigma) \cap p(t,z) & \text{otherwise.} \end{cases}$$

Then for all $\sigma \in \Sigma$, $z_\sigma \in p(x_\sigma, y_\sigma)$. Let $U \in \mathcal{U}(z)$. $\mathcal{B}(p(x,y))$ from (a) and $\mathcal{B}(p(t,z)) = \{B_t^* \cap B_z^* \mid B_t \in \mathcal{B}(t), B_z \in \mathcal{B}(z)\}$ are local bases at the points $p(x,y)$ and $p(t,z)$ of Π. According to (β) and since $z \in p(x,y) \cap p(t,z)$, there are $B_x \in \mathcal{B}(x), B_y \in \mathcal{B}(y), B_z \in \mathcal{B}(z)$ and $B_t \in \mathcal{B}(t)$ such that

$$\text{(2)} \qquad \forall y[(\exists r \in B_x^* \cap B_y^*, \quad \exists s \in B_z^* \cap B_t^*, \quad y \in r \cap s) \Rightarrow y \in U]$$

From the proof of (a) we have $< p(x_\sigma, y_\sigma) > \rightarrow p(x,y)$, so there is $\sigma_1 \in \Sigma$ such that for $\sigma \geq \sigma_1, p(x_\sigma, y_\sigma) \in B_x^* \cap B_y^*$. Also $p(t,z) \in B_t^* \cap B_z^*$, thus according to (2) we have that for each $\sigma \geq \sigma_1$ for which $p(x_\sigma, y_\sigma) \neq p(t,z)$ it holds $p(x_\sigma, y_\sigma) \cap p(t,z) \subset U$, that is $z_\sigma \in U$. Clearly, if $p(x_\sigma, y_\sigma) = p(t,z)$, then $z_\sigma = z \in U$. Since $U \in \mathcal{U}(z)$ is arbitrary we have $< z_\sigma > \rightarrow z$ □

Theorem 7. *If* (X, Π) *is a topological Euclidean plane, then* Π *is a topological* 2*−partition on* X.

Proof. As in the previous theorem Π is a 2-partition on X which satisfies (a).

(b) Let $z \in p(x,y)$ and $t \notin p(x,y)$. According to [4] Lemma 5, since $p(x,y) \neq p(t,z)$, these lines have totaly disjoint neighborhoods. Thus there are $B_x \in \mathcal{B}(x)$ and $B_y \in \mathcal{B}(y)$ such that for each $q \in B_x^* \cap B_y^*$ we have $q \cap p(t,z) \neq \emptyset$. As in the Theorem 4.1, $< p(x_\sigma, y_\sigma) > \rightarrow p(x,y)$ and there is $\sigma_1 \in \Sigma$ such that for $\sigma \geq \sigma_1, p(x_\sigma, y_\sigma) \in B_x^* \cap B_y^*$. Then $p(x_\sigma, y_\sigma)$ and $p(t,z)$ intersect and for z_σ defined as in the Theorem 4.1, because of (β) we have $< z_\sigma > \rightarrow z$. □

5. Other examples

There are some ways to make a new topological $n-$ partition of a given space from the old one. Here we mention two of them.

Theorem 8. *Let X and Y be Hausdorff spaces, $f : X \to Y$ a function, $\Pi_X \in TP_n(X)$ and $\Pi_Y = \{f(p) \mid p \in \Pi_X\}$. Then*

(i) If f is a bijection, then Π_Y is an $n-$partition on Y and for distinct points $x^1, ..., x^n \in X$ and $y^1, ..., y^n \in Y$ it holds:

$$f(p(x^1, ..., x^n)) = p(f(x^1), ..., f(x^n)),$$

$$f^{-1}(p(y^1, ..., y^n)) == p(f^{-1}(y^1), ..., f^{-1}(y^n)).$$

(ii) If f is a homeomorphism then $\Pi_Y \in TP_n(Y)$ and Π_X and Π_Y are homeomorphic spaces.

Proof. (i) The proof is completely straightforward.

(ii) Firstly, we will prove that $\Pi_Y \in TP_n(Y)$. Let $y^1, ..., y^n, t \in Y$ be distinct points and let $< y^i_\sigma > \to y^i$ for $i = 1, ..., n$. Then $x^i = f^{-1}(y^i), i = 1, ..., n$ and $z = f^{-1}(t)$ are distinct elements of X and, if $x^i_\sigma = f^{-1}(y^i_\sigma)$, we have $< x^i_\sigma > \to x^i$, $i = 1, ..., n$, (because f^{-1} is continuous).

(a) Let $< t_\sigma > \to t$ and $t_\sigma \in p(y^1_\sigma, ..., y^n_\sigma), \sigma \in \Sigma$. If $z_\sigma = f^{-1}(t_\sigma)$, continuity of f^{-1} gives $< z_\sigma > \to z$. Also, according to (i), $z_\sigma \in p(x^1_\sigma, ..., x^n_\sigma)$ for $\sigma \in \Sigma$. Since Π_X is a topological $n-$partition, we have $z = p(x^1, ..., x^n)$. Now (i) gives $t \in p(y^1, ..., y^n)$.

(b) If $t \in p(y^1, ..., y^n)$, then $z \in p(x^1, ..., x^n)$. Now, there is a net $< z_\sigma > \to z$ such that $z_\sigma \in p(x^1_\sigma, ..., x^n_\sigma)$, $\sigma \in \Sigma$. Since f is continuous, $< t_\sigma > \to t$, where $t_\sigma = f(z_\sigma)$. Also , by (i), $t_\sigma \in p(y^1_\sigma, ..., y^n_\sigma)$ for all $\sigma \in \Sigma$.

Let us prove $\Pi_Y \cong \Pi_X$. We define $\Psi : \Pi_X \to \Pi_Y$, where $\Psi(p) = f(p)$ for all $p \in \Pi_X$. Ψ is obviously a bijection, since f is. Continuity of Ψ and Ψ^{-1} follows from the relation

$$\Psi^{-1}(\bigcap_{i=1}^{n} B_i^*) = \bigcap_{i=1}^{n}(f^{-1}(B_i))^*$$

where $B_1, ..., B_n$ are basic open sets in Y, continuity of f and f^{-1} and the Theorem 3.1. (i) □

In the proof of the following statement we will use the next well-known fact.

Lemma 2. *Let A be a subspace of the space X, $< a_\sigma \mid \sigma \in \Sigma >$ a net in A and $a \in A$. Then $< a_\sigma > \rightarrow_X a$ iff $< a_\sigma > \rightarrow_A a$.*

Theorem 9. *If X is a Hausdorff space, $A \subset X$, where $\mid A \mid \geq n$ and $\Pi_X \in TP_n(X)$, then*

(i) $\Pi_A = \{p \cap A \mid p \in \Pi_X$ and $\mid p \cap A \mid \geq n\}$ is an $n-$partition of A.

(ii) If A is open, then $\Pi_A \in TP_n(A)$.

Proof. (i) Obviously.

(ii) Let $x^1, ..., x^n, z \in A$ be distinct points and $< x^i_\sigma \mid \sigma \in \Sigma >$ nets in A where $< x^i_\sigma > \rightarrow_A x^i$ for $i = 1, ..., n$.

(a) Let $< z_\sigma >$ be a net in A such that $< z_\sigma > \rightarrow_A z$ and $z_\sigma \in p_A(x^1_\sigma, ..., x^n_\sigma) = p(x^1_\sigma, ..., x^n_\sigma) \cap A$, $\sigma \in \Sigma$. According to the last lemma then $< x^i_\sigma > \rightarrow_X x^i$, $i = 1, ..., n$ and $< z_\sigma > \rightarrow_X z$. Since $z_\sigma \in p(x^1_\sigma, ..., x^n_\sigma)$ and $\Pi_X \in TP_n(X)$ we have $z \in p(x^1, ..., x^n)$. But $z \in A$, so $z \in p(x^1, ..., x^n) \cap A = p_A(x^1, ..., x^n)$.

(b) Suppose that $z \in p_A(x^1, ..., x^n)$. Then $z \in p(x^1, ..., x^n)$ and because of that, there is a net $< y_\sigma >$ in X such that $< y_\sigma > \rightarrow_X z$ and $y_\sigma \in p(x^1_\sigma, ..., x^n_\sigma)$, $\sigma \in \Sigma$. A is open, so there is $\sigma_1 \in \Sigma$ such that for $\sigma \geq \sigma_1$, $y_\sigma \in A$. We define

$$z_\sigma = \begin{cases} y_\sigma & \text{for } \sigma \geq \sigma_1 \\ x^1_\sigma & \text{otherwise} \end{cases}$$

Now, $z_\sigma \in A$ for all $\sigma \in \Sigma$ and $z_\sigma \in p_A(x^1_\sigma, ..., x^n_\sigma)$. Also $< z_\sigma > \rightarrow_X z$, and by Lemma 5.1, $< z_\sigma > \rightarrow_A z$. □

Example 1. Let $(X, \mathcal{O})$ be a Hausdorff space, where $\mid X \mid \geq n$. Then $[X]^n$ is an $n-$ partition of X. Trivially, it is a topological $n-$ partition (since $z \in p(x^1, ..., x^n)$ would imply $\mid p(x^1, ..., x^n) \mid > n$).

Example 2. Let $(X, \| \, \|)$ be a normed vector space. It is easy to check that

$$\Pi = \{p(x, y) \mid x, y \in X, \ \ x \neq y\}$$

where $p(x,y) = \{\lambda x + (1-\lambda)y \mid \lambda \in R\}$ is an one-dimensional manifold, is a 2-partition on X. For a proof that $\Pi \in TP_n(X)$ we will use Theorem 2.2. and

Lemma 3. *Let $a_n \in X$, $n \in N$; $a, b \in X$ and $\lambda_n \in R$ for $n \in N$. If $<\lambda_n a_n> \to b$ and $<a_n> \to a \neq 0$, then $<\lambda_n>$ is a convergent sequence.*

Proof. Since the norm is continuous we have $<|\lambda_n| \, \|a_n\|> \to \|b\|$ and $<\|a_n\|> \to \|a\|$. Also $\|a\| \neq 0$, and it follows:

$$<|\lambda_n|> = <\frac{|\lambda_n| \, \|a_n\|}{\|a_n\|}> \to \frac{\|b\|}{\|a\|} = \mu.$$

For $\mu = 0$ the proof is over. If $\mu > 0$, suppose that $<\lambda_n>$ does not converge. Then there are subsequences $<\lambda_{n_k} a_{n_k}> \to -\mu a$ and $<\lambda_{n_m} a_{n_m}> \to \mu a$. A contradiction . □

Let $x, y, z \in X$ be distinct points, $<x_n> \to x$ and $<y_n> \to y$.

(a) Let $<z_n> \to z$ and $z_n \in p(x_n, y_n)$, $n \in N$. Then $\lambda_n(y_n - x_n) = z_n - x_n$. By Lemma 5.2, since $<\lambda_n(y_n - x_n)> \to z - x$ and $<y_n - x_n> \to y - x \neq 0$, $<\lambda_n>$ converges. If $<\lambda_n> \to \lambda$, then $<z_n> \to x + \lambda(y - x)$, that is $z \in p(x, y)$.

(b) Let $z \in p(x,y)$. Then for some $\lambda \in R$, $z = x + \lambda(y - x)$. If we define $z_n = x_n + \lambda(y_n - x_n)$, then $z_n \in p(x_n, y_n)$ for $n \in N$ and $<z_n> \to z$.

Example 3. If in the previous example $X = R^2$, then we have the usual Euclidean plane. By Theorem 5.1, for each automorphism of R^2 we get a topological 2-partition of R^2. For example the family of curves $cy = (ax+b)^3$, where $a^2 + c^2 > 0$, which consists of cubic paraboles and horizontal and vertical lines is such a partition.

Also, for every open set $G \subset R^2$, according to Theorem 5.2, Π_G is a topological 2-partition.

Example 4. Let (R^2, Π) be the Euclidean plane from the last example and $A = [0,1]^2 \cup ([1,2]) \times \{0\})$. Then Π_A is a 2-partition of A, but it is not topological although A is compact, connected etc. (The condition (b) is not satisfied).

Example 5. Let X be the complex plane (or R^2, because these two spaces are homeomorphic). For distinct points $x, y, z \in X$ there is the unique cirsle

$\mathcal{K}(x,y,z) = \mathcal{K}(c,r) = \{c + re^{i\varphi} \mid \varphi \in [0,2\pi]\}$, where the center c and the radius r are given by

$$c(x,y,z) = \frac{|x|^2(y-z) + |y|^2(z-x) + |z|^2(x-y)}{2iIm\{\overline{(y-x)}(z-x)\}},$$

$$r(x,y,z) = \frac{|x-y||y-z||z-x|}{2|Im\{\overline{(y-x)}(z-x)\}|}$$

if x, y and z are not colinear; or there is the unique line $\ell(x,y) = \{x + \lambda(y-x) \mid \lambda \in R\}$, if these points are colinear. Thus the set of all circles and lines is a 3-partition of the plane. It is also easy to verify that the points x, y, z are colinear iff $Im\{\overline{(y-x)}(z-x)\} = 0$.

Let us show that this partition is topological. Since **C** is a first countable space, we can apply Theorem 2.2. Let x, y, z, t be distinct points, and $< x_n > \to x$, $< y_n > \to y$ and $< z_n > \to z$.

Lemma 4. *If $p(x_n, y_n, z_n)$ is a circle for all $n \in N$, then*

(a') if $< t_n > \to t$ and $t_n \in p(x_n, y_n, z_n)$, $n \in N$, then $t \in p(x,y,z)$

(b') if $t \in p(x,y,z)$, then there is a sequence $< t_n >$ such that $< t_n > \to t$ and $t_n \in p(x_n, y_n z_n)$, $n \in N$.

Proof. Assume $p(x_n, y_n, z_n) = \mathcal{K}(c_n, r_n)$, where $c_n = c(x_n, y_n, z_n)$ and $r_n = r(x_n, y_n, z_n)$, $n \in N$. We divide the proof in two parts:

1^0 x, y and z are not colinear.

Then $p(x,y,z) = \mathcal{K}(c,r)$, where $c = c(x,y,z)$, $r = r(x,y,z)$. Since $< (x_n, y_n, z_n) > \to (x,y,z)$, in C^3 and $c(x,y,z)$, $r(x,y,z)$ are continuous functions, we have $< c_n > \to c$ and $< r_n > \to r < \infty$.

(a') Let $< t_n > \to t$ and $t_n \in p(x_n, y_n z_n)$. Then $t_n = c_n + r_n e^{i\varphi_n}$ that is $|t_n - c_n| = r_n$ for all $n \in N$. By continuity of $|\,.\,|$ we have $|t - c| = r$, that is $t \in \mathcal{K}(c,r) = p(x,y,z)$.

(b') If $t \in p(x,y,z)$, then $t = c + re^{i\varphi_t}$. If we define $t_n = c_n + r_n e^{i\varphi_t}$, then $t_n \in p(x_n, y_n z_n)$ and $< t_n > \to t$.

2^0 x, y and z are colinear

Then $Im\{\overline{(y-x)}(z-x)\} = 0$ and $p(x,y,z) = \ell(x,y)$. Since

$$<|x_n - y_n||y_n - z_n||z_n - x_n|> \to |x-y||y-z||z-x|$$

we have $< r_n > \to \infty$.

(a') Let $< t_n > \to t$ and $t_n \in p(x_n, y_n, z_n)$. Suppose that $t \notin \ell(x,y)$. Then x, y and t are not colinear. Since $< x_n > \to x$, $< y_n > \to y$ and $< t_n > \to t$, where $x_n, y_n, t_n \in \mathcal{K}(c_n, r_n)$, by 1^0 we have $< r_n > \to r < \infty$. A contradiction. Thus $t \in \ell(x,y) = p(x,y,z)$.

(b') Let $t \in p(x,y,z)$. Then $t = x + \lambda(y - x)$ for some $\lambda \in R$. If $x_n = c_n + r_n e^{i\varphi_n^x}$ and $y_n = c_n + r_n e^{i\varphi_n^y}$, $n \in N$ we define

$$t_n = c_n + r_n \frac{1 + i\lambda \frac{|y_n - x_n|}{r_n}}{\mid 1 + i\lambda \frac{|y_n - x_n|}{r_n} \mid} e^{i\varphi_n^x}, \quad n \in N.$$

Now, since $\mid t_n - c_n \mid = r_n$ it holds $t_n \in p(x_n, y_n, z_n)$, $n \in N$. Since

$$t_n = x_n + [(\frac{1 + i\lambda \frac{|y_n - x_n|}{r_n}}{\mid 1 + i\lambda \frac{|y_n - x_n|}{r_n} \mid} - 1)\frac{r_n}{\mid y_n - x_n \mid}] \mid y_n - x_n \mid e^{i\varphi_n^x}$$

and since $\frac{|y_n - x_n|}{r_n} \to 0$, it can be shown that the part in the brackets tends to $i\lambda$ and that $\mid y_n - x_n \mid e^{i\varphi_n^x} \to \frac{y-x}{i}$, thus $< t_n > \to x + \lambda(y - x) = t$. □

Lemma 5. *If $p(x_n, y_n, z_n)$ is a line for all $n \in N$, then (a') and (b') from the previous lemma hold.*

Proof. Since $< x_n > \to x$, $< y_n > \to y$, $< z_n > \to z$ and $z_n \in \ell(x_n, y_n)$, according to Example 2 we have $z \in \ell(x,y)$ and $p(x,y,z)$ is a line. Now, it holds the conclusion from Example 2. □

Assume that in the family $\{p(x_n, y_n, z_n) \mid n \in N\}$ there are both lines and circles.

Let $p(x,y,z)$ be a circle. Suppose that infinitely many of $p(x_n, y_n, z_n)$ are lines. Then they make a convergent subsequence, and according to the last lemma $p(x,y,z)$ is a line. A contradiction. Thus there is at most finitely many lines and we can apply Lemma 5.3.

Let $p(x,y,z)$ be a line. For the discussion is interesting the case when $p(x_n, y_n, z_n)$ are lines for $n \in N'$, and circles for $n \in N''$, where $N = N' \cup N''$ and $\mid N' \mid, \mid N'' \mid \geq \aleph_0$. But application of Lemma 5.3. and Lemma 5.4. to these subsequences gives the desired conclusion.

References

[1] Engelking, R., General Topology. Warszawa: PWN - Polish Scientific Publishers 1977.

[2] Hartmanis, J., Generalized partitions and lattice embedding theorems. In: Proc. of Symposia in Pure Mathematics, Vol II, Lattice theory. pp. 22-30. Amer. Math. Soc. 1961.

[3] Picket, H.E., A note on generalized equivalence relations. Math. Notes, 860-861 (1966).

[4] Skornyakov, L.A., Topological projective planes. Trud. Mosk. Mat. Obc. 3, 347-373 (1954).

REZIME

$n-$ PARTICIJE TOPOLOŠKIH PROSTORA

$n-$particija skupa X je familija $\prod$ podskupova X, takva da je svakih n tačaka skupa X sadržano u tačno jednom elementu particije i svaki element particije sadrži najmanje n tačaka iz X. Ako je $(X, \mathcal{O})$ Hausdorffov prostor a $\prod$ zadovoljava odredjene topološke uslove, na $\prod$ je definisana topologija. Ispitane su osnovne kardinalne funkcije prostora $(\prod, \mathcal{O}_\Pi)$. Pokazano je da su topološke projektivne ravni kao i topološke Euklidske ravni specijalne topološke 2-particije. Date su neke konstrukcije novih od starih topoloških $n-$particija kao i primeri ovakvih particija u normiranom linearnom prostoru.

Received by the editors August 12, 1992