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ON GROUPOIDS HAVING n^2 ESSENTIALLY n-ARY POLYNOMIALS

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Abstract

In this paper we prove that only rectangular grupoids and normal bands have squares of natural numbers as numbers of polynomials depending on all their variables.

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1. Introduction

For an arbitrary algebra $\mathbf{A}$, by $p_n(\mathbf{A})$ is denoted the number of essentially n-ary polynomials i.e. those n-ary operations which are composed of projection operations using the basic operations of $\mathbf{A}$ and which depend on all variables.

In [1] it was proved that if a non-associative groupoid $\mathbf{G}$ satisfies identities $xx = x$ $(xy)z = xz$ and $x(y(zu)) = x(z(yu))$, then $p_n(\mathbf{G}) = n^2$, for all $n \geq 0$.

For a semigroup $\mathbf{S}$ we have that $p_n(\mathbf{S}) = n^2$ for all $n \geq 0$, if and only if $\mathbf{S}$ generates the variety of normal bands (see [2]). Normal bands are idempotent semigroups satisfying $xyzu = xzyu$.

In this paper we shall show that there are no other groupoids having n^2 essentially n-ary polynomials. Namely, we have the following

MAIN THEOREM

Let **G** *be a groupoid. Then* $p_n(\mathbf{G}) = n^2$ *for all* $n \geq 0$ *if and only if one of the following conditions hold*

(i) **G** *generates the variety of normal bands;*

(ii) **G** *is not a semigroup and satisfies*

$$xx = x$$
$$x(yz) = xz$$
$$((xy)z)u = ((xz)y)u;$$

(iii) **G** *is not a semigroup and satisfies*

$$\begin{aligned} xx &= x \\ (xy)z &= xz \\ x(y(zu)) &= x(z(yu)). \end{aligned}$$

In order to prove the Main theorem we shall prove the following theorems.

Theorem 1. *Let* **G** *be a groupoid for which the polynomial* $x(yz)$ *is not essentially 3-ary. Then* $p_n(\mathbf{G}) = n^2$ *for all* $n \geq 0$ *if and only if* **G** *is non-associative and satisfies*

$$\begin{aligned} xx &= x \\ x(yz) &= xz \\ ((xy)z)u &= ((xz)y)u. \end{aligned}$$

Theorem 2. *There is no non-associative groupoid* **G** *for which* $x(yz)$ *and* $(xy)z$ *are essentially 3-ary polynomials and* $p_n(\mathbf{G}) = n^2$ *for all* $n \geq 0$.

Before passing to the proofs of Theorems 1 and 2 we shall explain some notations and prove a general lemma.

If $p(x_1, x_2, \ldots, x_n)$ is an n-ary polynomial and $\sigma \in \mathbf{S}_n$ (the group of permutations), then by p^σ we denote the polynomial $p(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$. In what follows we use sometimes x , y , z , u instead of x_1 , x_2 , x_3 , x_4.

Also, we denote by p and q the polynomials $x(yz)$ and $(xy)z$ i.e. $p = x(yz)$ and $q = (xy)z$.

Lemma 1. *Let* $\mathbf{G}$ *be a groupoid for which* $p_n(\mathbf{G}) = n^2$, $n \geq 0$. *Then*
(i) $\mathbf{G}$ *is idempotent.*
(ii) xy , yx *are two different essentially binary polynomials.*

Proof. (i) Follows from $p_1(\mathbf{G}) = 1$.
(ii) If xy is not essentially binary, then $xy = x$ or $xy = y$ which implies $p_2(\mathbf{G}) = 0$. However, $p_2(\mathbf{G}) = 4$ by the assumption. Analogously for yx. Suppose $xy = yx$. If $\mathbf{G}$ is a semigroup, then $\mathbf{G}$ is a semilattice and $p_2(\mathbf{G}) = 1$ which contradicts $p_2(\mathbf{G}) = 4$. If $\mathbf{G}$ is not a semigroup, then

$$p_n(\mathbf{G}) \geq \frac{1}{3}(2^n - (-1)^n),\ n \geq 2$$

as it was shown in [4]. However,

$$100 = p_{10}(\mathbf{G}) \geq \frac{1}{3}(2^{10} - 1) = 341.$$

Contradiction. □

2. Groupoids for which $p = x(yz)$ is not an essentially ternary polynomial

Lemma 2. *There is no groupoid* $\mathbf{G}$, *which satisfies the identity* $x(yz) = xy$, *such that* $p_n(\mathbf{G}) = n^2$, $n \geq 0$.

Proof. Supose that $\mathbf{G}$ is such a groupoid.

Claim 1. *Each polynomial of the groupoid* $\mathbf{G}$ *is equal to a polynomial of the form* $(\ldots((x_{i_1}x_{i_2})x_{i_3})\ldots)x_{i_n}$, *where the variables are not necessarily different.*

Proof. Follows from $x(yz) = xy$. □

Claim 2. *The set* $\{(xy)x, (yx)y, (xy)y, (yx)x\}$ *is not a subset of the set* $\{x, y, xy, yx\}$.

Proof. The opposite implies that the set $\{x, y, xy, yx\}$ is closed under the multiplication of polynomials. This means that $p_2(\mathbf{G}) = 2$, which is a contradiction. □

Claim 3. *If* t *is a polynomial having* x *as its first variable, then* $tx = t$. *Especially,* $(xy)x = xy$ *and* $(yx)y = yx$.

Proof. From $x(yz) = xy$ it follows that $tx = tt = t$. □

Claim 4. *If* r *and* s *are two polynomials having different first variables, then* $r \neq s$. *Especially,* $(xy)y \neq (yx)x$.

Proof. Let x be the first variable of r and y the first variable of s. If $r = s$, then $xr = xs$ i.e. $x = xy$, which contradicts Lema 1. □

Claim 5. *The set* $\{xy, yx, (xy)y, (yx)x\}$ *contains four essentially binary polynomials.*

Proof. From Claim 2 and 3 it follows that $(xy)y, (yx)x \notin \{x, y, xy, yx\}$. The proof now follows from $(xy)y \neq (yx)x$ (Claim 4). □

Claim 6 *If* r, s *are polynomials such that* $r \neq s$ *and* z *is a variable which does not appear in* r *and* s, *then* $rz \neq sz$.

Proof. If r and s have different first variables, then this is Claim 4. If the first variable is the same for r and s, then from Claim 3 it follows that $rx \neq sx$ i.e. $rz \neq sz$. □

Now we can prove the Lemma.

Consider the polynomial $(xy)z$. According to Claim 4 every polynomial depends on the first variable. Hence, $(xy)z$ depends on x. For $y = x$ we obtain the polynomial xz, which is, according to Claim 5, essentially binary. Therefore $(xy)z$ depends on z. Analogously, for $z = x$, we have that $(xy)z$ depends on y.

In the same way we prove, by taking $y = x$ and $z = x$, that polynomials $((xy)z)z, ((xy)y)z, (((xy)y)z)z$ depend on all variables.

Let us show now that $(xy)z, ((xy)z)z, ((xy)y)z, (((xy)y)z)z$ are different. If we put in these polynomials $y = x$, we obtain $xz, (xz)z, xz, (xz)z$. Therefore

$$\begin{aligned} (xy)z &\neq ((xy)z)z, \\ (xy)z &\neq (((xy)y)z), \\ ((xy)y)z &\neq ((xy)z)z, \\ ((xy)y)z &\neq (((xy)y)z)z. \end{aligned}$$

If in the same polynomials we insert $z = x$ we obtain polynomials $xy, xy, (xy)y, (xy)y$ (Claim 3). This implies

$$\begin{aligned} (xy)z &\neq ((xy)y)z \\ ((xy)z)z &\neq (((xy)y)z)z \end{aligned}$$

(Claim 5).

The above arguments show that each one of the sets

$$\begin{aligned} A &= \{(xy)z, ((xy)z)z, ((xy)y)z, (((xy)y)z)z\} \\ B &= \{(yz)x, ((yz)x)x, ((yz)z)x, (((yz)z)x)x\} \\ C &= \{(zx)y, ((zx)y)y, ((zx)x)y, (((zx)x)y)y\} \end{aligned}$$

contain four essentially 3-ary polynomials. Claim 4 implies that

$$A \cap B = B \cap C = C \cap A = \emptyset.$$

Hence, the set $A \cup B \cup C$ contains 12 essentially 3-ary polynomials. This contradicts the assumption that $p_3(\mathbf{G}) = 9$. □

Proof of Theorem 1.
($\leftarrow$) The dual of this was proved in [1].
($\rightarrow$) Taking $z = y$ we see that $x(yz)$ depends on x. If the polynomial $x(yz)$ does not depend on y, then $x(yz) = x(zz) = xz$. It was proved in [1] (dual) that in that case **G** is an idempotent non-associative groupoid satisfying the identity $((xy)z)u = ((xz)y)u$. If the polynomial $x(yz)$ does not depend on z, then $x(yz) = x(yy) = xy$. However, from Lemma 2 it follows that this is not possible. □

3. Non–associative groupoids having both of the polynomials $x(yz)$ and $(xy)z$ essentially ternary

In the following lemmas of this section, **G** is a non-associative groupoid having $p = x(yz)$ and $q = (xy)z$ essentially 3-ary and $p_n(\mathbf{G}) = n^2$, for all $n \geq 0$.

Lemma 3. *At least one of the following identities is true on* **G**.

(I_1)	$x(yz) = x(zy)$	(I_9)	$x(yz) = (yz)x$
(I_2)	$x(yz) = y(xz)$	(I_{10})	$x(yz) = (zx)y$
(I_3)	$x(yz) = y(zx)$	(I_{11})	$x(yz) = (zy)x$
(I_4)	$x(yz) = z(xy)$	(I_{12})	$(xy)z = (xz)y$
(I_5)	$x(yz) = z(yx)$	(I_{13})	$(xy)z = (yx)z$
(I_6)	$x(yz) = (xy)z$	(I_{14})	$(xy)z = (yz)x$
(I_7)	$x(yz) = (xz)y$	(I_{15})	$(xy)z = (zx)y$
(I_8)	$x(yz) = (yx)z$	(I_{16})	$(xy)z = (zy)x.$

Proof. All the polynomials $p^\sigma, q^\sigma, \sigma \in \mathbf{S}_3$, are essentially 3-ary and there are 12 of them. Because of $p_3(\mathbf{G}) = 9$ two of them must be equal i.e. there are $\sigma, \tau \in \mathbf{S}_3$ such that

$$p^\sigma = p^\tau, \sigma \neq \tau, \text{or } q^\sigma = q^\tau, \sigma \neq \tau, \text{or } p^\sigma = q^\tau,$$

which implies

$$p = p^{\sigma^{-1}\tau}, \sigma^{-1}\tau \neq (1), \text{or } q = q^{\sigma^{-1}\tau}, \sigma^{-1}\tau \neq (1), \text{or } p = q^{\sigma^{-1}\tau}.$$

□

Lemma 4. *The following identities do not hold on* **G**

$$I_3, I_4, I_6, I_7, I_8, I_9, I_{10}, I_{11}, I_{14}, I_{15}.$$

Proof. I_6 is the associative law and from I_9 and I_{11} follows the law of commutativity. Therefore I_6, I_9, I_{11} are not true on **G**.

Suppose I_3 holds on **G** i.e. $x(yz) = y(zx)$. A simple argument shows that

$$x(yx) \;=\; y(xx) = yx$$

$$\begin{aligned} x(xy) &= x(yx) = yx \\ (xy)x &= x(x(xy)) = x(yx) = yx \\ (yx)x &= x(x(yx)) = x(yx) = yx \\ (xy)(yx) &= y(x(xy)) = y(yx) = xy, \end{aligned}$$

which means that the set $\{x, y, xy, yx\}$ is closed under the multiplication of polynomials. This is in contradiction with $p_2(\mathbf{G}) = 4$. A dual argument shows that I_{15} is not true on **G**. I_4 implies I_3 and hence I_4 is not true on **G**. A dual argument shows that I_{14} is not true on **G**.

Suppose I_7 holds on **G** i.e. $x(yz) = (xz)y$. We have

$$\begin{aligned} x(yx) &= (xx)y = xy \\ (xy)y &= x(yy) = xy \\ (xy)x &= ((xy)y)x = (xy)(xy) = xy \\ x(xy) &= (xy)x = xy \\ (xy)(yx) &= ((xy)x)y = (xy)y = xy, \end{aligned}$$

which contradicts $p_2(\mathbf{G}) = 4$. Hence, I_7 is not true on **G**.

Suppose I_8 holds on **G** i.e. $x(yz) = (yx)z$. Then

$$\begin{aligned} x(xy) &= (xx)y = xy \\ (xy)x &= y(xx) = yx \\ x(yx) &= x(y(yx)) = (yx)(yx) = yx \\ (yx)x &= x(yx) = yx \\ (xy)(yx) &= (y(xy))x = (xy)x = yx. \end{aligned}$$

This contradicts $p_2(\mathbf{G}) = 4$.

Suppose that I_{10} holds on **G** i.e. $x(yz) = (zx)y$. Then

$$\begin{aligned} x(yx) &= (xx)y = xy \\ (xy)x &= y(xx) = yx \\ x(xy) &= x((yx)y) = (yx)(yx) = yx \\ (yx)x &= x(xy) = yx \\ (xy)(yx) &= (x(xy))y = (yx)y = xy \end{aligned}$$

and therefore contradicts $p_2(\mathbf{G}) = 4$. □

Lemma 5.

(i) *The polynomial* $f=(xy)(zu)$ *is essentially 4-ary.*

(ii) $\mathbf{G}$ *satisfies* $f=f^\sigma$ *for some* $\sigma\in\mathbf{S}_4, \sigma\neq(1)$.

(iii) $f=f^\sigma$ *does not hold on* $\mathbf{G}$ *if* $\sigma(1)\neq 1$ *and* $\sigma(4)\neq 4$.

Proof. (i). Follows from the assumption that p and q are essentially 3-ary and substitutions of the form $y=x$ and $u=t$.

(ii). Follows from $p_4(\mathbf{G})=16$ and $|\mathbf{S}_4|=24$ similarly as in Lemma 3.

(iii). All identities $f=f^\sigma, \sigma(1)\neq 1$ and $\sigma(4)\neq 4$, imply commutativity. Namely,

$$
\begin{aligned}
(xy)(zu)=(yz)(ux) &\implies xy=yx \text{ for } z=x, u=y\\
(xy)(zu)=(yu)(zx) &\implies xy=yx \text{ for } z=x, u=y\\
(xy)(zu)=(yx)(uz) &\implies xy=yx \text{ for } z=x, u=y\\
(xy)(zu)=(yu)(xz) &\implies xy=yx \text{ for } z=x, u=y\\
(xy)(zu)=(zy)(ux) &\implies xz=zx \text{ for } y=x, u=z\\
(xy)(zu)=(zu)(yx) &\implies xz=zx \text{ for } y=x, u=z\\
(xy)(zu)=(zx)(uy) &\implies xz=zx \text{ for } y=x, u=z\\
(xy)(zu)=(zu)(xy) &\implies xz=zx \text{ for } y=x, u=z\\
(xy)(zu)=(uy)(zx) &\implies xz=zx \text{ for } y=x, u=z\\
(xy)(zu)=(uz)(yx) &\implies xz=zx \text{ for } y=x, u=z\\
(xy)(zu)=(ux)(zy) &\implies xz=zx \text{ for } y=x, u=z\\
(xy)(zu)=(uz)(xy) &\implies xz=zx \text{ for } y=x, u=z\\
(xy)(zu)=(ux)(yz) &\implies xy=yx \text{ for } z=x, u=y\\
(xy)(zu)=(uy)(xz) &\implies xy=yx \text{ for } z=x, u=y.
\end{aligned}
$$

□

Lemma 6. *The following pairs of identities do not hold on* $\mathbf{G}$.

(i)	I_1, I_2	*(iv)*	I_{12}, I_{13}	*(vii)*	I_1, I_{13}	*(x)*	I_2, I_{16}
(ii)	I_1, I_5	*(v)*	I_{12}, I_{16}	*(viii)*	I_1, I_{16}	*(xi)*	I_5, I_{12}
(iii)	I_2, I_5	*(vi)*	I_{13}, I_{16}	*(ix)*	I_2, I_{12}	*(xii)*	I_5, I_{13}.

Proof. It holds that

$$
\begin{aligned}
x(yz)=x(zy)\wedge x(yz)=y(xz) &\Rightarrow x(yz)=x(zy)=z(xy)\\
x(yz)=x(zy)\wedge x(yz)=z(yx) &\Rightarrow x(yz)=x(zy)=y(zx)\\
x(yz)=y(xz)\wedge x(yz)=z(yx) &\Rightarrow x(yz)=y(xz)=z(xy).
\end{aligned}
$$

The obtained contradiction to Lemma 4 proves (i), (ii) and (iii). (iv), (v) and (vi) are dual to (i), (iii) and (ii) respectively.

(vii) This case contradicts Lemma 1 because of

$$x(yz) = x(zy) \wedge (xy)z = (yx)z \Rightarrow xy = (xy)(xy) = (yx)(yx) = yx.$$

(viii) Suppose I_1 and I_{16} hold on **G** i.e. $x(yz) = x(zy)$ and $(xy)z = (zy)x$ Then

$$(yx)x = (xx)y = xy$$

and

$$z(yx) = z(xy) = z((yx)x) = z(x(yx)).$$

If in the last equality we put $z = x(yx)$, we have

$$(x(yx))(yx) = x(yx) \Rightarrow (yx)x = x(yx) \Rightarrow xy = x(yx).$$

It is routine to verify

$$\begin{aligned} x(xy) &= x(yx) = xy \\ (xy)x &= (xy)(x(xy)) = (xy)(xy) = xy \\ (xy)(yx) &= (xy)(xy) = xy. \end{aligned}$$

Contradiction with $p_2(\mathbf{G}) = 4$.

(ix) Suppose I_2 and I_{12} are valid on **G** i.e. $x(yz) = y(xz)$ and $(xy)z = (xz)y$. Then

$$\begin{aligned} x(yx) &= yx \\ (xy)x &= xy. \end{aligned}$$

Also

$$x(yz) = y(xz) = y((xz)x) = (xz)(yx) = (x(yx))z = (yx)z$$

which contradicts Lemma 4.

(x) This case contradicts Lemma 1 because of

$$x(yz) = y(xz) \wedge (xy)z = (zy)x \Rightarrow yx = x(yx) = x((xy)y) = (xy)(xy) = xy.$$

Cases (xi) and (xii) are dual to (x) and (viii). □

Lemma 7. *The pair of identities I_1, I_{12} is not true on* **G**.

Proof. Suppose the opposite i.e. on **G** we have

$$\begin{aligned} x(yz) &= x(zy) \\ (xy)z &= (xz)y. \end{aligned}$$

Claim 7. $x(xy) \notin \{x, y, xy, yx\}$.

Proof. Suppose the opposite. In that case we have the following possibilities:

1^0 $x(xy) = x$. This implies

$$\begin{aligned} &x(yx) = x(xy) = x \\ &(xy)x = (xx)y = xy \\ &(yx)x = (yx)(x(xy)) = (yx)((xy)x) = (yx)(xy) = (yx)(yx) = yx \\ &(xy)(yx = (xy)(xy) = xy. \end{aligned}$$

2^0 $x(xy) = y$. This implies

$$\begin{aligned} &x(yx) = x(xy) = y \\ &(xy)x = (xx)y = xy \\ &(yx)x = (yx)(y(yx)) = (yx((yx)y) = (yx)(yx) = yx \\ &(xy)(yx) = xy. \end{aligned}$$

3^0 $x(xy) = xy$. This implies

$$\begin{aligned} &x(yx) = x(xy) = xy \\ &(xy)x = (xx)y = xy \\ &(yx)x = (y(yx))x = (yx)(yx) = yx \\ &(xy)(yx) = xy \end{aligned}$$

4^0 $x(xy) = yx$. This implies

$$\begin{aligned} &x(yx) = yx \\ &(xy)x = xy \\ &(yx)x = (y(yx))x = (yx)(yx) = yx \\ &(xy)(yx) = xy. \end{aligned}$$

All these cases contradict the assumption that $p_2(\mathbf{G}) = 4$. □

Claim 8. $x(xy) \neq y(yx)$.

Proof. If in $x(xy) = y(yx)$ we put $y = yx$, we hawe $x(x(yx)) = (yx)((yx)x)$ which implies

$$\begin{aligned} x(xy) &= x((xy)x) = x(x(xy)) = x(x(yx)) = (yx)((yx)x) = (yx)(x(yx)) = \\ &= (yx)(x(xy)) = (yx)((xy)x) = (yx)(xy) = yx. \end{aligned}$$

This contradicts Claim 7. □

Claim 9. *The set $T = \{p^\sigma | \sigma \in S_3\}$ has 6 elements.*

Proof. From Lemmas 4 and 6 it follows that on **G** no identity from I_1-I_{16}, except I_1 and I_{12}, holds. This implies the assertion. □

Claim 10. *The set $T \cup \{x(x(yz)), y(x(yz)), z(x(yz))\}$ contains all 9 essentially 3-ary polynomials.*

Proof. Insert $y = z$ in the polynomial. We obtain an essentially binary polynomial $x(xz)$ (Claim 7). If in the same polynomial we put $y = x$ and $z = x$, we obtain the polynomials $x(x(xz)) = x(xz)$, $x(x(yx)) = x(xy)$ which implies that $x(x(yz))$ is essentially 3-ary. Since

$$y(x(yz)) = y(y(xz)) \ , \ z(x(yz)) = z(z(xy)),$$

it follows that these two polynomials are essentially 3-ary.

The following calculation

$$\begin{aligned} x(x(yz)) = x(yz) &\Rightarrow x(xy) = xy \text{ for } z = y \\ x(x(yz)) = y(xz) &\Rightarrow x(xy) = yx \text{ for } z = x \\ x(x(yz)) = z(xy) &\Rightarrow x(xz) = zx \text{ for } y = x \\ x(x(yz)) = (xy)z &\Rightarrow x(xz) = xz \text{ for } y = x \\ x(x(yz)) = (yx)z &\Rightarrow x(xz) = xz \text{ for } y = x \\ x(x(yz)) = (zx)y &\Rightarrow x(xy) = xy \text{ for } z = x \end{aligned}$$

shows that $x(x(yz)), y(x(yz)), z(x(yz)) \notin T$.

From $x(x(yz)) = y(x(yz))$, for $z = y$, it follows that $x(xy) = y(xy)$ which is impossible according to Claim 8. Hence $\{x(x(yz)), y(x(yz)), z(x(yz))\}$ has 3 elements. This proves Claim 10. □

Claim 11. *$(xy)(x(yz))$ is an essentially 3-ary polynomial and*

$$(xy)(x(yz)) \notin T \cup \{x(x(yz)), y(x(yz)), z(x(yz)).$$

Proof. In $(xy)(x(yz))$ we insert $y = x$ and $y = z$ and get polynomials $x(xz)$ and xz. Therefore, $(xy)(x(yz))$ depends on z and x. According to Claim 7 $x(xz) \neq xz$ which implies dependence on y.

The second part of the claim follows from

$$\begin{array}{rcl}
(xy)(x(yz)) = x(yz) & \Rightarrow & xy = x(xy) \text{ for } z = x \\
(xy)(x(yz)) = y(xz) & \Rightarrow & xy = y(yx) \text{ for } z = y \\
(xy)(x(yz)) = z(xy) & \Rightarrow & xy = x(xy) \text{ for } z = x \\
(xy)(x(yz)) = (xy)z & \Rightarrow & x(xz) = xz \text{ for } y = x \\
(xy)(x(yz)) = (yx)z & \Rightarrow & x(xz) = xz \text{ for } y = x \\
(xy)(x(yz)) = (zx)y & \Rightarrow & xy = yx \text{ for } z = y \\
(xy)(x(yz)) = x(x(yz)) & \Rightarrow & xy = x(xy) \text{ for } z = y \\
(xy)(x(yz)) = y(x(yz)) & \Rightarrow & xy = y(yx) \text{ for } z = y \\
(xy)(x(yz)) = z(x(yz)) & \Rightarrow & xy = y(yx) \text{ for } z = y
\end{array}$$

which is in contradiction with Claim 7 and Lemma 1. □

Claim 10 and Claim 11 contradict $p_3(\mathbf{G}) = 9$. This proves our lemma. □

Lemma 8. *The pair of identities I_2, I_{13} is not true on* **G**.

Proof. Dual to Lemma 7. □

Lemma 9. *The pair of identities I_5, I_{16} is not true on* **G**.

Proof. Suppose the opposite, i.e.

$$\begin{array}{rcl}
x(yz) & = & z(yx) \\
(xy)z & = & (zy)x
\end{array}$$

holds on **G**.

Claim 12. *$x(xy) = yx$ and $(yx)x = xy$.*

Proof. Consequence of I_5 and I_{16}. □

Claim 13. $x(yx) = (xy)x$.

Proof. $x(yx) = x(x(xy)) = (xy)(xx) = (xy)x$. □

Claim 14. *$x(yx)$ is essentially binary and $x(yx) \notin \{xy, yx\}$.*

Proof. Claim 13 and $(xy)(yx) = ((yx)y)x$ imply $x(yx) \notin \{x, y, xy, yx\}$, because the opposite means that $\{x, y, xy, yx\}$ is closed under the multiplication of polynomials. □

Claim 15. *$x((yz)x)$ is an essentially 3-ary polynomial.*

Proof. Follows from the substitutions $z = x$ and $z = y$, by applying Claim 14 and Lemma 1. □

Claim 16. *The set $T \cup \{x((yz)x), y((zx)y), z((xy)z), x((zy)x)\}$ contains 10 essentially 3-ary polynomials.*

Proof. The set T has 6 elements which can be proved in the same way as in Lemma 7, Claim 9. The set $\{x((yz)x, y((zx)y), z((xy)z)), x((zy)x)\}$ contains 4 essentially 3-ary polynomials. This follows from Claim 15 and

$$\begin{aligned}
&x((yz)x) = y((zx)y) \Rightarrow x((yx)x = y(xy) \ \text{(for } z = x) \ \Rightarrow yx = y(xy)\\
&x((zy)x) = x((yz)x) \Rightarrow x((zx)x) = x((xz)x) \ \text{(for } y = x) \Rightarrow\\
&\qquad \Rightarrow zx = (x(xz))x \ \text{(Claim 13)} \ \Rightarrow zx = xz\\
&x((zy)x) = y((zx)y) \Rightarrow x((xy)x) = y(xy) \ \text{(for } z = x) \ \Rightarrow\\
&\qquad \Rightarrow (x(xy))x = y(xy) \ \text{(Claim 13)} \ \Rightarrow xy = y(xy)\\
&x((zy)x) = z((xy)z) \Rightarrow zx = z(xz) \ \text{(for } y = x)
\end{aligned}$$

(we have contradictions with Claim 14 and Lemma 1).

To prove that

$$T \cap \{x((yz)x), y((zx)y), z((xy)z), x((zy)x)\} = \emptyset$$

it is sufficient to show that $x((yz)x) \notin T$. However, this follows from

$$\begin{aligned}
x((yz)x) = x(yz) &\Rightarrow x(yx) = xy \text{ for } z = y \\
x((yz)x) = x(zy) &\Rightarrow x(yx) = xy \text{ for } z = y \\
x((yz)x) = y(xz) &\Rightarrow x((xz)x) = x(xz) \text{ for } y = x \Rightarrow \\
&\Rightarrow (x(xz))x = x(xz) \Rightarrow xz = zx \\
x((yz)x) = (xy)z &\Rightarrow x(yx) = yx \text{ for } z = y \\
x((yz)x) = (yx)z &\Rightarrow yx = xy \text{ for } z = x \\
x((yz)x) = (xz)y &\Rightarrow x(yx) = yx \text{ for } z = y
\end{aligned}$$

(we have contradictions with Claim 14 and Lemma 1). □

Claim 16 contradicts the assumption that $p_3(\mathbf{G}) = 9$ which means that our supposition about the pair I_5, I_{16} is not true. □

Lemma 10. *Exactly one of the identities $I_1, I_2, I_5, I_{12}, I_{13}, I_{16}$ holds on* **G** *and no other from the list I_1-I_{16}. The set $\{p^\sigma | \sigma \in S_3\} \cup \{q^\sigma | \sigma \in S_3\}$ contains all 9 essentially 3-ary polynomials.*

Proof. The first assertions follows from Lemmas 3, 4, 6, 7, 8, 9. The second part is a direct consequence of the first. □

Lemma 11. **G** *does not have a commutative binary polynomial.*

Proof. Suppose $\circ$ is a commutative operation on **G** induced by the given commutative polynomial. If $\circ$ is non-associative, then by [4], we have

$$p_n(\mathbf{G}) \geq p_n(\mathbf{G}') \geq \frac{1}{3}(2^n - (-1)^n),$$

where $\mathbf{G}' = (G, \circ)$. This contradicts $p_n(\mathbf{G}) = n^2$. If $\circ$ is associative, then $\mathbf{G}'$ is a semilattice and $x \circ y \circ z$ iz an essentially 3-ary polynomial. However, from Lemma 10 it follows that

$$x \circ y \circ z \in \{p^\sigma | \sigma \in S_3\} \cup \{q^\sigma | \sigma \in S_3\}$$

which implies $x \circ y = xy$ (e.g. if $x \circ y \circ z = (zx)y$, then we put $z = x$, etc). This contradicts Lemma 1. □

Lemma 12. *I_1 does not hold on* **G**.

Proof. Suppose the opposite i.e.

$$x(yz) = x(zy)$$

holds on **G**

Claim 17. $x(yx) = x(xy)$.

Proof. Obvious. □

Claim 18. $x(xy) \notin \{x, y, xy, yx\}$.

Proof. Suppose the opposite. Then we have the following 4 cases:

1^0 $x(xy) = x$. This implies

$$\begin{aligned}
&x(yx) = x \\
&(yx)x = (yx)(x(xy)) = (yx)((yx)x) = yx \\
&(xy)x = (xy)(x(xy)) = (xy)((yx)x) = (xy)(yx) = (xy)(xy) = xy \\
&(xy)(yx) = xy;
\end{aligned}$$

2^0 $x(xy) = y$. This implies

$$\begin{aligned}
&x(yx) = y \\
&(yx)x = (yx)(y(yx)) = (yx)((yx)y) = y \\
&(xy)x = (xy)(y(yx)) = (xy)((xy)y) = y \\
&(xy)(yx) = xy;
\end{aligned}$$

3^0 $x(xy) = xy$. This implies

$$\begin{aligned}
&x(yx) = xy \\
&(xy)x = (xy)((xy)x) = (xy)(x(xy)) = xy \\
&(yx)x = (yx)((yx)x) = (yx)(x(yx)) = (yx)(xy) = yx \\
&(xy)(yx) = xy;
\end{aligned}$$

4^0 $x(xy) = yx$. This implies

$$\begin{aligned}
&x(yx) = yx \\
&(yx)x = x(x(yx)) = x(yx) = yx \\
&(xy)x = x(x(xy)) = x(yx) = yx \\
&(xy)(yx) = xy.
\end{aligned}$$

All these cases are in contradiction with $p_2(\mathbf{G}) = 4$. □

Claim 19. *The polynomial $x(x(yz))$ is essentially 3-ary.*

Proof. For $z = y$ we have an essentially binary polynomial $x(xy)$ (Claim 18), so that$x(x(yz))$ depends on x and at least one of the variables y, z. However, since $x(x(yz) = x(x(zy))$ the assertion follows. □

Claim 20. $x(x(yz)) \notin \{p^\sigma | \sigma \in S_3\}$.

Proof. Follows from

$$\begin{aligned} x(x(yz)) = x(yz) &\Rightarrow x(xy) = xy \quad \text{for} \quad z = y \\ x(x(yz)) = y(xz) &\Rightarrow x(xy) = y(yx) \quad \text{for} \quad z = y \\ x(x(yz)) = z(xy) &\Rightarrow x(xy) = y(yx) \quad \text{for} \quad z = y \end{aligned}$$

(we obtain contradictions with Lemma 11 and Claim 18). □

Claim 21. $x(x(yz)) \notin \{q^\sigma | \sigma \in S_3\}$.

Proof. From $x(x(yz)) = q^\sigma$ it follows

$$q^\sigma = x(x(yz)) = x(x(zy)) = (x(x(yz)))^{(23)} = (q^\sigma)^{(23)} = q^{(23)\sigma}$$

i.e.

$$q = q^{\sigma^{-1}(23)\sigma}$$

which contradicts Lemma 10 because of

$$\sigma^{-1}(23)\sigma \neq (1). \square$$

Claims 19, 20, 21 contradict Lemma 10, which proves that I_2 does not hold on **G**. □

Lemma 13. *I_{13} is not true on* **G**.

Proof. Dual of the proof of Lemma 12. □

Lemma 14. *If I_2 holds on* **G** *then*

$$x(xy) \neq y.$$

Proof. Suppose the opposite, i.e. let

$$x(yz) = y(xz)$$
$$x(xy) = y$$

hold on **G**.

Claim 22. $(yx)x = x$.

Proof. $(yx)x = (yx)(y(yx)) = y((yx)(yx)) = y(yx) = x$.□

Claim 23. $(xy)x \notin \{x, y, xy, yx\}$.

Proof. Obiously, $x(yx) = yx$ and if

$$(xy)x \in \{x, y, xy, yx\},$$

then

$$(xy)(yx) = y((xy)x) \in \{x, y, xy, yx\},$$

i.e. the set $\{x, y, xy, yx\}$ is closed under the multiplication of polynomials which is impossible since $p_2(\mathbf{G}) = 4$. □

Claim 24. $(xy)(xz)$ *is an essentially 3-ary polynomial.*

Proof. Follows from the substitutions $y = x$ and $y = z$, and Lemma 1. □

Claim 25. $(xy)(xz) \notin \{p^\sigma | \sigma \in S_3\} \cup \{q^\sigma | \sigma \in S_3\}$.

Proof. Follows from

$$(xy)(xz) = x(yz) \Rightarrow (xy)x = yx \text{ for } z = x$$
$$(xy)(xz) = x(zy) \Rightarrow (xy)x = y \text{ for } z = x$$
$$(xy)(xz) = y(zx) \Rightarrow (xy)x = yx \text{ for } z = x$$
$$(xy)(xz) = (xy)z \Rightarrow z = xz \text{ for } y = x$$
$$(xy)(xz) = (xz)y \Rightarrow (xy)x = xy \text{ for } z = x$$
$$(xy)(xz) = (yx)z \Rightarrow (xy)x = x \text{ for } z = x$$
$$(xy)(xz) = (yz)x \Rightarrow (xy)x = x \text{ for } z = x$$
$$(xy)(xz) = (zx)y \Rightarrow (xy)x = xy \text{ for } z = x$$
$$(xy)(xz) = (zy)x \Rightarrow z = x \text{ for } y = x$$

(contradicts Claim 23 and Lemma 1). □

Claims 3 and 4 contradict Lemma 10. □

Lemma 15. *If I_2 holds on* **G** *then*

$$x(xy) \in \{x, y, xy, yx\}.$$

Proof. Suppose the opposite, i.e. let

$$x(yz) = y(xz)$$
$$x(xy) \notin \{x, y, xy, yx\}.$$

Claim 26. *The polynomial $x(x(yz))$ is essentially 3-ary.*

Proof. For $z = x$ and $z = y$ we obtain the polynomials yx and $x(xy)$ which imply dependence on y and x. From $yx \neq x(xy)$ it follows dependence on z. □

Claim 27. $x(x(yz)) = (xy)z$.

Proof. According to Lemma 10

$$x(x(yz)) \in \{p^\sigma | \sigma \in S_3\} \cup \{q^\sigma | \sigma \in S_3\}.$$

The cases

$$x(x(yz)) = x(yz) \Rightarrow x(xy) = xy \text{ for } z = y$$
$$x(x(yz)) = x(zy) \Rightarrow x(xy) = xy \text{ for } z = y$$
$$x(x(yz)) = y(zx) \Rightarrow x(xy) = y(yx) \text{ for } z = y$$
$$x(x(yz)) = (xz)y \Rightarrow yx = xy \text{ for } z = x$$
$$x(x(yz)) = (yx)z \Rightarrow yx = (yx)x \text{ (for } z = x) \Rightarrow$$
$$\Rightarrow x(xy) = x((xy)y) = (xy)(xy) = xy$$
$$x(x(yz)) = (yz)x \Rightarrow x(xy) = yx \text{ for } z = y$$
$$x(x(yz)) = (zx)y \Rightarrow yx = xy \text{ for } z = x$$
$$x(x(yz)) = (zy)x \Rightarrow x(xy) = yx \text{ for } z = y$$

lead to a contradiction with the assumption on $x(xy)$, Lemma 11 and Lemma 1. So it has to be

$$x(x(yz)) = (xy)z. \square$$

Claim 28. $x(xy) = (xy)y$, $(xy)x = yx$, $x(x(xy)) = xy$.

Proof. Follows from Claim 27 for $z = y, z = x, y = x$ respectively. □

Claim 29. *The polynomial* $(yz)x)x$ *is essentially 3-ary.*

Proof. For $y = z$ and $y = x$ we obtain twice the polynomial $z(zx)$ from which follows the dependence on x and z. If $((yz)x)x$ does not depend on y then

$$((yx)x)x = x.$$

This implies

$$x = ((yx)x)x = (yx)((yx)x) = (yx)(y(yx)) = y((yx)(yx)) = y(yx)$$

(we use Claim 28) which contradicts the assumption. □

Claim 30. $((yz)x)x \notin \{p\sigma|\sigma \in S_3\} \cup \{q^\sigma|\sigma \in S_3\}$.

Proof. Suppose the opposite. Then one of the following cases hold

$$\begin{aligned}
((yz)x)x = x(yz) &\Rightarrow y(yx) = xy \text{ for } z = y\\
((yz)x)x = x(zy) &\Rightarrow y(yx) = xy \text{ for } z = y\\
((yz)x)x = y(zx) &\Rightarrow z(zx) = zx \text{ for } y = x\\
((yz)x)x = (xy)z &\Rightarrow y(yx) = x(xy) \text{ for } z = y\\
((yz)x)x = (xz)y &\Rightarrow y(yx) = x(xy) \text{ for } z = y\\
((yz)x)x = (yx)z &\Rightarrow y(yx) = xy \text{ for } z = y\\
((yz)x)x = (yz)x &\Rightarrow y(yx) = yx \text{ for } z = y\\
((yz)x)x = (zx)y &\Rightarrow y(yx) = xy \text{ for } z = y\\
((yz)x)x = (zy)x &\Rightarrow y(yx) = yx \text{ for } z = y
\end{aligned}$$

(we use Claim 28). However, all these cases lead to contradictions with the assumption $x(xy) \notin \{x, y, xy, yx\}$ and Lemma 11. □

Claim 29 and Claim 30 contradict Lemma 10 so that assertion of our lemma follows. □

Lemma 16. *If I_2 holds on* **G** *then* $x(xy) = xy$.

Proof. From Lemma 15 we have

$$x(xy) \in \{x, y, xy, yx\}$$

and from Lemma 14

$$x(xy) \neq y.$$

We eliminate the other two posibilities for $x(xy)$ in the following way

$$x(xy) = x \Rightarrow x = x(x(yz)) = x(y(xz)) = y(x(xz)) = yx$$
$$x(xy) = yx \Rightarrow y(yx) = y(x(xy)) = x(y(xy)) = x(x(yy)) = x(xy)$$

(contradicts Lemma 1 and Lemma 11). □

Lemma 17. *If I_2 holds on* **G** *then*

$$(xy)x \notin \{x, y, xy, yx\}.$$

Proof. Suppose the opposite, i.e. that on **G** we have

$$x(yz) = y(xz)$$
$$(xy)x \notin \{x, y, xy, yx\}.$$

Claim 31. *The polynomial $(x(yz))x$ is not essentially 3-ary.*

Proof. Suppose the opposite. Then

$$(x(yz))x \in \{p^\sigma | \sigma \in S_3\} \cup \{q^\sigma | \sigma \in S_3\}$$

(Lemma 10). However, all the cases

$$(x(yz))x = x(yz) \Rightarrow (xy)x = xy \text{ for } z = y$$
$$(x(yz))x = x(zy) \Rightarrow (xy)x = xy \text{ for } z = y$$
$$(x(yz))x = y(zx) \Rightarrow (xy)x = y(yx) = yx \text{ for } z = y$$
$$(x(yz))x = (xy)z \Rightarrow (xz)x = xz \text{ for } y = x$$
$$(x(yz))x = (xz)y \Rightarrow (yx)x = xy \text{ (for } z = x) \Rightarrow$$
$$\Rightarrow yx = (yx)(yx) = y((yx)x) = y(xy) = xy$$
$$(x(yz))x = (yx)z \Rightarrow (xy)x = (yx)y \text{ for } z = y$$
$$(x(yz))x = (yz)x \Rightarrow (xy)x = yx \text{ for } z = y$$
$$(x(yz))x = (zx)y \Rightarrow (yx)x = xa \text{ (for } z = x) \Rightarrow$$
$$\Rightarrow yx = (yx)(yx) = y((yx)x) = xy$$
$$(x(yz))x = (zy)x \Rightarrow (xy)x = yx \text{ for } z = y$$

lead to a contradiction with the assumption about $(xy)x$ and Lemma 11. □

Claim 32. $(yx)x = x$.

Proof. for $y = z$ and $y = x$ we obtain the polynomial $(xz)x$, which is essentially binary by the assumption. Therefore, the polynomial $(x(yz))x$ depends on x and z so that, according to Claim 31 does not depend on y. It follows that

$$(x(yz))x = (xz)x$$

(for $y = z$), i.e.

$$(yx)x = x$$

(for $z = x$). □

Claim 33. $(xy)(zy) = zy$.

Proof. $(xy)(zy) = z((xy)y) = zy$. □

Claim 34. *No identity of the form $f = f^\sigma$, where $f = (xy)(zu)$, $\sigma \in S_4$, $\sigma \neq (1)$, holds on* **G**.

Proof. Suppose the opposite. According to Lemma 5(iii) one of the following identities is true on **G**

$$(xy)(zu) = (xy)(uz) \Rightarrow x(zu) = x(uz) \text{ for } y = x$$
$$(xy)(zu) = (xz)(yu) \Rightarrow zy = (xy)(zy) = (xz)(yy) = (xz)y$$
$$(xy)(zu) = (xz)(uy) \Rightarrow zy = (xy)(zy) = (xz)(yy) = (xz)y$$
$$(xy)(zu) = (xu)(yz) \Rightarrow (xy)z = (xy)(zz) = (xz)(yz) = yz$$
$$(xy)(zu) = (xu)(zy) \Rightarrow (xy)x = xy \text{ for } u = x, z = x$$
$$(xy)(zu) = (yx)(zu) \Rightarrow (xy)z = (yx)z \text{ for } u = z$$
$$(xy)(zu) = (yz)(xu) \Rightarrow (xy)z = (xy)(zz) = (yz)(xz) = xz$$
$$(xy)(zu) = (zx)(yu) \Rightarrow zy = (xy)(zy) = (zx)(yy) = (zx)y$$
$$(xy)(zu) = (zy)(xu) \Rightarrow zy = (xy)(zy) = (zy)(xy) = xy$$

However, it is easy to see that all these cases lead to a contradiction with the fact that no identity I_1-I_{16}, except I_2, holds on **G** (Lemma 10), or with the assumption that p and q are essentially 3-ary polynomials, or with $(xy)x \notin \{x, y, xy, yx\}$, or with Lemma 1. □

Since Claim 34 contradicts Lemma 5 the assertion of our Lemma holds. □

Lemma 18. *If I_2 is true on* **G**, *then*

$$(yx)x \notin \{x, y, xy, yx\}.$$

Proof. Follows from $p_2(\mathbf{G}) = 4$ and Lemmas 16, 17, $x(yx) = yx$ and $(xy)(yx) = y((xy)x) \in \{x, y, xy, yx\}$. □

Lemma 19. *If I_2 holds on* **G** *then*

$$(xy)x = yx.$$

Proof. According to Lemma 17

$$(xy)x \in \{x, y, xy, yx\}.$$

The posibilities

$$(xy)x = x \Rightarrow (yx)x = (x(yx))x = x$$
$$(xy)x = y \Rightarrow y = (xy)x = x((xy)x) = xy$$
$$(xy)x = xy \Rightarrow (yx)x = (x(yx))x = x(yx) = yx$$

do not hold because of the contradiction with Lemma 18 and Lemma 1, so that

$$(xy)x = yx.$$

□

Lemma 20. *I_2 does not hold on* **G**.

Proof. Suppose the opposite, i.e. let

$$x(yz) = y(xz)$$

be true on **G**. From Lemmas 16, 18, 19 and identity I_2 it follows that

$$x(xy) = xy$$
$$x(yx) = yx$$
$$(xy)x = yx$$
$$(yx)x \notin \{x, y, xy, yx\}.$$

Consider the polynomial $(x(zy))y$. We prove that it is essentially 3-ary by substituting $y = x$ and $y = z$ and using Lemma 11.

On the other hand

$$(x(zy))y \notin \{p^\sigma | \sigma \in S_3\} \cup \{q^\sigma | \sigma \in S_3\}$$

because in all the cases

$$\begin{aligned}
(x(zy))y = x(yz) &\Rightarrow (xy)y = xy \text{ for } z = y\\
(x(zy))y = x(zy) &\Rightarrow (xy)y = xy \text{ for } z = y\\
(x(zy))y = y(zx) &\Rightarrow (xy)y = yx \text{ for } z = x\\
(x(zy))y = (xy)z &\Rightarrow (zx)x = xz \text{ for } y = x\\
(x(zy))y = (xz)y &\Rightarrow (xy)y = xy \text{ for } z = x\\
(x(zy))y = (yx)z &\Rightarrow (zx)x = xz \text{ for } y = x\\
(x(zy))y = (yz)x &\Rightarrow (xy)y = yx \text{ for } z = y\\
(x(zy))y = (zx)y &\Rightarrow (xy)y = xy \text{ for } z = x\\
(x(zy))y = (zy)x &\Rightarrow (xy)y = yx \text{ for } z = y
\end{aligned}$$

we obtain

$$(yx)x \in \{x, y, xy, yx\}.$$

This contradiction with Lemma 18 proves our lemma. □

Lemma 21. *I_{12} is not true on* **G**.

Proof. This is the dual of Lemma 20. □

Lemma 22. *I_5 is not true on* **G**.

Proof. Suppose I_5 holds on **G**, i.e.

$$x(yz) = z(yx).$$

Claim 35. $x(xy) = yx$, $(xy)x = x(yx)$.

Proof.

$$\begin{aligned}
x(xy) &= y(xx) = yx\\
(xy)x &= x(x(xy)) = x(yx). \square
\end{aligned}$$

Claim 36. $x(yx) \notin \{x, y, xy, yx\}$.

Proof. In the opposite case we have the following possibilities

$$x(yx) = x \Rightarrow (yx)x = x(x(yx)) = xx = x$$
$$x(yx) = y \Rightarrow (yx)x = x(x(yx)) = xy$$
$$x(yx) = xy \Rightarrow (yx)x = x(x(yx)) = x(xy) = yx$$
$$x(yx) = yx \Rightarrow (yx)x = x(x(yx)) = x(yx) = yx$$

which, according to Claim 35, lead to the conclusion that the set $\{x, y, xy, yx\}$ is closed under the multiplication of polynomials. □

Claim 37. *The polynomial $(zy)(x(yz))$ is essentially 3-ary.*

Proof. Use the substitutions $y = x$ and $y = z$ and Claim 36. □

Claim 38. $(zy)(x(yz)) = y(xz)$.

Proof. According to Lemma 10

$$(zy)(x(yz)) \in \{p^\sigma | \sigma \in S_3\} \cup \{q^\sigma | \sigma \in S_3\}.$$

All the other possibilities are not true because they contradict Claim 36 or Lemma 1 in the following way

$$(zy)(x(yz)) = x(yz) \Rightarrow y(xy) = xy \text{ for } z = y$$
$$(zy)(x(yz)) = x(zy) \Rightarrow y(xy) = xy \text{ for } z = y$$
$$(zy)(x(yz)) = (xy)z \Rightarrow yx = (xy)x = x(yx) \text{ for } z = x$$
$$(zy)(x(yz)) = (xz)y \Rightarrow yx = xy \text{ for } z = x$$
$$(zy)(x(yz)) = (yx)z \Rightarrow zx = xz \text{ for } y = x$$
$$(zy)(x(yz)) = (yz)x \Rightarrow y(xy) = yx \text{ for } z = y$$
$$(zy)(x(yz)) = (zx)y \Rightarrow yx = xy \text{ for } z = x$$
$$(zy)(x(yz)) = (zy)x \Rightarrow y(xy) = yx \text{ for } z = y. \square$$

Claim 39. *The polynomial $(xy)((xz)z)$ is essentially 3-ary.*

Proof. Use substitutions $z = x$ and $z = y$ and apply Claim 36 and Lemma 11. □

Claim 40. *$(xy)((xz)z) = (xy)z$ or $(xy)((xz)z) = (yx)z$.*

Proof. Lemma 10 implies

$$(xy)((xz)z) \in \{p^\sigma | \sigma \in S_3\} \cup \{q^\sigma | \sigma \in S_3\}.$$

All the other possibilities do not hold since they contradict Claim 36 in the following way

$$\begin{aligned}
(xy)((xz)z) = x(yz) &\Rightarrow y(xy) = xy \text{ for } z = y \\
(xy)((xz)z) = x(zy) &\Rightarrow y(xy) = xy \text{ for } z = y \\
(xy)((xz)z) = y(xz) &\Rightarrow (xy)x = yx \text{ for } z = x \\
(xy)((xz)z) = (xz)y &\Rightarrow (xy)x = xy \text{ for } z = x \\
(xy)((xz)z) = (yz)x &\Rightarrow y(xy) = yx \text{ for } z = y \\
(xy)((xz)z) = (zx)y &\Rightarrow (xy)x = xy \text{ for } z = x \\
(xy)((xz)z) = (zy)x &\Rightarrow y(xy) = yx \text{ for } z = y. \square
\end{aligned}$$

Claim 41. *$(xy)y = y(xy)$.*

Proof. This is a consequence of Claim 40 for $z = y$ in the first case and $z = x$ in the second. □

Claim 42. *$(xy)(yx) = (yx)(xy)$.*

Proof. According to Claim 38 and Claim 41 we have

$$(xy)(yx) = (xy)((yx)(yx)) = y((yx)x) = y(x(yx)) = (yx)(xy),$$

which proves this Claim. □

The assertion of Claim 42 is in contradiction with Lemma 11 and this proves assertion of Lemma. □

Lemma 23. *I_{16} is not true on* **G**.

Proof. Dual of Lemma 22. □

Proof of Theorem 2.
Follows from Lemmas 10, 12, 13, 20, 21, 22, 23. □

4. Proof of the Main Theorem

It was proved in [2] that from (i) follows $p_n(\mathbf{G}) = n^2$, $n \geq 0$. In [1] it is proved that (iii) implies $p_n(\mathbf{G}) = n^2$, $n \geq 0$. The proof that (ii) implies $p_n(\mathbf{G}) = n^2$, $n \geq 0$ is the dual of the proof given in [1].

Suppose $p_n(\mathbf{G}) = n^2$. If the polynomial $x(yz)$ is not essentially 3-ary then (ii) holds (Theorem 1). If the polynomial $(xy)z$ is not essentially 3-ary then (iii) holds (dual of Theorem 1). If **G** is a semigroup then (i) holds (see [1]). Theorem 2 claims that it is not possible for **G** not to be a semigroup and both of the polynomials $x(yz)$ and $(xy)z$ to be essentially 3-ary. □

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REZIME

O GRUPOIDIMA KOJI IMAJU n^2 ESENCIJALNIH n-ARNIH POLINOMA

U radu je pokazano da samo pravougaoni grupoidi i normalne trake imaju p_n nizove oblika $(0, 1, 4, \ldots, n^2, \ldots)$. Time je data potpuna karakterizacija grupoida sa osobinom $p_n(\mathbf{G}) = n^2$.

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