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g - CALCULUS

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Abstract

This paper is restricted to the strict pseudo - addition $\oplus$ and the corresponding generator g. g -calculus is developed in a similar way as was done for the usual calculus. The obtained results are applied to some nonlinear differential equations.

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1. Introduction

The triangular conorm was elaborated by Schweizer and Sklar [13] as an important tool in probabilistic metric spaces. The notion of t-conorm decomposable measures was introduced by Dubois and Prade [3] and extensively investigated by many mathematicians among others by S.Weber [14] and [15], E.Pap [6] - [12], L.D'Appuzzo,M.Squillante, A.G.S.Ventre [1]. In this paper we shall move from the range-interval [0,1] where t-conorms act to an arbitrary interval [a,b] (in some cases a semiclosed interval) contained in $[-\infty, \infty]$ ([4],[11],[12]) under the name pseudo - addition. In this paper we restrict to strict pseudo - addition $\oplus$, which by Aczel's theorem ([1],[5]) itselves has a strictly monotone generator g. This enables us to develop a

calculus (g - derivative and g - integral),a so called g - calculus, in a similar way as for the usual calculus. We apply the obtained results to some simple nonlinear differential equations.
In subsequent papers we will apply the g - calculus to the wider class of nonlinear differential equations.

2. Pseudo-addition and pseudo-multiplication

We shall work in this paper with the extended real number system,i.e. the ordinary finite real numbers together with - ∞ and ∞ .Let [a,b] be a closed (in some cases semiclosed, see Remark 1) real interval. The operation $\oplus$ (pseudo-addition) is a function $\oplus : [a,b] \times [a,b] \to [a,b]$ which is commutative, nondecreasing, associative and has a zero element, denoted by **0** which is either a or b.In this paper we shall consider only strict pseudo - addition, i.e. such that the function $\oplus$ is continuous and strictly increasing in $(a,b) \times (a,b)$.

Remark 1. The requirement that the interval [a,b] be closed does not restrict us in any way.If we begin with (a,b] or (a,b) we can replace the condition $\mathbf{0} \oplus x = x$ by the condition: for all y in (a,b] or in (a,b)

$$\lim_{x \to a} \oplus(x,y) = y, \qquad \lim_{x \to a} \oplus(x,x) = a.$$

Adjoining the endpoint a to (a,b], or both endpoints a and b to (a,b), we reduce, by continuity of $\oplus$, to the previous case.
By Aczel's theorem for each strict pseudo - addition $\oplus$ there exists a monotone function g (generator for $\oplus$) , $g : [a,b] \to [0,\infty]$ such that either $g(a) = 0$ or $g(b) = 0$ and

$$u \oplus v = g^{-1}(g(u) + g(v)).$$

Remark 2. Hence g is an isomorphism of semigroup $([a,b],\oplus)$ with the semigroup $([0,\infty],+)$.
The operation $\otimes$ (pseudo-multiplication) is a function $\otimes : [a,b] \times [a,b] \to [a,b]$ which is commutative, nondecreasing, associative and has an unit element **1** . We suppose further that the pseudo - multiplication $\otimes$ is defined as

$$u \otimes v = g^{-1}(g(u)g(v)),$$

where g is a generator of strict pseudo - addition $\oplus$. $\otimes$ is distributive with respect to $\oplus$. Examples can be found in [4],[11] and [12].

3. g - derivative

Let g be the additive generator of the strict pseudo - addition$\oplus$ on [a,b] such that g is continuously differentiable on (a,b). The corresponding pseudo - multiplication $\otimes$ will be always defined as

$$u \otimes v = g^{-1}(g(u)g(v)).$$

Let f and h be two functions defined on the interval [c,d] and with values in [a,b]. Then, we define for any $x \in [c,d]$

$$(f \oplus h)(x) = f(x) \oplus h(x),$$

$$(f \otimes h)(x) = f(x) \otimes h(x)$$

and for any $\lambda \in [a,b]$

$$(\lambda \otimes f)(x) = \lambda \otimes f(x).$$

If the function f is differentiable on (c,d) and has the same monotonicity as the function g , then we define the g - derivative of f at the point $x \in (c,d)$ as

$$\frac{d^{\oplus} f(x)}{dx} := g^{-1}(\frac{d}{dx} g(f(x))),$$

when the right part is meaningfull.
Using the representations of the operations $\oplus$ and $\otimes$ by g and the preceding definition we have

Theorem 1. *Let f_1 and f_2 be two g - differentiable functions on (c,d) and with the values in [a,b]. Then we have for $\lambda \in [a,b]$*

$$(i) \qquad \frac{d^{\oplus}(f_1 \oplus f_2)}{dx} = \frac{d^{\oplus} f_1}{dx} \oplus \frac{d^{\oplus} f_2}{dx};$$

$$(ii) \qquad \frac{d^{\oplus}(\lambda \otimes f_1)}{dx} = \lambda \otimes \frac{d^{\oplus} f}{dx};$$

$$(iii) \qquad \frac{d^{\oplus} \lambda}{dx} = \mathbf{0}.$$

Example 1. Taking $g(u) = e^{-u}$ we obtain

$$u \oplus v = -ln(e^{-u} + e^{-v})$$

and

$$u \otimes v = u + v.$$

g - derivative is given as

$$\frac{d^{\oplus} f(x)}{dx} = -ln(\frac{d}{dx} e^{-f(x)}) = f(x) - ln(-f'(x))$$

for $f'(x) < 0$.

Example 2. Let

$$g(u) = ln\frac{1+u}{1-u}.$$

Then we have

$$g^{-1}(u) = \frac{e^u - 1}{e^u + 1}$$

and the g - derivative is given by

$$\frac{d^{\oplus} f(x)}{dx} = \frac{\exp \frac{2f'(x)}{1-f^2(x)} - 1}{\exp \frac{2f'(x)}{1-f^2(x)} + 1}.$$

Example 3. Let $g(u) = \frac{u}{1-u}$.Then we have

$$g^{-1}(u) = \frac{u}{1+u}$$

and

$$u \oplus v = \frac{u + v - 2uv}{1 - uv}.$$

The g - derivative is given by

$$\frac{d^{\oplus} f(x)}{dx} = \frac{f'(x)}{(1 - f(x))^2 + f'(x)}.$$

Example 4. Let $g(u) = -ln(1 - u)$. Then we have $g^{-1}(u) = 1 - e^{-u}$ and

$$u \oplus v = u + v - uv.$$

The g - derivative is given by

$$\frac{d^{\oplus} f(x)}{dx} = 1 - \exp \frac{f'(x)}{1 - f(x)}.$$

The usual derivative could have an unusual behavior with respect to the pseudo - product. For example, if $\otimes$ is from Example 1 , then we have

$$(f_1 \otimes f_2)' = f_1' \otimes f_2'.$$

In general case we have

Theorem 2. *Let f_1 and f_2 be two functions defined on $[c,d]$ and with values in $[a,b]$. If both functions are differentiable, then we have*

$$\frac{d^{\oplus}(f_1 \otimes f_2)}{dx} = (\frac{d^{\oplus} f_1}{dx} \otimes f_2) \oplus (f_1 \otimes \frac{d^{\oplus} f_2}{dx}).$$

Proof. From one side we obtain

$$\frac{d^{\oplus}(f_1 \otimes f_2)}{dx} = g^{-1}(\frac{d}{dx} g(f_1 \otimes f_2)) = g^{-1}(\frac{d}{dx} g(g^{-1}(g(f_1)g(f_2))) =$$
$$g^{-1}(g'(f_1)f_1' g(f_2) + g(f_1)g'(f_2)f_2').$$

On the other side we have

$$((\frac{d^{\oplus} f_1}{dx}) \otimes f_2) \oplus (f_1 \otimes \frac{d^{\oplus} f_2}{dx}) =$$
$$g^{-1}(g(\frac{d^{\oplus} f_1}{dx} \otimes f_2) + g(f_1 \otimes \frac{d^{\oplus} f_2}{dx})) =$$
$$g^{-1}(g(g^{-1}(g(\frac{d^{\oplus} f_1}{dx})g(f_2))) + g(g^{-1}(g(f_1)g(\frac{d^{\oplus} f_2}{dx})))) =$$
$$g^{-1}(g(g^{-1}(\frac{dg(f_1)}{dx}))g(f_2) + g(f_1)g(g^{-1}(\frac{dg(f_2)}{dx}))) =$$
$$g^{-1}(g'(f_1)f_1' g(f_2) + g(f_1)g'(f_2)f_2').$$

If there exists the (n-1) - g -derivative of a function $f : [c,d] \to [a,b]$, then we define n - g - derivative of f (if it exists)

$$\frac{d^{(n)\oplus} f}{dx^n} := \frac{d^{\oplus}}{dx}(\frac{d^{(n-1)\oplus} f}{dx^{n-1}}).$$

Theorem 3. *If there exists an n -g - derivative of f, then we have*

$$\frac{d^{(n)\oplus} f}{dx^n} = g^{-1}(\frac{d^n}{dx^n} g(f)).$$

Proof. We shall prove the theorem by induction. For n=1, with the convention

$$\frac{d^{(0)\oplus} f}{dx} = f,$$

the equality is obvious. Suppose that the theorem is true for n - 1, i.e.

$$\frac{d^{(n-1)\oplus} f}{dx^{n-1}} = g^{-1}(\frac{d^{n-1}}{dx^{n-1}} g(f)).$$

Then we have

$$\frac{d^{(n)\oplus} f}{dx^n} = \frac{d^{\oplus}}{dx}(\frac{d^{(n-1)\oplus} f}{dx^{n-1}}) =$$

$$\frac{d^{\oplus}}{dx}(g^{-1}(\frac{d^{(n-1)}}{dx^{n-1}} g(f))) = g^{-1}(\frac{d}{dx} g(g^{-1}(\frac{d^{(n-1)}}{dx^{n-1}} g(f)))) =$$

$$g^{-1}(\frac{d^{(n)}}{dx^n} g(f)).$$

Example 5. For pseudo - addition $\oplus$ as in Example 3 we have

$$\frac{d^{(2)\oplus} f}{dx^2} = \frac{f" - f"f + 2f'^2}{(1-f)^3 + f" - f"f + 2f'^2}.$$

We shall use in the next theorem the following notation:
the value of function f at the point x_0 will be denoted by $f[x_0]$. With this notation $\frac{d^{\oplus} f[x_0]}{dx}$ means the value of g - derivative of f at the point x_0. By this convention we have

$$\frac{d^{\oplus}[x_0]}{dx} = \frac{d^{\oplus} i[x_0]}{dx},$$

where $i(x) = x$.

Theorem 4. *Suppose that h is a function on* $[a, b] \subset [0, \infty]$ *and* $f : [c, d] \to [a, b]$ *and*

$$F(x) = h(f(x)).$$

Suppose that f has derivative at $x_0 \in (c,d)$ and that h has a derivative at $f(x_0)$. Then $\frac{d^{\oplus}F[x_0]}{dx}$ exists and

$$\frac{d^{\oplus}F[x_0]}{dx} = \frac{d^{\oplus}[h(f(x_o))]}{dx} \otimes \frac{d^{\oplus}}{dx} g^{-1}(h[f(x_0)]) \otimes \frac{d^{\oplus} g^{-1}(f[x_0])}{dx}.$$

Proof. We have

$$\frac{d^{\oplus}F[x_0]}{dx} = g^{-1}(\frac{d}{dx}g(F[x_0])) = g^{-1}(\frac{d}{dx}g[h(f(x_0))]\frac{d}{dx}h[f(x_0)]\frac{d}{dx}f[x_0]) =$$

$$\frac{d^{\oplus}[h(f(x_0))]}{dx} \otimes \frac{d^{\oplus}}{dx} g^{-1}(h[f(x_0)]) \otimes \frac{d^{\oplus}}{dx} g^{-1}(f[x_0]),$$

where we have used the equality

$$\frac{d}{dx}s[x] = \frac{d}{dx}g(g^{-1}(s[x])).$$

Remark 3. For $g(x) = x$,using the fact $g'(x) = 1$ and hence $g'[h(f(x_0))] = 1$,we obtain the usual Chain rule.

4. g - integral

For any measurable function $f : [c,d] \to [a,b]$ we define

$$\int_{[c,d]}^{\oplus} f dx := g^{-1}(\int_c^d g(f)dx).$$

Remark 4. Let $\mathcal{F}$ be a family of subsets of X. A set function $m : \mathcal{F} \to [a,b]$ is a $\oplus$ -decomposable measure if

$$m(A \cup B) = m(A) + m(B)$$

holds for $A, B \in \mathcal{F}$ such that $A \cap B = \emptyset$ and $A \cup B \in \mathcal{F}$.
We have investigated the Saks type decomposition and regular Borel $\oplus$ -decomposable measures in [11]. We have introduced in [12] an integral with respect to a decomposable measure.In particular, if $\oplus$ is a strict pseudo-addition with a monotone generator g we have for measurable function f and $\sigma - \oplus$ -decomposable measure m

$$\int^{\oplus} f \otimes dm = g^{-1}\Big(\int (g \circ f)dx\Big).$$

We have investigated also the extensions of decomposable measures.

We obtain by Theorem 4.2 from [4]

Theorem 5. *We have*

$$(i) \qquad \int_{[c,d]}^{\oplus} (f_1 \oplus f_2)dx = \int_{[c,d]}^{\oplus} f_1 dx \oplus \int_{[c,d]}^{\oplus} f_2 dx;$$

$$(ii) \qquad \int_{[c,d]}^{\oplus} (\lambda \otimes f)dx = \lambda \otimes \int_{[c,d]}^{\oplus} f dx;$$

$$(iii) \qquad f_1 \leq f_2 \Longrightarrow \int_{[c,d]}^{\oplus} f_1 dx \leq \int_{[c,d]}^{\oplus} f_2 dx;$$

$$(iv) \qquad \int_{[c,d]\cup[e,f]}^{\oplus} f dx = \int_{[c,d]}^{\oplus} f dx \oplus \int_{[e,f]}^{\oplus} f dx.$$

Example 6. For $\oplus$ and $\otimes$ as in Example 1 we have

$$\int_{[0,x]}^{\oplus} x dx = -ln\Big(\int_0^x e^{-x} dx\Big) = x$$

and

$$\int_{[1,x]}^{\oplus} \ln x dx = -\ln \ln x.$$

Example 7. For $\oplus$ and $\otimes$ as in Example 3 we have

$$\int^{\oplus} x dx = g^{-1}\Big(\int g(x)dx\Big) = g^{-1}\Big(\int \frac{x}{1-x} dx\Big) = \frac{x + \ln(1-x)}{x + \ln(1-x) - 1} \oplus C.$$

Example 8. Let $g(x) = x^p$ for $p > 0$.Then we have

$$\int_{[0,x]}^{\oplus} x dx = \Big(\int_0^x x^p dx\Big)^{1/p} = p^{1/p} x^{1/q},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 6. *Suppose that f is continuous on $[c, d]$. Then we have*

$$\frac{d^{\oplus}}{dx}\left(\int_{[c,x]}^{\oplus} f dx\right) = f(x)$$

for each $x \in (c, d)$.

Proof. We have for $x \in (c, d)$

$$\frac{d^{\oplus}}{dx}\left(\int_{[c,x]}^{\oplus} f(x)dx\right) = \frac{d^{\oplus}}{dx}\left(g^{-1}\left(\int_{c}^{x} g((f(x))dx\right)\right) =$$

$$g^{-1}\left(\frac{d}{dx}g\left(g^{-1}\left(\int_{c}^{x} g(f(x))dx\right)\right)\right) = g^{-1}\left(\frac{d}{dx}\int_{c}^{x} g(f(x))dx\right) = f(x),$$

where we have used the fundamental theorem of the usual calculus.

Theorem 7. *Suppose that f has continuous g-derivative on (c,d). Then we have*

$$\int_{[c,x]}^{\oplus} \frac{d^{\oplus} f}{dx} dx \oplus f(c) = f(x)$$

for $x \in (c, d)$.

5. Applications

We shall apply the g - derivative and g - integral on nonlinear differential equations.

Example 9. Let $\oplus$ and $\otimes$ be as in Example 1. Then we can consider the simple g - differential equation

$$\frac{d^{\oplus} y}{dx} = -x, i.e.$$

$$y - \ln(-y') = -x$$

for $y' < 0$. We can obtain in a simple way the general solution of this equation, only applying on both sides the corresponding g - integral (i.e. for $g(u) = e^{-u}$).Then we obtain

$$\int^{\oplus} \frac{d^{\oplus} y}{dx} dx = \int^{\oplus} (-x) dx$$

and so by Theorem 7 we have

$$y = -x \oplus C_1 = -\ln(e^x + e^{-C_1}) = -\ln(e^x + C),$$

where $C = e^{-C_1}$.
In the general case we have

Theorem 8. *Suppose that $f : [c,d] \times [a,b] \to [a,b]$ is continuous, that ψ is defined and continuous on $J = \{x : x_0 - h < x < x_0 + h\} \subset [c,d]$ with values in $[a,b]$, and that $(x_0, y_0) \in [c,d] \times [a,b]$ with $\psi(x_0) = y_0$. Then the necessary and sufficient condition that ψ be a solution of*

$$\text{(1)} \qquad \frac{d^{\oplus}\psi}{dx} = f(x, \psi(x))$$

on J is that ψ satisfies the g - integral equation

$$\text{(2)} \qquad \psi(x) = y_0 \oplus \int_{[x_0,x]}^{\oplus} f(t, \psi(t))dt$$

for $x \in J$.

Proof. We apply the g-integral for $[x_0, x]$ on both sides of (1) and pseudo-adding $\psi(x_0)$ to the both sides we obtain

$$\int_{[x_0,x]}^{\oplus} \frac{d^{\oplus}\psi}{dx} dx \oplus \psi(x_0) = \int_{[x_0,x]}^{\oplus} f(t, \psi(t))dt \oplus \psi(x_0).$$

Hence, by Theorem 7, relation (2).
By verification we obtain that ψ given by (2) satisfies (1) and the initial condition $\psi(x_0)$.
We consider Burgers equation (see [11])

$$\frac{\partial u}{\partial t} + 1/2\left(\frac{\partial u}{\partial x}\right)^2 - c/2\frac{\partial^2 u}{\partial x^2} = 0$$

for $x \in \mathbf{R}$ and $t > 0$ with the initial condition $u(x,0) = u_0(x)$, where c is the given positive constant. If we introduce the pseudo-addition

$$u \oplus v := -c\ln(e^{-u/c} + e^{-v/c})$$

with the generator $g(u) = e^{-u/c}$ and the distributive pseudo- multiplication

$$u \otimes v := u + v,$$

then for solutions u_1 and u_2 the function $(\lambda_1 \otimes u_1) \oplus (\lambda_2 \otimes u_2)$ is also a solution of Burgers equation.
The solution of the given initial problem is

$$u(x,t) = \frac{c}{2}\ln(2\pi ct) \otimes \int^{\oplus} [\frac{(x-s)^2}{2t}] \otimes u_0(s)ds,$$

where the integral is given by

$$\int^{\oplus} f(x)dx := -c\ln(\int e^{-f(x)/c}dx).$$

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REZIME

g - RAČUN

U radu se pomoću aditivnog generatora g za striktno pseudo- sabiranje $\oplus$ razvija tzv. g - diferencijalni i g - integralni račun. Dobijeni rezultati se primenjuju na rešavanje nelinearnih diferencijalnih jednačina.

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