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ON NUMERICAL METHODS FOR QUASILINEAR SINGULAR PERTURBATION PROBLEMS WITHOUT TURNING POINTS

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Abstract

A brief survey of numerical methods for quasilinear singularly perturbed boundary value problems without turning points is given. A new method is proposed for which the first order pointwise accuracy uniform in the perturbation parameter is proved.

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1. Introduction

We shall consider the following singularly perturbed boundary value problem:

$$-\varepsilon u'' - b(u)u' + c(x,u) = 0, \quad x \in I = [0,1], \tag{1.a}$$

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$$(1.b) \qquad u(0) = U_0, \quad u(1) = U_1,$$

with the hypotheses which will be assumed throughout the paper:

$$\begin{aligned} &\text{H1.} && \varepsilon \in (0,1], \quad U_0, U_1 \in \mathbf{R}, \\ & && b \in C^3(R), \quad c \in C^3(I \times \mathbf{R}), \\ &\text{H.2} && c_u(x,u) \geq c_* > 0, \quad x \in I, \quad u \in \mathbf{R}, \\ &\text{H.3} && b(u) \geq b_* > 0, \quad u \in W := [u_*, u^*], \end{aligned}$$

where $u_* \leq u^*$ and

$$(2) \qquad c(x,u^*) \geq 0 \geq c(x,u_*), \quad x \in I, \quad u^* \geq U_j \geq u_*, \quad j = 0,1.$$

(Note that u_* and u^* exist because of H2.) Thus u^* and u_* are upper and lower solutions, respectively, to the problem (1) and it follows that (1) has a unique solution $u_\varepsilon \in C^5(I)$ satisfying

$$u_\varepsilon(x) \in W, \quad x \in I.$$

Because of H3 the following estimates hold, see [6]:

$$(3) \qquad |u_\varepsilon^{(k)}| \leq M(1 + \varepsilon^{-k} \exp(-b_* x/\varepsilon)), \quad x \in I, \quad k = 0(1)4.$$

(Here and throughout the paper M denotes any positive constant independent of ε. When dealing with numerical methods these constant will be independent of h as well, where $h = 1/n, n$ being the number of steps of discretization meshes.) Since the estimates (3) are sharp, we can see that u_ε has a boundary layer at the origin. Because of that some special methods for numerical solution of the problem (1) should be applied.

All our references deal with numerical solution of problems of type (1) (in some of the papers assumptions are somewhat different). A survey of results of these papers will be given in the next section. Although in practice there are several numerical methods for the problem (1), we might say that there is a gap in the theory: with the exception of [11] there are no proofs of the uniform (= uniform in ε) pointwise convergence (i.e. the convergence of the numerical solution towards the discretization of u_ε). Since the method from [11] has not been analysed completely, the question of the uniform pointwise convergence is still interesting from theoretical point of view. In this paper in Section 3 we shall present a numerical method for which we shall prove the first order uniform pointwise convergence. Since this is of theoretical interest only, we choose not to give numerical results.

2. A survey of numerical methods

We shall briefly describe main results of paper [1–11].

Papers [4] and [5] do not deal with the uniform convergence in the above sense: the second and third (respectively) order pointwise convergence is shown outside of the layer only.

In [3] the global (not pointwise) uniform convergence was proved in the L^1-norm. The same result was proved in [6] but numerical experiments showed the first order pointwise convergence as well, which is not the case for the method from [3]. A result similar to [6] is obtained in [7] by a somewhat different method. Both [6] and [7] use the upwind finite-difference scheme and a discretization mesh dense in the layer. Even more satisfactory from practical point of view is paper [8] since there numerical results show the second order pointwise convergence. However, only the uniform stability of the method was proved in [8].

In [10] an exponentially fitted scheme was proposed but only existence and uniqueness of a solution to the discrete problem was investigated. Another approach was used in [11]: the continuous problem was approximated by a problem with piecewise constant coefficients for which an exact scheme was derived. The first order uniform pointwise convergence was proved. However, the resulting discrete problem is complicated and some open questions remain, e.g. how to solve it.

Papers [1], [2] and [9] make use of the reduced problem

$$-b(u)u' + c(x,u) = 0, \quad x \in I, \quad u(1) = U_1, \tag{4}$$

with a unique solution u_0 which is a good approximation to u_ε outside of the layer. In [2] the point $k\varepsilon$ was introduced, $[0, k\varepsilon]$ representing the layer. Then (4) was solved in $[k\varepsilon, 1]$ by the fourth order Runge–Kutta method, and finally (1a) was solved in $[0, k\varepsilon]$ (with the boundary conditions $u(0) = U_0$ and $u(k\varepsilon) = U_0(k\varepsilon)$) by using the central differences. This resulted in an error with the terms $\exp(-kb_*)$ and $k^2\varepsilon$, hence the uniform pointwise convergence was not proved.

A different approach was used in [1] and a $O(\varepsilon + h)$–error was proved. This result is improved to $O(\varepsilon + h^2)$ in [9]. The exponential fitting and equidistant meshes were used in [1], while [9] uses a switching scheme and a special non-equidistant mesh.

The method which will be presented here is essentially a combination of approaches from [1] and [6]. Now we shall give its basic discription. Details will be given in the next section.

The idea from [1] was to consider an initial value problem related to (1). In [9], the initial value problem was written in the following form:

$$\varepsilon u' + f(u) = f(U_0(x)), \quad x \in I, \quad u(0) = U_0, \tag{5}$$

(a different form was used in [2]). Here and throughout the paper we take:

$$f(u) = \int_{u_*}^{u} b(s)\, ds. \tag{6}$$

In [2] and [9] it was shown that

$$|\tilde{u}_\varepsilon(x) - u_\varepsilon(x)| \le M\varepsilon, \quad x \in I, \tag{7}$$

where $\tilde{u}_\varepsilon$ is a unique solution to the problem (5).
Thus, if (5) is solved with a pointwise accuracy $O(h)$, we get the total pointwise error $O(\varepsilon + h)$. This is quite satisfactory in practice since usually $\varepsilon \ll h$. However, if we consider all values of ε from (0,1] the error gets bad for greater values of ε, and obviously, this result does not mean the uniform pointwise convergence. It was noted in [1] that the method described should be combined with another method when ε is not small, but such a method was not specified. (Note that the combination cannot have accuracy of order greater than 1, and that is exactly what we shall get here.) We shall propose a particular method which will give the error $O(h)$ when $h \le M^*\varepsilon$, where M^* is a constant bounded both from above and below indepenedently of ε and h. The method is essentially the same as the one from [6] except that here we shall use the central scheme instead of the upwind scheme. We shall introduce a new independent variable t, transform the problem (1) and solve it numerically on equidistant t-mesh. The problem (5) will be treated in a similar way when $h > M_*\varepsilon$, and then we shall get the error $O(\varepsilon + h) = O(h)$. In this way, the combination of the two methods gives the first order uniform pointwise convergence.

3. The uniform pointwise convergence result

Let us rewrite the equation (1a) in the conservation form:

$$-\varepsilon u'' - f(u)' + c(x, u) = 0,$$

where f is given in (6). Let us then introduce new variables t and y:

$$x = \lambda(t), \quad y(t) = u(\lambda(t)),$$

with

$$\lambda(t) = \begin{cases} \omega(t) := a\varepsilon[(\frac{Q}{Q-t})^8 - 1], & t \in [0, \alpha], \\ \pi(t) := A(t-\alpha)^3 + \frac{1}{2}\omega''(\alpha)(t-\alpha)^2 + \omega'(\alpha)(t-\alpha) & \\ +\omega(\alpha), & t \in [\alpha, 1], \end{cases}$$

where $\alpha \in (0,1)$ and $Q = \alpha + \varepsilon^{1/12}$. The coefficient A is determined from the condition $\pi(1) = 1$ and $a > 0$ has to be chosen so that $A \geq 0$. Thus $\lambda \in C^2(I)$ and

(8) $$\lambda^{(k)}(t) > 0, \quad k = 0, 1, 2, \quad t \in I.$$

The problem (1) is transformed to the following problem, c.f. [6]:

(1') $$-\varepsilon(\mu(t)y')' - f(y)' + q(t,y) = 0, \quad t \in I, \quad y(0) = U_0, \quad y(1) = U_1,$$
where now $I' = d/dt$, and:

$$\mu(t) = 1/\lambda'(t), \quad q(t,y) = c(\lambda(t), y)\lambda'(t).$$

In the same way we transform (5):

(5') $$\varepsilon\mu(t)y' + a(t,y) = 0, \quad t \in I, \quad y(0) = U_0, \text{ where}$$

$$a(t,y) = f(y) - f(U_0(\lambda(t))).$$

Let I^h be an equidistant t-mesh with the points

$$t_i = ih, \quad i = 0(1)n, \quad h = 1/n, \quad n \in \mathbf{N}\backslash\{1\}.$$

By w^h, v^h etc. we shall denote mesh functions on $I^h\backslash\{0,1\}$. They will be identified with $\mathbf{R}^{n-1}$–vectors.
In particular, we take:

$$u_\varepsilon^h = [u_\varepsilon(\lambda(t_1)), u_\varepsilon(\lambda(t_2)), \ldots, u_\varepsilon(\lambda(t_{n-1}))]^T,$$

$$\tilde{u}_\varepsilon^h = [\tilde{u}_\varepsilon(\lambda(t_1)), \tilde{u}_\varepsilon(\lambda(t_2)), \ldots, \tilde{u}_\varepsilon(\lambda(t_{n-1}))]^T,$$

$$e^h = [1, 1, \ldots, 1]^T.$$

Let

$$\|w^h\|_\infty = \max_{1 \leq i \leq n-1} |w_i|, \quad (w_i := w_i^h),$$

$$\|w^h\|_1 = h\sum_{i=1}^{n-1}|w_i|.$$

The corresponding matrix norms will be denoted in the same way. Let

$$W^h = \{w^h \in \mathbf{R}^{n-1} : u_* e^h \le w^h \le u^* e^h\},$$

(the inequality sign in $\mathbf{R}^{n-1}$ should be understood componentwise). Finally, let

$$M^* = \frac{2}{\lambda'(1)b^*}$$

where

$$b(u) \le b^*, \quad u \in W.$$

It is obvious that

$$M_1 \le M^* \le M_2,$$

where M_1 and M_2 are positive constants independent of ε and h.

We shall use the central discretization of (1'):

$$\begin{aligned}(9)\quad T_i w^h : &= -\varepsilon h^{-2}[\mu_{i-1/2}w_{i-1} - (\mu_{i-1/2}+\mu_{i+1/2})w_i \\ &+ \mu_{i+1/2}w_{i+1}] - [f(w_{i+1}) - f(w_{i-1})]/2h + 2(t_i, w_i) = 0, \\ i &= 1(1)n-1,\end{aligned}$$

where

$$\mu_{i\pm 1/2} = \mu(t_i \pm h/2)$$

and we take formally:

$$w_0 := U_0, \quad w_n := U_1.$$

The problem (5') will be discretized by the backward Euler scheme:

$$(10)\qquad S_i w^h := \varepsilon\mu(t_i)(w_i - w_{i-1})/h + a(t_i, w_i) = 0, \quad i = 1(1)n-1,$$

where again $w_0 := U_0$.

Lemma 1. *Let $h \le M^*\varepsilon$. Then the discrete problem (9) has a unique solution $z^h \in W^h$, and the following stability inequality holds for any w^h, $v^h \in W^h$:*

$$(11)\qquad \|w^h - v^h\|_1 \le b_*^{-1}\|Tw^h - Tv^h\|_1.$$

Proof. It holds that

$$(12) \qquad Tw^h - Tv^h = B(w^h - v^h, \quad B = \int_0^1 T'(v^h + s(w^h - v^h))ds,$$

where $T'(w^h)$ denotes the Fréchet derivative of the operator T at w^h. The condition $h \leq M^*\varepsilon$ guarantees that B is an L-matrix. Moreover, we have, cf. [6]:

$$B^T t^h \geq b_* e^h, \quad t^h = [t_1, t_2, \ldots, t_{n-1}]^T.$$

This means that B is an M-matrix and that

$$\|B^{-1}\|_1 \leq b_*^{-1}.$$

From here and (12) we get (11).

The existence follows from

$$Tu^* e^h \geq 0 \geq Tu_* e^h$$

which is satisfied because of (2). □

Similarly we have, cf. [9]:

Lemma 2. *The discrete problem (10) has a unique solution* $\tilde{z}^h \in W^h$ *and the following stability inequality holds for any* w^h, $v^h \in W^h$*:*

$$\|w^h - v^h\|_\infty \leq b_*^{-1} \|Sw^h - Sv^h\|_\infty.$$

Let

$$\bar{z}^h = \begin{cases} z^h & \text{if } h \leq M^* \\ \tilde{z}^h & \text{otherwise} \end{cases}.$$

Then we have

Theorem 1. *Let the function* λ *be given with a fixed* $\alpha = t_j$ *for some* $j \in \{1, 2, \ldots, n-1\}$*. Then it holds that*

$$\|\bar{z}^h - u_\varepsilon^h\|_\infty \leq Mh.$$

Proof. Let $h \leq M^*\varepsilon$, thus $\bar{z}^h = z^h$. We shall show that

$$(13) \qquad \|Tu_\varepsilon^h\|_1 \leq Mh^2$$

and from (11) it will follow

$$\|z^h - u_\varepsilon^h\|_1 \le Mh^2,$$

i.e.

$$\|z^h - u_\varepsilon^h\|_\infty \le Mh.$$

The estimate (13) follows from

$$|T_i u_\varepsilon^h| \le Mh^2, \quad i = 1(1)n-1, \tag{14}$$

which can be proved because of the special choice of the function λ, cf. [6]. In order to prove (14) expand the truncation error $T_i u_\varepsilon^h$ and use the following facts:

– function ω is smooth in $[0,\alpha]$ and π is smooth in $[\alpha,1]$,
– in addition to (8) it holds that

$$0 \le \lambda^{(k)}(t) \le M, \quad k = 3,4 \quad t \in I\backslash\{\alpha\},$$

– using (3) we have

$$|y^{(k)}(t)| \le M, \quad k = 0(1)4, \quad t \in I\backslash\{\alpha\},$$

– since $\omega'(\alpha \ge M\varepsilon^{1/4}$, for $t \in [\alpha,1]$ we have

$$\varepsilon|\mu^{(k)}(t)| \le M, \quad k = 0(1)3, \tag{15}$$

– for $t \in [0,\alpha]$ (15) can be checked directly.
(Note that because of $\alpha = t_j$ it holds that all the above quantities occur in the expansion of $T_i u_\varepsilon^h$ at points which are different from α.)

Similarly we can prove, cf. [6], [9]:

$$\|\tilde{z}^h - \tilde{u}_\varepsilon^h\|_\infty \le Mh,$$

which together with (7) gives

$$\|\tilde{z}^h - u_\varepsilon^h\|_\infty \le M(\varepsilon + h). \tag{16}$$

Since we use $\tilde{z}^h$ when $h > M^*\varepsilon$, from (16) we get

$$\|\tilde{z}^h - u_\varepsilon^h\|_\infty \le Mh. \square$$

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REZIME

O NUMERIČKIM METODIMA ZA KVAZILINEARNE SINGULARNE PERTURBACIONE PROBLEME BEZ POVRATNIH TAČAKA

Dat je kratak pregled numeričkih metoda za kvazilinearne singularno perturbovane konturne probleme bez povratnih tačaka. Predložen je jedan novi metod za koji je dokazana tačnost prvog reda u maksimum normi, uniformna po perturbacionom parametru.

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