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ON WEAK PARTIAL £-VALUED CONGRUENCE ALGEBRAS

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Abstract

In this paper for a given algebra $\mathcal{A}$, the notation of a weak partial £-valued (fuzzy) congruence algebra $\overline{K_\omega(\mathcal{A})} = (\overline{C_\omega(A)}, \wedge, \vee, \circ, \sigma, \Delta, A^2)$, is introduced. $\overline{C_\omega(A)}$ is a weak £-valued (fuzzy) congruence relation on an $\mathcal{A}$, defined in [1]. $\overline{K_\omega(\mathcal{A})}$ gives more information on $\mathcal{A}$ just lattice $\overline{C_\omega(\mathcal{A})}$.

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1.

1. Let $\mathcal{A} = (A, F)$ be an algebra and $K \subseteq A$ be the set of its constants (if $K \neq \emptyset$, then we consider the empty set as a subalgebra of $\mathcal{A}$). Let $\pounds = (L, \wedge, \vee, 1, 0)$ be a complete lattice. All £-valued sets here are mappings from A (or A^2 in the case of £-valued relations) to L. The set A and its subsets are identified with their characteristic functions (0, and 1 are from L). Thus, $K : A \to L$, and $K(x) = 1$ if $x \in K$. Otherwise, $K(x) = 0$. A £-valued subalgebra of $\mathcal{A}$ is any mapping $\bar{B} : A \to L$, such that

(a) $K \subseteq \bar{B}$

(b) $\bar{B}(f(x_1,\ldots,x_n)) \geq \bar{B}(x_1) \wedge \ldots \wedge \bar{B}(x_n)$,

for all $x_1,\ldots,x_n \in A, f \in F_n \subseteq F,\ n \in N$.
The set of all £-valued subalgebras on $\mathcal{A}$ is denoted by $\overline{S(\mathcal{A})}$. A weak £-valued congruence on $\mathcal{A}$ is a mapping $\bar{\rho} : A^2 \to L$, such that ([1]):

i. For every $c \in K, \bar{\rho}(c,c) = 1$ (reflexivity);
ii. For all $x, y \in A, \bar{\rho}(x,y)$ (symmetry);
iii. For all $(x,y) \in A, \bar{\rho}(x,y) \geq \bigvee_{z\in A} (\bar{\rho}(x,z) \wedge \bar{\rho}(z,y))$ (transivity)
iv. For all $x_1,\ldots,x_n,\ y_1,\ldots,y_n \in A,\ f \in F_n \subseteq F,\ n \geq 1$,

$$\bar{\rho}(f(x_1,\ldots,x_n),\ f(y_1,\ldots,y_n)) \geq \bigwedge_{i=1}^{n} \bar{\rho}(x_i,y_i) \text{ substitution}$$

The set of all weak £-valued congruence relations on $\mathcal{A}$ is denoted by $\overline{C_\omega(\mathcal{A})}$. If (i) is replaced by

(i'): for every $x \in A,\ \bar{\rho}(x,x) = 1$ (reflexivity),
then $\bar{\rho}$ is a £-valued congruence relation on $\mathcal{A}$, and the set of all such relations on $\mathcal{A}$ is denoted by $C(\mathcal{A})$.

2.

$(\overline{C_\omega(\mathcal{A})}, \leq)$ is a complete lattice (where $\bar{\rho} \leq \bar{\theta}$, iff for every $x, y \in A,\ \bar{\rho}(x,y) \leq \bar{\theta}(x,y)$), having as a sublattice the lattice $(\overline{C(\mathcal{A})}, \leq)$ and the lattice $(\overline{S(\mathcal{A})}, \leq)$. ([1]).

c) If $\bar{\rho}$ and $\bar{\theta}$ are two arbitrary fuzzy relations on A, then $\bar{\rho} \circ \bar{\theta} : A^2 \to L$, and for $x, y \in A$,

$$\bar{\rho} \circ \bar{\theta}(x,y) = \bigvee_z (\bar{\rho}(x,y) \bigwedge \bar{\theta}(z,y)).$$

3.

If £ is complete and infinitely distributive, then $\circ$ is an associative operation in the set of all £-valued fuzzy on A, (see [4]).

In the following, £ is complete and infinitely distributive.

2. Let us give an algebra $\mathcal{A}$. By its weak partial £-valued congruence algebra we mean a partial algebra:

$$\overline{K_\omega(\mathcal{A})} = (\overline{C_\omega(A)}, \wedge, \vee, \circ, \bar{\sigma}, \Delta, A^2)$$

where:

$$(\overline{C_\omega(A)}, \wedge, \vee) = \overline{C_\omega(A)},$$

$\circ$- is a composition of £-valued relations (binary partial operation on $\overline{C_\omega(A)}$, $\bar{\sigma}$) is a weak £-valued congruence relation on A, such that for $x, y \in A$:

$$\bar{\sigma}(x,y) \stackrel{def}{=} \begin{cases} \bar{B}_m(x), & x = y \\ 0, & \text{othervise} \end{cases}$$

where $\bar{B}_m$ is the least £-valued subalgebra of $\mathcal{A}$ and

$$\Delta(x,y) \stackrel{def}{=} \begin{cases} 1, & x = y \\ 0, & \text{otherwise.} \end{cases}$$

The next example shows that $\overline{K_\omega(\mathcal{A})}$ gives more information on a just lattice $\overline{C_\omega(\mathcal{A})}$.

Example 1. *Let us consider the following algebras $\mathcal{A}_1, \mathcal{A}_2$ both with the same domanian $B = \{0,1,2,3,4,5\}$ which is also the set of constants,*

$$\mathcal{A}_1 = (B, *_1, f_1, B);\ \mathcal{A}_2 = (B, *_2, f_2, B)$$

$*_1$	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

f_1	0	1	2	3	4	5
	0	1	2	3	4	5

$*_2$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	0	0	0	0	0	0
4	0	0	0	0	0	0
5	0	0	0	0	0	0

f_2	0	1	2	3	4	5
	1	2	3	4	5	1

and where $\mathcal{L} = (\{1,0)\}, \wedge, \vee, 0, 1)$.

It is easy to verify that

$$\overline{C_\omega(\mathcal{A}_1)} \cong \overline{C_\omega(\mathcal{A}_2)},$$

while

$$\overline{K_\omega(\mathcal{A}_1)} \ncong \overline{K_\omega(\mathcal{A}_2)}.$$

The following theorem holds:

Theorem 1. *Let $\mathcal{A}$ be an algebra and $\overline{K_\omega(\mathcal{A})}$ its weak partial $\mathcal{L}$-valued congruence algebra. Then for any $\bar{\rho}, \bar{\theta} \in \overline{C_\omega(\mathcal{A})}$ it holds that:*

(1) $\Delta \circ (\Delta \vee \bar{\rho}) = \Delta \vee \bar{\rho}$,

(2) $\Delta \circ (\Delta \wedge \bar{\rho}) = \Delta \wedge \bar{\rho}$,

(3) $\bar{\rho} \circ \bar{\theta} \in \overline{C_\omega(\mathcal{A})}$, *iff* $\bar{\rho} \circ \bar{\theta} = \bar{\theta} \circ \bar{\rho}$

Proof. (1) for all $x, y \in A$ we have:

$$(\Delta \circ (\Delta \vee \bar{\rho}))(x,y) = \bigvee_{z \in A} (\Delta(x,z) \wedge (\Delta \vee \bar{\rho}(z,y)) =$$

$$= (\Delta(x,x) \wedge (\Delta \vee \bar{\rho})(x,y)) \vee \bigvee_{z \neq x} (\Delta(x,z) \wedge (\Delta \vee \bar{\rho})(z,y)) = (\Delta \vee \bar{\rho})(x,y):$$

(2) similarly as (1);

(3) see p.2.1. ([2]). □.

Now, we shall give the necessary and sufficient conditions that weak partial $\mathcal{L}$-valued congruence algebra $\overline{K_\omega(\mathcal{A})}$ of a given algebra $\mathcal{A}$ is just an algebra.

Theorem 2. *A weak partial $\mathcal{L}$-valued congruence algebra $\overline{K_\omega(\mathcal{A})}$ is an algebra iff following is satisfied:*

(1) for each $\bar{\rho}, \bar{\theta} \in \overline{C(\mathcal{A})}$ there holds: $\bar{\rho} \circ \bar{\theta} = \bar{\theta} \circ \bar{\rho}$;

(2) there holds: either $\overline{S(\mathcal{A})} = \{A\}$ or $\overline{S(\mathcal{A})} = \{\emptyset, A\}$.

Proof.

($\Leftarrow$) If conditions (1) and (2) are satisfied, then it follows immediately that $\overline{K_\omega(\mathcal{A})}$ is an algebra.

($\Rightarrow$) If $\overline{S(\mathcal{A})} \neq \{A\}$ and $\overline{S(\mathcal{A})} \neq \{\emptyset, A\}$, then there exists $\bar{B} \in \overline{S(\mathcal{A})}$ such that $\bar{B} \leq A$. By L 1.4. ([1]) we have $\bar{B}^2 \in \bar{C}_\omega(A)$. Now, it is easy to verify that $\bar{B} \circ A^2 \neq A^2 \circ \overline{B^2}$, and so, by Theorem 1. (3)., $\overline{B^2} \circ A^2 \notin \overline{C_\omega(A)}$. □

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REZIME

O SLABIM PARCIJALNIM £-VREDNOSNIM KONGRUENCIJSKIM ALGEBRAMA

U radu se uvodi slaba parcijalna £-vrednosna (rasplinuta) kongruencijska algebra $\overline{K_\omega(\mathcal{A})}$ za datu algebru. Ispitivana su neka svojstva takve parcijalne algebre.

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