Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 23, 2 (1993), 11 - 22

Review of Research
Faculty of Science
Mathematics Series

ON THE COINCIDENCE THEOREMS AND A GENERALIZATION OF KAKUTANI-KY FAN THEOREM

Shih-sen Chang
Department of Mathematics, Sichuan University
China

Abstract

Coincidence theorems generalizing the coincidence theorems of [11, 9, 6, 3] and extending fixed point theorems for multivalued mappings of [7, 4, 1] and single-valued mappings of [5] are established. Moreover a generalization of the famous Kakutani-Ky Fan's theorem is given too.

AMS Mathematics Subject Classification (1991): 47H10
Key words and phrases: multivalued mappings,fixed point, coincidence theorems.

1. Introduction

Recently, some coincidence theorems have been considered by several authors (cf.[11, 9, 12, 6, 8, 3]). The purpose of this paper is to give some new coincidence theorems for multivalued or single valued mappings. These results generalize some of the main results of [11, 9, 6, 3, 7, 4, 1, 5]. Moreover, in §4 we give a coincidence theorem for multivalued mapping in locally convex Hausdorff linear topological space which generalizes the famous Kakutani-Ky Fan theorem.

2. Coincidence Theorems for Multivalued Mappings

In this section we shall always assume that X is an aribitrary nonempty set, (M,d) a metric space, I_M the identity on M, CB(M) the family off all nonempty closed and bounded subsets of M, and H the Hausdorff metric on CB(M) induced by the metric d. Moreover, in this section we assume that the function $\Phi : [\,0,\infty)^5 \to [0,\infty)$ satisfies the following conditions (Φ_1) and (Φ_2) (or (Φ_1) and (Φ_3)) :

(Φ_1) Φ is upper semi-continuous and nondecreasing for each variable.

(Φ_2) $\max\{\Phi(t,t,t,at,bt),\ a\,,b\ =\ 0,1,2,\ a+b=\ 2\} \le \varphi(t),\ \ \forall t \ge 0$, where $\varphi(t)\ :\ [0,\infty)\ \to [0,\infty),\ \varphi(0)=0,\ \varphi(t)<t,\ \forall t\ >0.$

(Φ_3) $\Phi(t,t,t,at,bt) \le \gamma t$,

where $\gamma \in (0,1)$ is a constant , $a,b=0,1,2,$ and $a\ +\ b\ =2.$

In what follows we need the following

Lemma. *(Nadler [7]). Let $A,B \in CB(M)$, then for arbitrarily given $a \in A$ and $\beta > 1$ there exsists point $b \in B$ such that*

$$d(a,b) \le \beta \cdot H(A,B).$$

Theorem 1. *Let $P_1,P_2\ : X \to CB(M)$ and $T : X \to M$ be such that $T(X)$ is a comlete subspace of M and $P_i(X) \subset T(X),\ \ i=1,2$. Let us then assume that the following conditions are satisfied*

(i) *For each $u \in T(X)$*

(2.1) $$P_i(x)=P_i(y),\quad \forall\ x,y\ \in\ T^{-1}u,\quad i=1,2;$$

(ii) *For any $x,y \in X$*

(2.2) $$H(P_1(x),P_2(y)) \le$$
$$\le \Phi(d(Tx,Ty),d(Tx,P_1(x)),d(Ty,P_2(y)),d(Tx,P_2(y)),d(Ty,P_1(x))),$$

where the function Φ satisfies the conditions (Φ_1) and (Φ_2).

(iii) *Let $\beta > 1, u_0 \in T(x), u_1 \in P_1(T^{-1}u_0)$, and let $\{t_k\}$ be a sequence of nonnegative real numbers which is defined by*

(2.3) $$\left.\begin{array}{c} t_0=0,\ \ t_1 > d(u_0,u_1), \\ t_{k+1}=t_k+\varphi(\beta(t_k-t_{k-1})),\ \ k=1,2.... \end{array}\right\}$$

where φ is the function which appears in the condition (Φ_2). If $t_k \to t_* < \infty$ $(k \to \infty)$, then T, P_1 and P_2 have a coincidence point, that is, there exsists $x_* \in X$ such that

$$Tx_* \in P_1(x_*) \cap P_2(x_*).$$

Proof. First we define the mapping $F_i, i = 1,2$ as follows:

$$F_i : T(X) \to CB(T(X)), u \to P_i(T^{-1}u).$$

By condition (i) we have

$$F_i(u) = P_i(T^{-1}u) = P_i(x), \ \ \forall x \in T^{-1}u. \tag{2.4}$$

By condition (ii) and (2.4), for any $u, v \in T(X)$ and any $x \in T^{-1}u$, $y \in T^{-1}v$ we have

$$H(F_1(u), F_2(v)) = H(P_1(x), P_2(y)) \leq$$
$$\leq \Phi(d(u,v), d(u, F_1(u)), d(v, F_2(v), d(u, F_2(v)), d(v, F_1(u))). \tag{2.5}$$

Therefore by using condition (iii) and Lemma, for the given $\beta > 1$, $u_0 \in T(X)$, $u_1 \in P_1(T^{-1}u) = F_1(u_0)$ there exsists $u_2 \in F_2(u_1)$ such that

$$d(u_1, u_2) \leq \beta \cdot H(F_1(u_0), F_2(u_1)).$$

By using Lemma again there exsists $u_3 \in F_1(u_2)$ such that

$$d(u_2, u_3) \leq \beta \cdot H(F_2(u_1), F_1(u_2)).$$

Continuing in this way we can produce a sequence $\{u_n\}_{n=0}^{\infty} \subset T(X)$ such that

$$\left.\begin{array}{r} u_{2n+1} \in F_1(u_{2n}), u_{2n+2} \in F_2(u_{2n+1}), \quad n = 0,1,2,.... \\ d(u_{2n+1}, u_{2n}) \leq \beta \cdot H(F_1(u_{2n}), F_2(u_{2n-1})), \quad n = 1,2,... \\ d(u_{2n+2}, u_{2n+1}) \leq \beta \cdot \ H(F_2(u_{2n+1}), F_1(u_{2n})), \quad n = 0,1,2,... \end{array}\right\} \tag{2.6}$$

Now we prove that the following inequalities are true:

$$\left.\begin{array}{r} d(u_{2n}, F_1(u_{2n})) \leq d(u_{2n-1}, u_{2n}), n = 1,2,.... \\ d(u_{2n+1}, F_2(u_{2n+1})) \leq d(u_{2n}, u_{2n+1}, n = 0,1,2... \end{array}\right\} \tag{2.7}$$

In fact, it follows from (2.5) and (2.6) that

$$\begin{aligned} d(u_{2n}, F_1(u_{2n})) \leq \ & H(F_2(u_{2n-1}), F_1(u_{2n})) \\ \leq \ & \Phi(d(u_{2n}, u_{2n-1}), d(u_{2n}, F_1(u_{2n})), d(u_{2n-1}, u_{2n}), 0, \\ & \quad d(u_{2n-1}, u_{2n}) + d(u_{2n}, F_1(u_{2n}))). \end{aligned}$$

If $d(u_{2n}, F_1(u_{2n})) > d(u_{2n-1}, u_{2n})$ then we have

$$d(u_{2n}, F_1(u_{2n})) \leq \varphi(d(u_{2n}, F_1(u_{2n})) < d(u_{2n}, F_1(u_{2n})),$$

this is a contradiction. Therefore we have

$$d(u_{2n}, F_1(u_{2n})) \leq d(u_{2n-1}, u_{2n}), \qquad n = 1, 2, \ldots.$$

In the same way we can prove that the second inequality in (2.7) is true.

Next we show by induction that

$$d(u_j, u_{j-1}) \leq \beta \cdot (t_j - t_{j-1}), \quad j = 1, 2, \ldots. \tag{2.8}$$

In fact, by assumptions it is obvious that (2.8) is true for $j = 1$. Suppose that (2.8) is true for $j = k$, and now we prove it remains true for $j = k + 1$.

If k is even, from condition (2.3), and (2.5), (2.6) we have

$$\begin{aligned} d(u_{k+1}, u_k) &\leq \beta \cdot H(F_1(u_k), F_2(u_{k-1})) \\ &\leq \beta \cdot \Phi(d(u_k, u_{k-1}), d(u_k, u_{k-1}), d(u_{k-1}, u_k), 0, 2d(u_{k-1}, u_k)) \\ &\leq \beta \cdot \varphi(d(u_k, u_{k-1})) \leq \beta \cdot \varphi(\beta(t_k - t_{k-1})) = \beta \cdot (t_{k+1} - t_k). \end{aligned}$$

If k is odd we can prove that the same inequality remains true. This comletes the proof of (2.8).

Since $t_k \to t_* < \infty$, hence for any positive integers k, m we have

$$\begin{aligned} d(u_{k+m}, u_k) &\leq \sum_{j=k}^{k+m-1} d(u_{j+1}, u_j) \leq \beta \cdot \sum_{j=k}^{k+m-1} (t_{j+1} - t_j) \\ &= \beta \cdot (t_{k+m} - t_k). \end{aligned}$$

This implies that $\{u_n\}$ is a Cauchy sequence of T(X). By the completeness of $T(X)$ we can suppose that $u_n \to u_* \in T(X)$.

Now we prove that u_* is the common fixed point of F_1 and F_2. In fact, we have

$$\begin{aligned} d(u_*, F_1(u_*)) &\leq d(u_*, u_{2n}) + d(u_{2n}, F_1(u_*)) \\ &\leq d(u_*, u_{2n}) + H(F_2(u_{2n-1}), F_1(u_*)) \\ &\leq d(u_*, u_{2n}) + \Phi(d(u_*, u_{2n-1}), d(u_*, F_1(u_*)), d(u_{2n-1}, u_{2n}), \\ &\quad d(u_*, u_{2n}) + d(u_{2n}, F_2(u_{2n-1})), d(u_{2n-1}, u_*) + d(u_*, F_1(u_*))) \end{aligned}$$

Letting $n \to \infty$ on the right side and nothing the upper semi-continuity of Φ we have

$$\begin{aligned} d(u_*, F_1(u_*)) &\leq \Phi(0, d(u_*, F_1(u_*)), 0, 0, d(u_*, F_1(u_*))) \\ &\leq \varphi(d(u_*, F_1(u_*))). \end{aligned}$$

Hence we have $d(u_*, F_1(u_*)) = 0$. That is $u_* \in F_1(u_*)$.

Similarly, we can prove that $u_* \in F_2(u_*)$ i.e.

$$u_* \in F_1(u_*) \cap F_2(u_*).$$

By virtue of (2.4), for any $x_* \in T^{-1}u_*$ we have

$$Tx_* = u_* \in F_1(u_*) \cap F_2(u_*) = P_1(x_*) \cap P_2(x_*)$$

This implies that each $x_* \in T^{-1}u_*$ is the coincidence point of T, P_1 and P_2.

This completes the proof of Theorem 1. □

Theorem 2. *Let $P_1, P_2 : X \to CB(M)$ and $T : X \to M$ be such that $T(X)$ is a complete subspace of M and $P_i(X) \subset T(X), i = 1, 2$. Let us further assume that the conditions (i) and (ii) of Theorem 1 are satisfied, where the function Φ satisfies the conditions (Φ_1) and (Φ_3). Then the conclusion of Theorem 1 still holds.*

Proof. Taking $t_0 = 0, u_0 \in T(X), u_1 \in P_1(T^{-1}u_0), t_1 > d(u_0, u_1)$ we define a sequence of nonnegative real numbers $\{t_k\}$ as follows:

$$t_{k+1} = t_k + \gamma \cdot \beta\,(t_k - t_{k-1}), \quad k = 1, 2, ..., \tag{2.9}$$

where γ is a constant which appears in condition $(\Phi_3), \gamma \in (0, 1)$, and $\beta > 1, \gamma \cdot \beta < 1$. It follows from (2.9) that

$$t_{k+1} - t_k = \gamma \cdot \beta(t_k - t_{k-1}) = \cdots = (\gamma \cdot \beta)^k t_1,$$

and

$$\lim_{k \to \infty} t_k = \lim_{k \to \infty} \sum_{i=1}^{k} (t_i - t_{i-1}) = \frac{t_1}{1 - \gamma \cdot \beta} < \infty.$$

This shows that condition (iii) of Theorem 1 is true. Hence the conclusion of Theorem 2 follows from Theorem 1 immediately.□

Corollary 1. *Let $P_1, P_2 : X \to CB(M)$, $T : X \to M$ be such that $T(X)$ is complete subspace of M, $P_i(X) \subset T(X)$, $i = 1, 2$. Let us further assume that the condition (i) of Theorem 1 and the following condition (iv) are satisfied:*

(iv) *For any $x, y \in X$*

$$H(P_1(x), P_2(y)) \leq q \cdot \max\{d(Tx, Ty), d(Tx, P_1(x)), d(Ty, P_2(y)), \tfrac{1}{2}[d(Tx, P_2(y)) + d(Ty, P_1(x))]\}, \tag{2.10}$$

where $q \in (0, 1)$. Then the cunclusion of Theorem 2 still holds.

Proof. Taking

$$\Phi(t_1, t_2, t_3, t_4, t_5) = q \cdot \max\{t_1, t_2, t_3, \frac{1}{2}(t_4 + t_5)\},$$

we have

$$\Phi(t, t, t, at, bt) = q \cdot t,$$

where $a, b = 0, 1, 2$, and $a + b = 2$. Therefore it satisfies conditions (Φ_1) and (Φ_3), and the conclusion of Corollary 1 follows from Theorem 2. □

Corollary 2. *Let $P : X \to CB(M)$, $T : X \to M$ be such that $T(X)$ is a complete subspace of M, $P(X) \subset T(X)$. Let us further assume that there exists q, $0 < q < 1$, such that the following holds*

(v) $H(P(x), P(y)) \leq q \cdot d(Tx, Ty), \ \forall x, y \in X.$

Then T and P have a coincidence point in X.

Proof. Taking $P = P_1 = P_2$ in Corollary 1. and using condition (v) we see the condition (iv) is satisfied. Moreover, for any $u \in T(X)$ and any $x, y \in T^{-1}(u)$ it follows condition (v) that

$$H(Px, Py) \leq q \cdot d(Tx, Ty) = q \cdot d(u, u) = 0.$$

This yields $P(x) = P(y)$, $\forall x, y \in T^{-1}u$. Therefore the condition (i) of Theorem 1 is satisfied. Hence the conclusion of Corollary 2 follows from Corollary 1.

Remark 1. Theorem 1, Theorem 2 and Corollary 1 can be extended to the case that $\{P_i\}$ is a sequence of mapings. For the sake of saving space we omit the statement here.

Remark 2. Theorem 1 improves the results of [11] and [4]. Theorem 2 is an improvement and generalization of some of the main results of [6, 3, 7, 1].

Remark 3. Corollary 2 is first proved in [6]. Here we obtain it as an immediate consequence of Corollary 1. Even in such a simple case it still generalizes the results of [7] and [1].

3. Coincidence Theorems for Single Valued Mappings

Theorem 3. *Let X be an arbitrary nonempty set, (M, d) a metric space. Let $P : X \to M$ and $T : X \to M$ be such that $T(X)$ is a complete subspace of M, $P(X) \subset T(X)$. Let us further assume that the following conditions are satisfied*

(i) *for each $u \in T(X)$*

$$P(x) = P(y), \ \forall x, y \in T^{-1}(u); \tag{3.1}$$

(ii) *for a given $\varepsilon > 0$ there exsists $\delta(\varepsilon) > 0$ such that for $x, y \in X$*

$$\begin{aligned} \varepsilon &\le max\{d(Tx, Ty), d(Tx, Px), d(Ty, Py), \frac{1}{2}[d(Tx, Py) + d(Ty, Px)]\} \\ &< \varepsilon + \delta \end{aligned}$$

implies $d(P(x), P(y)) < \varepsilon$.

Then there exists an $x_* \in X$ such that $Tx_* = P(x_*)$ and for all $u \in T(X), (PT^{-1})^n(u) \to Tx_*$.

Proof. Define a mapping F as follows:

$$T(X) \to T(X), \ u \to (PT^{-1})(u).$$

In view of condition (i), for each $u \in T(X)$ we have

$$(3.2) \qquad F(u) = (PT^{-1})(u) = P(x), \forall x \in T^{-1}(u).$$

Now we prove that for given $\varepsilon > 0$, there exsists a $\delta(\varepsilon) > 0$ such that for any $u, v \in T(X)$

$$\begin{aligned} \varepsilon &\leq \max\{d(u,v), d(u,F(u)), d(v,F(v)), \frac{1}{2}[d(u,F(v)) + d(v,F(u))]\} < \\ (3.3) \quad &< \varepsilon + \delta \end{aligned}$$

implies $d(F(u), F(V)) < \varepsilon$:
In fact, for any $u, v \in T(X)$ and any $x \in T^{-1}(u), y \in T^{-1}(v)$, from (3.3) and (3.2) we have

$$\begin{aligned} \varepsilon \leq &\max\{d(Tx,Ty), d(Tx,P(x)), d(Ty,P(y)), \tfrac{1}{2}\lfloor d(Tx,P(y)) + d(Ty,P(x))\rfloor\} \\ < &\varepsilon + \delta . \end{aligned}$$

By condition (ii) we have

$$d(P(x), P(y)) = d(F(u), F(v)) < \varepsilon.$$

Therefore the conclusion is true. By using Theorem 4 of [10] there exsists a unique $u_* \in T(X)$ such that $u_* = F(u_*)$, and for any $u \in T(X)$ the iterative sequence $F^n(u) \to u_*$ $(n \to \infty)$. Hence for any $x_* \in T^{-1}u_*$, it gets

$$Tx_* = u_* = F(u_*) = (PT^{-1})(u_*) = P(x_*),$$

and

$$(PT^{-1})^n(u) = F^n(u) \to Tx*, n \to \infty.$$

This completes the proof of Theorem 3. □

Remark 4. Theorem 3 generalizes the main results of Park [9]. By virtue of Theorem 3 we can obtain the following results.

Theorem 4. *Let (M,d) be a metric space, f a self-mapping on M and $\alpha, \beta, |\alpha| \neq |\beta|$ two arbitrary real numbers. Denote*

$$T = \alpha I_M + \beta f, P = \beta I_M + \alpha f,$$

where I_M is the identity on M, and assume that $P(M) \subset T(M)$ and $T(M)$ is a complete subspace of M. Then assume the following conditions are satisfied:

(i) *For each* $u \in T(M)$

$$P(x) = P(y), \ \forall x, y \in T^{-1}(u)$$

.

(ii) *For a given* $\varepsilon > 0$*, there exsists* $\delta(\varepsilon) > 0$ *such that for* $x, y \in M$

$$\begin{aligned}\varepsilon &\leq \max\{d(Tx,Ty), d(Tx,P(x)), d(Ty,P(y)), \frac{1}{2}[d(Tx,P(y)) + d(Ty,P(x))]\} \\ &< \varepsilon + \delta\end{aligned}$$

implies $d(P(x), P(y)) < \varepsilon$.

Then there exists a unique fixed point x_* *of* f *in* M *, and for* $u \in M$ *the iterative sequence* $f^n(u) \to x_*$ *,* $n \to \infty$ *.*

Proof. Taking $\alpha = 1$, $\beta = 0$ we have $T = I_M, P + f$, and $f(M) \subset M$. By Theorem 3 there exists $x_* \in M$ such that $x_* = f(x_*)$, and for any $u \in M$, $f^n(u) \to x_*$, $n \to \infty$.□

Remark 5. Theorem 3 extends the results of Park [9] and Meir-Keeler [5].

4. Coincidence Theorem on Locally Convex Linear Topological Spaces A Generalization of Kakutani-Ku Fan's Theorem

Theorem 5. *Let* X *be an arbitrary nonempty set* M *a locally convex Hausdoff linear topological space . Let* $P : X \to CL(M)$ *(the family of all nonempty closed sets of* M*) and* $T : X \to M$ *be such that* $P(X) \subset T(X)$ *and* $T(X)$ *is a nonempty compact convex set of* M*. Next assume the following conditions are satisfied.*

(i) *For all* $u \in T(X)$

$$P(x) = P(y), \ \forall x, y \in T^{-1}(u).$$

(ii) *For each* $x \in X, P(x)$ *is a nonempty closed convex set of* $T(X)$*.*

(iii) *The set*

$$\bigcup_{u \in T(X)} \{(u,y), \ \ y \in P(T^{-1}u) \ \}$$

is closed set of $M \times M$.

Then there exsists $x_* \in X$ *such that* $Tx_* \in P(x_*)$.

Proof. We first define a mapping F as follows:

$$F : T(X) \to CL(T(X)), \ \ u \to P(T^{-1}u).$$

By condition (i) we have

$$F(u) = P(T^{-1}u) = P(x), \ \ \forall x \in T^{-1}(u). \tag{4.1}$$

By condition (ii) for each $u \in T(X)$, $F(u)$ is a nonempty compact convex subset of $T(X)$.
By condition (iii) the graph of F

$$\text{Graph } \ F \ = \bigcup_{u \in T(X)} \{(u,y), \ y \in F(u) = P(T^{-1}u)\}$$

is a closed set of $M \times M$. It follows from Ky Fan's theorem (cf.[2]) that there exsists a $u_* \in T(X)$ such that $u_* \in F(u_*)$. Therefore for each $x_* \in T^{-1}u_*$, from (4.1) we have

$$Tx_* = u_* \in F(u_*) = P(x_*).$$

This completes the proof of Theorem 5. □

Remark 6. Take $X = M$, $T = I_M$ in Theorem 5 and asssume that M is a nonempty compact convex Hausdorff linear topological space, then the famous Kakutani-Ky Fan's theorem is obtained.

References

[1] Covitz, H., Nadler,S.B. , Jr., Multi-valued contraction mappings in generalized metric spaces, Israel J. Math., 8(1980), 5-11.

[2] Fan, K., Fixed point and minimax theorem in locally convex topological linear spaces, Proc. Nat. Acad. Sci. USA, 38 (1952), 121-126.

[3] Goebel, K., A coincidence theorems, Bull. Acad. Polon. Sci. Ser.Sci. Math., 16(1968), 733-735.

[4] Guay,M.D. ,Singh, K.L., Whitfield,J.H.M., Common fixed points for set-valued mappings, Bull. Acad, Polon. Sci. Ser. Sci. Math., V.30(1982), 545-552.

[5] Meir, A., Keeler, E., A theorem on contraction mappings, J. Math. Anal. Appl., 2(1969), 526-529.

[6] Naimpally, S.A., Singh, S.L., Whitfield, J.H.M., Coincidence theorems (to appear).

[7] Nadler,Jr. S.B., Multivalued contraction mapping, Pacific J. Math., 30(1969), 475-488.

[8] Okada,T., Coincidence theorems on L-spaces, Math. Japonica 26(1981), 291-295.

[9] Park,S., A coincidence theorem, Bull. Acad. Polon. Sci. Ser.Math., 29(1981), 487-489.

[10] Park, S., Rhoades, B.E., Meir-Keeler type contractive conditions Math. Japonica, No. 1(1981), 13-30.

[11] Rhoades,B.E. , Singh, S.L., Kulshrestha,C., Coincidence theorems for some multivalued mappings, Internat. J. Math. Sci., V.7, No. 3(1984), 429-434.

[12] Singh , S.L., Kulshrestha, C., Coincidence theorems, Indian J.Phy. Natur. Sci., 2(B)(1982), 32-35.

[13] Smithson,R.E., Fixed points for contractive multi-functions, Proc. Amer. Math. Soc., 27(1971), 192-194.

REZIME

O NEKIM TEOREMAMA KOINCIDENCIJE I UOPŠTENJE KAKUTANI-KI FANOVE TEOREME

Dokazane su teoreme koincidencije koje uopštavaju teoreme koincidencije [11, 9, 6, 3] i proširuju teoreme o nepokretnoj tački za višeznačna preslikavanja iz [7, 4, 1] i jednoznačna preslikavanja iz [5]. Takodje je dokazano jedno uopštenje poznate Kakutani-Ki Fanove teoreme.

Received by the editors Februar 17, 1988