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FILTERS IN A WEAK-CONGRUENCE LATTICE

Branimir Šešelja, Andreja Tepavčević[1]
Institute of Mathematics, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

The necessary and sufficient conditions are given under which a weak-congruence lattice of a factor-algebra of an algebra is isomorphic with the suitable filter in a weak-congruence lattice of the same algebra.

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1. Introduction

If $\mathcal{A} = (A, F)$ is an algebra and Θ is a congruence on $\mathcal{A}$, then the filter $[\Theta)$ is isomorphic with the lattice $\mathrm{Con}\frac{\mathcal{A}}{\Theta}$, under the mapping $\rho \to \frac{\rho}{\Theta}$, where for $\rho \geq \Theta$ and $x, y \in A$, $[x]_\Theta \frac{\rho}{\Theta} [y]_\Theta$ if and only if $x\rho y$.

If $C_W\mathcal{A}$ is the weak-congruence lattice of $\mathcal{A}$ (i.e. the lattice of all the congruences on all the subalgebras of $\mathcal{A}$ under $\subseteq$), then for $\Theta \in \mathrm{Con}\mathcal{A}$, $C_W\frac{\mathcal{A}}{\Theta}$ is not, in general, isomorphic with a filter in $C_W\mathcal{A}$, not even with its sublattice (see [3] for the latter).

Recall that $C_W\mathcal{A}$ is the lattice of all the symmetric and transitive subalgebras of $\mathcal{A}^2$. If $\Delta = \{(x,x) | x \in A\}$, then $\mathrm{Con}\mathcal{A} = [\Delta)$ in $C_W\mathcal{A}$, and $\mathrm{Sub}\mathcal{A}$ is isomorphic with $(\Delta]$ under $\rho \longmapsto \{x | x\rho x\}$.

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Recall that an algebra $\mathcal{A}$ has the congruence extension property (CEP) if every congruence on a subalgebra of $\mathcal{A}$ is a restriction of a congruence on $\mathcal{A}$.

Some lattice characterizations of the CEP were given in [3].

$\mathcal{A}$ is said to have the congruence intersection property (CIP) if for $\rho \in \mathrm{Con}\mathcal{B}$, $\Theta \in \mathrm{Con}\mathcal{C}$, $\mathcal{B}$, $\mathcal{C} \in \mathrm{Sub}\mathcal{A}$

$$(\rho \cap \Theta)_A = \rho_A \cap \Theta_A,$$

where ρ_A is a least congruence on $\mathcal{A}$ extending ρ ([5]).

For all $\rho, \Theta \in C_W\mathcal{A}$,

$$(\rho \vee \Theta) \wedge \Delta = (\rho \wedge \Delta) \vee (\Theta \wedge \Delta),$$

which means that Δ is always a codistributive element of $C_W\mathcal{A}$ ([5]).

$\mathcal{A}$ has the CIP if and only if Δ is a distributive element of $C_W\mathcal{A}$, i.e. if and only if for $\rho, \Theta \in C_W\mathcal{A}$

$$(\rho \wedge \Theta) \vee \Delta = (\rho \vee \Delta) \wedge (\Theta \vee \Delta),$$

since $\rho_A = \rho \vee \Delta$ in $C_W\mathcal{A}$ (see again [5]).

$\mathcal{A}$ is said to have the infinite congruence intersection property (*CIP) (see [2]), if for $\{\Theta_i, i \in I\} \subseteq C_W\mathcal{A}$

$$(\bigcap_{i\in I} \Theta_i)_A = \bigcap_{i\in I} (\Theta_i)_A,$$

i.e. if Δ is an infinitely distributive element in $C_W\mathcal{A}$:

$$\Delta \vee \bigwedge_{i\in I} \Theta_i = \bigwedge_{i\in I} (\Delta \vee \Theta_i).$$

Some characterizations of the *CIP were given in [2].

2. Results

In paper [3], $C_W\frac{\mathcal{A}}{\Theta}$, for $\Theta \in \mathrm{Con}\mathcal{A}$, is located in the lattice $C_W\mathcal{A}$ as (up to the mapping $\rho \longmapsto \frac{\rho}{\Theta}$) the set

$$\bigcup([B^2 \wedge \Theta, B^2] | \mathcal{B} \in \mathrm{Sub}\mathcal{A} \text{ and } B[\Theta] = B),$$

where $[B^2 \wedge \Theta, B^2]$ is an interval sublattice of $C_W\mathcal{A}$, and

$$B[\Theta] = \{x \in A | x\Theta b \text{ for some } b \in B\}.$$

Conditions under which $C_W\frac{\mathcal{A}}{\Theta}$ is a sublattice of $C_W\mathcal{A}$ were given in the above-mentioned paper.

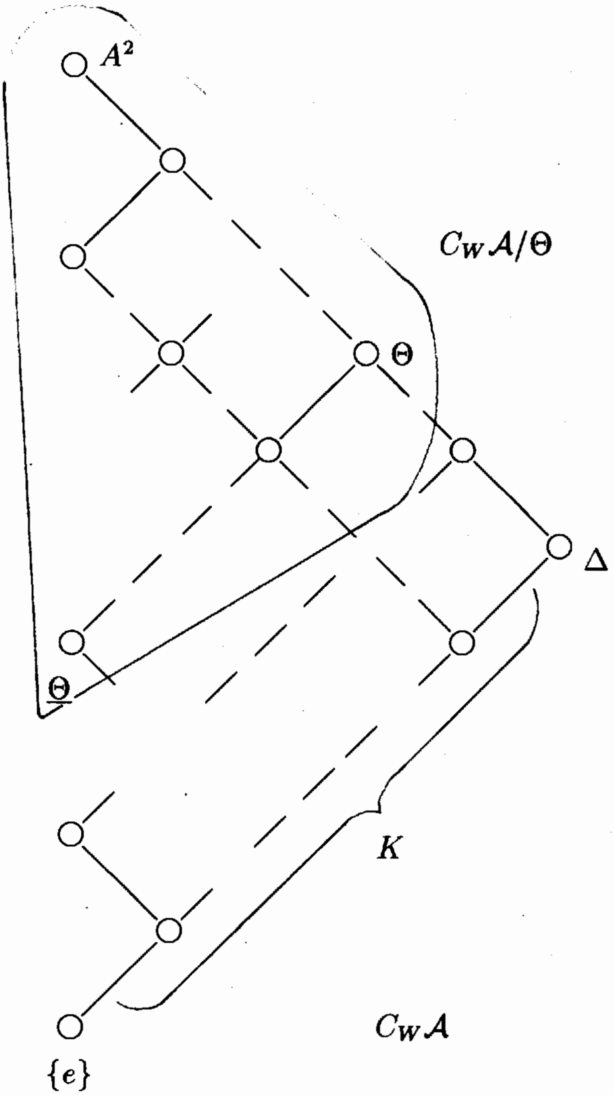

Fig.1

In the following example, where $\mathcal{A}$ is a cyclic group of order $k \in N$, $C_W\frac{\mathcal{A}}{\Theta}$ is, for every $\Theta \in \mathrm{Con}\mathcal{A}$, a filter-sublattice of $C_W\mathcal{A}$.

Namely,

$$C_W\frac{\mathcal{A}}{\Theta} \cong [\underline{\Theta}),$$

under $\rho \longmapsto \frac{\rho}{\Theta}$, where

$$\underline{\Theta} = \wedge\{\rho | \rho_A \geq \Theta\}$$

(recall that in $C_W\mathcal{A}$ $\rho_A = \rho \vee \Delta$).

This example is the motivation for the following problem, which will be the subject of our consideration for the rest of the paper:

Characterize the algebras $\mathcal{A}$, having a property that for every $\Theta \in \mathrm{Con}\mathcal{A}$,

$$C_W\frac{\mathcal{A}}{\Theta} \cong [\underline{\Theta}), \tag{1}$$

under the mapping $\rho \longmapsto \frac{\rho}{\Theta}$, where $[\underline{\Theta})$ is the principal filter in $C_W\mathcal{A}$ generated by the weak congruence

$$\underline{\Theta} = \bigwedge(\rho \in C_W\mathcal{A} | \rho \vee \Delta \geq \Theta).$$

Proposition 1. *Let $\mathcal{A}$ be an algebra satisfying (1). Then*

*a) $\mathcal{A}$ has the *CIP;*

b) $\mathcal{A}$ has the CEP;

c) if $\mathcal{B} \in Sub\mathcal{A}$, then

$$B[\Theta] = B \text{ if and only if } (B^2 \wedge \Theta) \vee \Delta = \Theta.$$

Proof.

a) It was proved in [2] that an algebra $\mathcal{A}$ has the *CIP if and only if the set $\{\rho | \rho \vee \Delta \geq \Theta\}$ has a minimum element $\underline{\Theta}$ for every $\Theta \in \mathrm{Con}\mathcal{A}$. Since by assumption $\underline{\Theta}$ exists for every $\Theta \in \mathrm{Con}\mathcal{A}$, $\mathcal{A}$ has the *CIP.

b) Suppose that $\mathcal{A}$ does not satisfy the CEP. Then, there are $\rho_1, \rho_2 \in \mathrm{Con}\mathcal{B}$, for some $\mathcal{B} \in \mathrm{Sub}\mathcal{A}$, such that $\rho_1 \neq \rho_2$, and $\rho_1 \vee \Delta = \rho_2 \vee \Delta = \Theta \in \mathrm{Con}\mathcal{A}$. We can assume that ρ_1 and ρ_2 are comparable, say $\rho_1 < \rho_2$ (otherwise, we could always consider e.g. $\rho_1 \wedge \rho_2$ and ρ_2). Then, $B^2 \wedge \Theta \geq \rho_2$, and $B^2 \wedge \Theta > \rho_1$. Thus, $\rho_1 \notin Cw\frac{\mathcal{A}}{\Theta}$, but $\rho_1 \in [\underline{\Theta})$, which is impossible because of (1). Hence, $\mathcal{A}$ has the CEP.

c) Let $\mathcal{B}$ be a subalgebra of $\mathcal{A}$.

If $B[\Theta] = B$, then $B^2 \wedge \Theta$ is, up to the isomorphism, in $Cw\frac{\mathcal{A}}{\Theta}$, i.e. it belongs to $[\underline{\Theta})$. Hence, $(B^2 \wedge \Theta) \vee \Delta = \Theta$.

On the other hand, if $(B^2 \wedge \Theta) \vee \Delta = \Theta$, then $B^2 \wedge \Theta$ belongs to $[\underline{\Theta})$, and B is a union of some classes of Θ, i.e. $B[\Theta] = B$, which was to be proved. □

The converse of Proposition 1 also holds.

Proposition 2. *If $\mathcal{A}$ is an algebra satisfying the conditions a), b) and c) from Proposition 1, then for every $\Theta \in Con\mathcal{A}$,*

$$Cw\frac{\mathcal{A}}{\Theta} \cong [\underline{\Theta}),$$

under $\rho \longmapsto \frac{\rho}{\Theta}$.

Proof. For every $\Theta \in \mathrm{Con}\mathcal{A}$, $\underline{\Theta}$ exists because of the *CIP. We shall prove that

$$\bigcup([B^2 \wedge \Theta,\ B^2]\|\ B[\Theta] = B,\ \mathcal{B} \in \mathrm{Sub}\mathcal{A}) = [\underline{\Theta}),$$

for every $\Theta \in \mathrm{Con}\mathcal{A}$.

Let $\rho \in [B^2 \wedge \Theta,\ B^2]$, where $B[\Theta] = B$. By c), $B^2 \wedge \Theta$ belongs to $[\underline{\Theta})$, and thus $\rho \in [\underline{\Theta})$. Hence,

$$\bigcup([B^2 \wedge \Theta,\ B^2]\|\ B[\Theta] = B,\ \ \mathcal{B} \in \mathrm{Sub}\mathcal{A}) \subseteq [\underline{\Theta}).$$

On the other hand, if $\rho \in [\underline{\Theta})$, $\rho \in \mathrm{Con}\mathcal{B}$, for some $\mathcal{B} \in \mathrm{Sub}\mathcal{A}$, then $\rho \vee \Delta \geq \Theta$, and thereby $B^2 \vee \Delta \geq \Theta$. Hence, by a)

$$(B^2 \wedge \Theta) \vee \Delta = (B^2 \vee \Delta) \wedge \Theta = \Theta,$$

which, by c), implies that

$$(i) \qquad B[\Theta] = B.$$

Moreover, $\rho \geq B^2 \wedge \Theta$. Indeed, by a),

$$(\rho \wedge B^2 \wedge \Theta) \vee \Delta = (\rho \wedge \Theta) \vee \Delta = (\Delta \vee \rho) \wedge \Theta = \Theta,$$

since $\Delta \vee \rho \geq \Theta$. Thus,

$(\rho \wedge \Theta) \vee \Delta = (B^2 \wedge \Theta) \vee \Delta = \Theta$, and since

$(\rho \wedge \Theta) \wedge \Delta = (B^2 \wedge \Theta) \wedge \Delta$, i.e. $\rho \wedge \Theta$ and $B^2 \wedge \Theta$ are congruences on the same subalgebra of $\mathcal{A}$, it follows by b) that

$\rho \wedge \Theta = B^2 \wedge \Theta$, i.e.

$(B^2 \wedge \rho) \wedge \Theta = B^2 \wedge \Theta$, and finally

(*ii*) $\rho \geq B^2 \wedge \Theta$.

By (*i*) and (*ii*)

$$[\underline{\Theta}) \subseteq \bigcup([B^2 \wedge \Theta, B^2] \mid B = B[\Theta],\ \mathcal{B} \in \mathrm{Sub}\mathcal{A}),$$

and the proof is complete. □

Summing up Propositions 1 and 2, we obtain the following theorem.

Theorem 1. *Necessary and sufficient conditions under which for every congruence* Θ *of an algebra* $\mathcal{A}$

(1) $$C_W \frac{\mathcal{A}}{\Theta} \cong [\underline{\Theta})$$

under $\rho \longmapsto \frac{\rho}{\Theta}$, *where* $\underline{\Theta} = \bigwedge\{\rho \in C_W\mathcal{A} | \rho \vee \Delta \geq \Theta\}$ *are:*

a) $\mathcal{A}$ *has the *CIP;*

b) has the CEP;

c) if $\mathcal{B} \in Sub\mathcal{A}$ *and* $\Theta \in Con\mathcal{A}$ *then*

$$\mathcal{B}[\Theta] = B \text{ if and only if } (B^2 \wedge \Theta) \vee \Delta = \Theta. \quad \square$$

For some particular classes of algebras, not all of the listed conditions are necessary in order that (1) should hold.

Recall that an algebra $\mathcal{A}$ is **coherent** if it satisfies the following:

If $\mathcal{B} \in \mathrm{Sub}\mathcal{A}$, and $\mathcal{B}$ contains a class of a congruence Θ on $\mathcal{A}$, then $\mathcal{B}$ is a union of classes of Θ, i.e. $B = B[\Theta]$.

An algebra $\mathcal{A}$ is **regular** if every congruence on $\mathcal{A}$ is uniquely determined by any of its classes.

Proposition 3. *If* $\mathcal{A}$ *is a coherent algebra, then (1) holds if and only if* $\mathcal{A}$ *satisfies a), b) and*

c') for every $\mathcal{B} \in Sub\mathcal{A}$ *and* $\Theta \in Con\mathcal{A}$,

$$\mathcal{B} \text{ contains a class of } \Theta \text{ iff } (B^2 \wedge \Theta) \vee \Delta = \Theta.$$

Proof. Straightforward.□

Proposition 4. *If $\mathcal{A}$ is a regular algebra, then (1) holds if and only if $\mathcal{A}$ satisfies a), b) and*

c") if $\mathcal{B} \in Sub\mathcal{A}$ and $\Theta \in Con\mathcal{A}$, then

$$(B^2 \wedge \Theta) \vee \Delta = \Theta \text{ implies } B[\Theta] = B.$$

Proof. To prove the "only if" part of c), let $B[\Theta] = B$. Then, since $(B^2 \wedge \Theta) \vee \Delta \leq \Theta$, the congruence $(B^2 \wedge \Theta) \vee \Delta$ contains a class of Θ. Hence, by the regularity of $\mathcal{A}$, $(B^2 \wedge \Theta) \vee \Delta$ is not different from Θ. □

It is known ([1]) that any coherent variety is congruence regular. Thus, Propositions 3 and 4 yield the following.

Corollary 1. *Let $\mathcal{A}$ be an algebra in a coherent variety, with conditions*

$$(B^2 \wedge \Theta) \vee \Delta = \Theta \text{ implies } B[\Theta] = B.$$

Then, for every $\Theta \in Con\mathcal{A}$,

$$C_W \frac{\mathcal{A}}{\Theta} \cong [\underline{\Theta})$$

*if and only if $\mathcal{A}$ has the *CIP and the CEP.* □

It was proved in [2] that the *CIP is a necessary and sufficient condition under which a group is Hamiltonian, and thus Hamiltonian groups are the only ones having the property (1), since they have the CEP, as well. Obviously, if all groups in a variety satisfy (1), then this variety consists of Abelian groups.

As a negative example for the algebras in a coherent variety (all in the light of Corollary 1), we have Boolean algebras. Among bounded lattices, the only one satisfying the CIP (and thus the *CIP) is a two-element chain ([4]). Hence, no other Boolean algebra satisfies (1).

The following proposition is from [2]. Note that $\mathcal{A}_m$ is a minimal subalgebra of an algebra $\mathcal{A}$, and that $\mathcal{A}$ is $\mathcal{A}_m-$**regular** if every congruence on $\mathcal{A}$ is uniquely determined by the class containing $\mathcal{A}_m$.

Theorem 2. **([2])** *If $|A_m| = 1$, then the following are equivalent for an algebra $\mathcal{A}$:*

(i) $\mathcal{A}$ is Hamiltonian and every subalgebra of $\mathcal{A}$ is A_m−regular;

(ii) the mapping $C \to C^2 \vee \Delta_B$ ($\mathcal{C} \in Sub\mathcal{B}$, $\Delta_B = \{(x,x)|x \in B\}$) is an isomorphism from $Sub\mathcal{B}$ to $Con\mathcal{B}$, for every $\mathcal{B} \in Sub\mathcal{A}$;

*(iii) $\mathcal{A}$ satisfies the *CIP, the CEP, and the set of minimal weak congruences in the classes induced by $\rho \longmapsto \rho \vee \Delta$ is the set of all the squares B^2, $\mathcal{B} \in Sub\mathcal{A}$.* □

The algebras characterized in the last theorem have property (1). To prove this, we need the following two propositions.

Lemma 1. *If $\mathcal{A}$ is a Hamiltonian algebra, then $\mathcal{A}$ satisfies c") (from Proposition 4).*

Proof. Let $\mathcal{B}$ be a subalgebra of $\mathcal{A}$, $\Theta \in \mathrm{Con}\mathcal{A}$, and let

$$(B^2 \wedge \Theta) \vee \Delta = \Theta.$$

Obviously, $B^2 \vee \Delta \geq \Theta$. $\mathcal{A}$ is Hamiltonian, i.e. B is a class of *a* congruence on $\mathcal{A}$, namely of $B^2 \vee \Delta$, and thus $B[\Theta] = B$. □

Lemma 2. *If $\mathcal{A}$ is Hamiltonian and $\mathcal{A}_m$−regular for which $|A_m| = 1$, and has the CIP, then it satisfies c) (from Theorem 1).*

Proof. By Lemma 1, we have to prove the converse of c"). Let $B[\Theta] = B$, then

$$(B^2 \wedge \Theta) \vee \Delta = (B^2 \vee \Delta) \wedge \Theta,$$

by the CIP. Obviously,

$$(B^2 \vee \Delta) \wedge \Theta \leq \Theta,$$

and since $\mathcal{A}$ is Hamiltonian, $B[B^2 \vee \Delta] = B$. By $\mathcal{A}_m$−regularity it follows that $(B^2 \vee \Delta) \wedge \Theta$ can not be different from Θ, that is

$$(B^2 \wedge \Theta) \vee \Delta = \Theta. \quad \Box$$

Theorem 3. *If $\mathcal{A}$ is an algebra for which $|A_m| = 1$, and with isomorphic lattices of subalgebras and congruences for every $\mathcal{B} \in Sub\mathcal{A}$, then $\mathcal{A}$ has the property* (1) *i.e. for every $\Theta \in Con\mathcal{A}$*

$$C_W \frac{\mathcal{A}}{\Theta} \cong [\underline{\Theta}).$$

Proof. By Theorem 1, since $\mathcal{A}$ has the *CIP, the CEP (Theorem 2), and since it satisfies c) (Lemmas 1 and 2). □

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REZIME

FILTRI U MREŽI SLABIH KONGRUENCIJA

Posmatraju se algebre koje imaju svojstvo da je mreža slabih kongruencija svake faktor-algebre izomorfna sa odgovarajućim filtrom u mreži slabih kongruencija te algebre. Daju se potrebni i dovoljni uslovi pod kojima to svojstvo važi.

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