Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 23, 2 (1993), 1-9

Review of Research
Faculty of Science
Mathematics Series

A GENERALIZATION OF S.ITOH'S FIXED POINT THEOREM IN PROBABILISTIC METRIC SPACES

Olga Hadžić
Institute of Mathematics, University of Novi Sad,
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

In this paper a generalization of S. Itoh's fixed point theorem in probabilistic metric spaces is proved.

AMS Mathematics Subject Classifications (1980): 47H10
Key words and phrases: Multivalued mappings, fixed point, probalistic metric spaces.

1. Introduction and Preliminaries

In [3] Sh. Itoh proved the following fixed point theorem.

Theorem A. *Let (X,d) be a complete metric space, A a condensing mapping of X into $CB(X)$. Suppose that there is a nonempty bounded subset K of X such that $A(K)$ is bounded and* $\inf_{x\in K} d(x,Ax)=0$. *Then there exists a fixed point $z\in\overline{K}$ of A.*

In this paper a generalization of Theorem A in probabilistic metric spaces is proved.

First, we shall give some definitions and notations.

By Δ we shall denote the set of all distribution functions F such that $F(0)=0$ (F is a nondecreasing, leftcontinuous mapping from $\mathbf{R}$ into $[0,1]$ so that $\sup_{x\in\mathbf{R}} F(x)=1$).

The ordered pair $(S, \mathcal{F})$ is a *probabilistic metric space* [5] if S is a nonempty set and $\mathcal{F} : S \times S \to \Delta(\mathcal{F}(p,q)$ is denoted by $F_{p,q}$, for every $(p,q) \in S \times S)$ satisfies the following conditions:

1. $F_{u,v}(x) = 1$, for every $x > 0 \Rightarrow u = v (u, v \in S)$.

2. $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1 \Rightarrow F_{u,w}(x+y) = 1$ for $u, v, w \in S$ and $x, y \in \mathbf{R}^+$.

A Menger space is a triple $(S, \mathcal{F}, T)$, where $(S, \mathcal{F})$ is a probabilistic metric space and T is a $t-$ norm [5].

The $(\epsilon, \lambda)-$ topology in S is introduced by the family of neighbourhoods given by

$$\mathcal{U} = \{U_v(\epsilon, \lambda)\}_{(v,\epsilon,\lambda) \in S \times \mathbf{R}^+ \times (0,1)}, \quad \text{where}$$
$$U_v(\epsilon, \lambda) = \{u;\ F_{u,v}(\epsilon) > 1 - \lambda\}.$$

If $t-$ norm T is continuous then S is, in the (ϵ, λ) topology, a metrizable topological space.

Let $(S, \mathcal{F})$ be a probabilistic metric space. In [1] the notions of probabilistic diameter and the Kuratowski function is given.

Definition 1. *Let A be a nonempty subset of S. The function $D_A(\cdot)$, defined on $\mathbf{R}^+$ by*

$$D_A(u) = \sup_{s<u} \inf_{p,q \in A} F_{p,q}(s), \quad u \in \mathbf{R}^+,$$

is called the probabilistic diameter of the set A and the set A is probabilistic bounded if and only if

$$\sup_{u \in \mathbf{R}^+} D_A(u) = 1.$$

Definition 2. *Let A be a probabilistic bounded subset of S. The Kuratowski function $\alpha_A : \mathbf{R}^+ \to [0,1]$ is defined by $\alpha_A(u) = \sup\{\epsilon;\ \epsilon > 0,$ there is a finite family $\{A_j\}_{j \in J}$ in S such that $A = \bigcup_{j \in J} A_j$ and $D_{A_j}(u) \geq \epsilon$, for every $j \in J\}$.*

The Kuratowski function has the following properties:

1) $\alpha_A \in \Delta$.

2) $\alpha_A(u) \geq D_A(u)$, for every $u \in \mathbf{R}^+$.

3) $\emptyset \neq A \subset B \subset S \Rightarrow \alpha_A(u) \geq \alpha_B(u)$, for every $u \in \mathbf{R}^+$.

4) $\alpha_{A\cup B}(u) = min\{\alpha_A(u), \alpha_B(u)\}$, for every $u \in \mathbf{R}^+$.

5) $\alpha_A(u) = \alpha_{\bar{A}}(u)(u \in \mathbf{R}^+)$, where $\overline{A}$ is the closure of A.

6) $\alpha_A = H \iff A$ is precompact, where

$$H(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0. \end{cases}$$

The function $\beta_A : \mathbf{R}^+ \to [0,1]$ is defined by

$\beta_A(u) = \sup\{\epsilon;\ \epsilon > 0$, there exists a finite subset A_f of S such that $\tilde{F}_{A,A_f}(u) \geq \epsilon\}$ where

$$\tilde{F}_{A,B}(u) = \sup_{s<u} \inf_{x\in A} \sup_{y\in B} F_{x,y}(s),$$

for probabilistic bounded subsets $A, B \subset S$.

Let $(S, \mathcal{F})$ be a probabilistic metric space, K a probabilistic bounded subset of S and $A : K \to 2^S \setminus \emptyset$. If $A(K)$ is probabilistic bounded subset of S and for every $B \subset K$:

$\gamma_{A(B)}(u) \leq \gamma_B(u)$, for every $u > 0 \Rightarrow B$ is precompact,
where γ_B is α_B or β_B for $B \subset S$ then we say that A is *densifyng on the set K in respect to the function* γ.

If $A : S \to 2^S \setminus \emptyset$ by $Fix(A)$ we shall denote the set $\{x;\ x \in S,\ x \in Ax\}$.

By $\mathcal{M}$ we shall denote the set of all functios $m : \mathbf{R}^+ \to \mathbf{R}^+$, which are continuous and $m(0) = 0$. If $m \in \mathcal{M}$ satisfies the condition $m(t+s) \geq m(t) + m(s)(t, s \in \mathbf{R}^+)$ we say that $m \in \mathcal{M}'$.

V. Radu proved the following result.

Theorem B. *If $t-$ norm T is such that $T \geq T_f$, where T_f is an Archimedean $t-$ norm with the additive generator f, then by*

$$d_{m_1,m_2}(p,q) = \sup\{u;\ u \geq 0,\ m_1(u) \leq f \circ F_{p,q}(m_2(u))\}$$

$(p, q \in S;\ m_1, m_2 \in \mathcal{M}')$ a metric on Menger space $(S, \mathcal{F}, T)$ is defined and d_{m_1,m_2} induces the $(\epsilon, \lambda)-$ topology on S.

Theorem 1. *Let $(S, \mathcal{F}, T)$ be a complete Menger space with a continuous $t-$ norm $T, A : S \to CB(S)$ a closed mapping and there exists a nonempty* *probabilistic bounded subset K of S such that $A(K)$ is probabilistic bounded and the following conditions are satisfied:*

a) There exist $m_1, m_2 \in \mathcal{M}$ and a decreasing function $f : [0,1] \to [0,b]$ $(b > 0)$ such that

$$\inf_{x\in K} \inf_{y\in Ax} \sup\{u;\ u \geq 0,\ f \circ F_{x,y}(m_2(u)) \geq m_1(u)\} = 0.$$

b) The mapping A is densifyng on K in respect to γ, where $\gamma \in \{\alpha, \beta\}$. Then $Fix(A) \neq \emptyset$.

Proof. From a) it follows that for every $n \in \mathbf{N}$ there exists $x_n \in K$ and $y_n \in Ax_n$ so that

$$\sup\{u;\ u > 0, f \circ F_{x_n,y_n}(m_2(u)) \geq m_1(u)\} < 2^{-n}. \tag{1}$$

From (1) it follows that

$$f \circ F_{x_n,y_n}(m_2(2^{-n})) < m_1(2^{-n}). \tag{2}$$

We shall prove that (2) implies that for every $\epsilon > 0$, $\lim_{n\to\infty} F_{x_n,y_n}(\epsilon) = 1$, which means that for every $\epsilon > 0$ and $\lambda \in (0,1)$ there exists $n_0(\epsilon, \lambda) \in \mathbf{N}$ so that $F_{x_n,y_n}(\epsilon) > 1 - \lambda$, for every $n \geq n_0(\epsilon, \lambda)$. Since m_1 is continuous and $m_1(0) = 0$ there exists $n_0(b) \in \mathbf{N}$ so that $m_1(2^{-n}) < b$, for every $n \geq n_0(b)$. Then, for $n \geq n_0(b)$, from (2) it follows that

$$F_{x_n,y_n}(m_2(2^{-n})) > f^{-1}[m_1(2^{-n})].$$

Let $n_1(\epsilon, \lambda) \in \mathbf{N}$ be such that

$$m_1(2^{-n}) < f(1-\lambda),\ m_2(2^{-n}) < \epsilon, \quad \text{for } n \geq n_1(\epsilon, \lambda).$$

Then

$$F_{x_n,y_n}(\epsilon) \geq F_{x_n,y_n}(m_2(2^{-n})) > f^{-1}[m_1(2^{-n})] > 1 - \lambda$$

for every $n \geq n_2(\epsilon, \lambda, b) = \max\{n_0(b), n_1(\epsilon, \lambda)\}$ which means that $\lim_{n\to\infty} F_{x_n,y_n}(\epsilon) = 1$.

We shall prove that

$$\gamma_{\{x_n; n\in\mathbf{N}\}} = \gamma_{\{y_n; n\in\mathbf{N}\}}.$$

In fact, we shall prove that

$$\gamma_{\{y_n;n\in\mathbf{N}\}} \leq \gamma_{\{x_n;n\in\mathbf{N}\}}$$

which means that for every $u > 0$:

$$\gamma_{\{y_n;n\in\mathbf{N}\}}(u) \leq \gamma_{\{x_n;n\in\mathbf{N}\}}(u). \tag{3}$$

First, we shall suppose that $\gamma = \beta$. Inequality (3) holds if for every $0 < \epsilon < u$:

$$\beta_{\{y_n;n\in\mathbf{N}\}}(u-\epsilon) \leq \beta_{\{x_n;n\in\mathbf{N}\}}(u), \tag{4}$$

since β is leftcontinuous. If $\beta_{\{y_n;n\in\mathbf{N}\}}(u-\epsilon) = 0$ then (4) holds and we shall suppose that $\beta_{\{y_n;n\in\mathbf{N}\}}(u-\epsilon) > 0$.

Let $r < \beta_{\{y_n;n\in\mathbf{N}\}}(u-\epsilon)$. Then there exists a finite set $A_f \subset S$ such that

$$\sup_{s<u-\epsilon}\ \inf_{n\in\mathbf{N}}\ \max_{z\in A_f} F_{y_n,z}(s) \geq r.$$

which implies that $\inf_{n\in\mathbf{N}} \max_{z\in A_f} F_{y_n,z}(u-\epsilon) \geq r$ and so for every $n \in \mathbf{N}, \max_{z\in A_f} F_{y_n,z}(u-\epsilon) \geq r$.

Let $z_n \in A_f$ be such that $F_{y_n,z_n}(u-\epsilon) \geq r$, for every $n \in \mathbf{N}$. Let $\delta \in (0,r)$. From the continuity of T and relation $T(1,r) = r$ it follows that there exists $\tilde{\delta} \in (0,1)$ such that

$$1 \geq s > 1-\tilde{\delta} \Rightarrow T(s,r) > r-\delta.$$

Since $\lim_{n\to\infty} F_{x_n,y_n}(\frac{\epsilon}{2}) = 1$ there exists $n_0(\epsilon,\tilde{\delta}) \in \mathbf{N}$ so that $F_{x_n,y_n}(\frac{\epsilon}{2}) > 1-\tilde{\delta}$, for every $n \geq n_0(\epsilon,\tilde{\delta})$. From this it follows that

$$F_{x_n,z_n}(u-\frac{\epsilon}{2}) \geq T(F_{x_n,y_n}(\frac{\epsilon}{2}),\ F_{y_n,z_n}(u-\epsilon)) \geq$$

$$T(F_{x_n,y_n}(\frac{\epsilon}{2}),r) > r-\delta$$

for every $n \geq n_0(\epsilon,\tilde{\delta})$. This implies that

$$r-\delta \leq \beta_{\{x_n;n\geq n_0(\epsilon,\tilde{\delta})\}}(u) =$$

$$\beta_{\{x_n;n\in\mathbf{N}\}}(u).$$

Since δ is an arbitrary number from $(0,r)$ it follows that $\beta_{\{x_n;n\in\mathbf{N}\}} \geq r$. Hence (4) holds.

Similarly, $\beta_{\{x_n;n\in\mathbf{N}\}}(u) \leq \beta_{\{y_n;n\in\mathbf{N}\}}(u)$, for every $u > 0$ and so

$$\beta_{\{y_n;n\in\mathbf{N}\}}(u) = \beta_{\{x_n;n\in\mathbf{N}\}}(u),\ u > 0.$$

Since $\{y_n; n \in \mathbf{N}\} \subseteq \bigcup_{n\in\mathbf{N}} Ax_n$ it follows that

$$\beta_{\bigcup_{n\in\mathbf{N}} Ax_n}(u) \leq \beta_{\{y_n;n\in\mathbf{N}\}}(u) = \beta_{\{x_n;n\in\mathbf{N}\}}(u)$$

for every $u > 0$. From b) it follows that $\{x_n; n \in \mathbf{N}\}$ is compact i.e. there exists a convergent subsequence $\{x_{n_k}\}_{k\in\mathbf{N}}$. If $z = \lim_{k\to\infty} x_{n_k}$ then $\lim_{k\to\infty} y_{n_k} = z$ and since $y_{n_k} \in Ax_{n_k}$ $(k \in \mathbf{N})$ and A is closed we conclude that $z \in Az$. Let $\gamma = \alpha$. We shall prove that for every $u > 0$:

$$\alpha_{\{x_n;n\in\mathbf{N}\}}(u) = \alpha_{\{y_n;n\in\mathbf{N}\}}(u). \tag{5}$$

Let $\epsilon \in (0,u)$ and $\alpha_{\{y_n;n\in\mathbf{N}\}}(u-\epsilon) > 0$. We shall prove that

$$\alpha_{\{y_n;n\in\mathbf{N}\}}(u-\epsilon) \leq \alpha_{\{x_n;n\in\mathbf{N}\}}(u).$$

Let $r < \alpha_{\{y_n;n\in\mathbf{N}\}}(u-\epsilon)$. Then there exists $A_1, A_2, ..., A_n \subset S$ so that: $\{y_n; n \in \mathbf{N}\} = \bigcup_{j=1}^{n} A_j$, $D_{A_j}(u-\epsilon) \geq r$, for every $j \in \{1,2,...,n\}$. Then $\inf_{x,y\in A_j} F_{x,y}(u-\epsilon) \geq r$ and so $F_{x,y}(u-\epsilon) \geq r$, for every $x, y \in A_j$.

Let δ be an arbitrary number from the interval $(0,r)$ and $\tilde{\delta} \in (0,1)$ such that

$$1 \geq u, w > 1-\tilde{\delta} \Rightarrow T(u, T(r,w)) > r - \delta.$$

Since the mapping $(u,w) \mapsto T(u,T(r,w))$ is continuous and $T(1,T(r,1)) = r$ such a number $\tilde{\delta}$ exists. Let for every $j \in \{1,2,...,n\}$:

$$B_j = \{z; F_{z,y}(\frac{\epsilon}{4}) > 1-\tilde{\delta},\ \text{ for some } y \in A_j\}.$$

If $n_1(\epsilon,\tilde{\delta}) \in \mathbf{N}$ is such that

$$F_{x_n,y_n}(\frac{\epsilon}{4}) > 1-\tilde{\delta},\ \text{ for every }\ n \geq n_1(\epsilon,\tilde{\delta})$$

then

$$\{x_n; n \geq n_1(\epsilon,\tilde{\delta})\} \subseteq \bigcup_{j=1}^{n} B_j.$$

We shall prove that

$$\sup_{s<u} \inf_{x,y \in B_j} F_{x,y}(s) \geq r - \delta.$$

If $x \in B_j$ and $y \in B_j$, then there exists $\tilde{x} \in A_j$ and $\tilde{y} \in A_j$ so that:

$$F_{x,\tilde{x}}(\frac{\epsilon}{4}) > 1 - \tilde{\delta}, \ F_{y,\tilde{y}}(\frac{\epsilon}{4}) > 1 - \tilde{\delta}.$$

Since $F_{\tilde{x},\tilde{y}}(u-\epsilon) \geq r$ we have that

$$F_{x,y}(u - \frac{\epsilon}{2}) \geq T(F_{x,\tilde{x}}(\frac{\epsilon}{4}), \ T(F_{\tilde{x},\tilde{y}}(u-\epsilon), \ F_{\tilde{y},y}(\frac{\epsilon}{4}))) \geq$$

$$\geq T(F_{x,\tilde{x}}(\frac{\epsilon}{4}), \ T(r, F_{\tilde{y},y}(\frac{\epsilon}{4}))) > r - \delta$$

which implies

$$\sup_{s<u} \inf_{x,y \in B_j} F_{x,y}(s) \geq r - \delta$$

and so

$$\alpha_{\{x_n; n \geq n_1(\epsilon,\tilde{\delta}\}}(u) \geq r - \delta.$$

Since $\alpha_{\{x_n; n \in \mathbf{N}\}}(u) = \alpha_{\{x_n; n \geq n_1(\epsilon,\tilde{\delta})\}}(u)$ we obtain that

$$\alpha_{\{x_n; n \in \mathbf{N}\}}(u) \geq r.$$

Hence

$$\alpha_{\{y_n; n \in \mathbf{N}\}}(u) \leq \alpha_{\{x_n; n \in \mathbf{N}\}}(u)$$

for every $u > 0$ and similarly

$$\alpha_{\{x_n; n \in \mathbf{N}\}}(u) \leq \alpha_{\{y_n; n \in \mathbf{N}\}}(u).$$

So, we proved that

$$\alpha_{\{x_n; n \in \mathbf{N}\}}(u) = \alpha_{\{y_n; n \in \mathbf{N}\}}(u)$$

for every $u > 0$. The rest of the proof is as in the case $\gamma = \beta$.

Corollary 1. *Let $(S, \mathcal{F}, T)$, T, A and K be as in the Theorem so that instead of a) we have that*

(6) $$\inf_{x \in K} \inf_{y \in Ax} \sup\{u; u \geq 0, \ 1 - u \geq F_{x,y}(u)\} = 0.$$

If b) holds then $Fix(A) \neq \emptyset$.

Proof. The Corollary follows from the Theorem if we take that $m_1(s) = m_2(s) = s$, for every $s \geq 0$ and $f(s) = 1 - s$, for every $s \geq 0$.

We can prove Theorem A using Corollary 1. It is well known that (X, d) may be considered as the Menger space $(X, \mathcal{F}, min)$, where $\mathcal{F}$ is defined by

$$F_{x,y}(u) = \begin{cases} 0, & d(x,y) \geq u \\ 1, & d(x,y) < u \end{cases} \quad (u \in \mathbf{R}^+; x, y \in X)$$

and the (ϵ, λ) topology is the same as the topology induced by the metric d. From the condition $\inf_{x \in K} d(x, Ax) = 0$ it follows the existence of two sequences $\{x_n\}_{n \in \mathbf{N}}$ and $\{y_n\}_{n \in \mathbf{N}} (x_n \in K,\ y_n \in Ax_n,\ n \in \mathbf{N})$ such that $\lim_{n \to \infty} d(x_n, y_n) = 0$. Then for every $u > 0$ there exists $n_0(u) \in \mathbf{N}$ such that $F_{x_n, y_n}(u) = 1$ for every $n \geq n_0(u)$.

References

[1] Boscan, Gh., Constantin, Gh., The Kuratowski function and some applications to the probabilistic metric spaces, Sem. Teor. Funct. si. Mat. Apl. Timisoara, RS Romania, 1, 1973.

[2] Hadžić, O., Generalized contractions for multivalued mappings in probabilistic metric spaces, Univ. u Novom Sadu Zb. Rad. Prirod. - Mat. Fak. Ser. Mat. 22,2 (1992), 39 - 45.

[3] Itoh, Sh., Some fixed point theorems in metric spaces, Fund. Math. 102, 2 (1979), 109 - 117.

[4] Radu, V., A family of deterministic metrics on Menger spaces, Sem. Ter. Prob. si. Apl., 78, Timisoara, RS Romania, 1985.

[5] Schweizer, B., Sklar, A., Probabilistic metric spaces, North - Holland, 1983.

REZIME

GENERALIZACIJA TEOREME O NEPOKRETNOJ TAČKI S. ITOHA U VEROVATNOSNIM METRIČKIM PROSTORIMA

U ovom radu je dokazana generalizacija teoreme o nepokretnoj tački S. Itoha u verovatnosnim metričkim prostorima.

Received by the editors December 14, 1993