Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 23, 2 (1993), 39 - 60

Review of Research
Faculty of Science
Mathematics Series

SOME GENERALIZATIONS OF PROBABILISTIC METRIC SPACES

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Abstract

The purpose of this paper is to exhibit some $m-$ metrization theorems of $m-$ uniform spaces. Furthermore generalized probabilistic $m-$ spaces are introduced. Some preface remarks on contractions in $m-$ uniform spaces are given.

AMS Mathematics Subject Classification (1991): 54H25
Key words and phrases: probabilistic metric space.

Introduction

The concept of an $m-$ uniform space has been investigated by S. Gähler [5]. For definitions of an $m-$ metric space and a generalized $m-$ metric space see S. Gähler [5], [6] and [7]. Probabilistic metric spaces were introduced by Menger [13]. For the reference to definitions and basic facts of the theory of probabilistic metric spaces see for example [1], [2], [8], [9], [20], [21] and [22].

At first in our paper are formulated definitions and lemmas on an $m-$ uniform structure (see § 1). The topology induced by the $m-$ uniformity is defined in § 2. In § 3 we establish the answer to the following question:

when the topology $\mathcal{T}_{\mathcal{U}}$ of the $m-$ uniform space is completely regular and when $(X, \mathcal{U}, \mathcal{T}_{\mathcal{U}})$ is a Hausdorff's space ? (see Theorem 3.1).

In papers [14] and [15] examples of fixed - point theorems for selfmappings in some $m-$ uniform spaces are given. Definitions of such "special" $m-$ uniform spaces as for example $m-\mathcal{H}-$ spaces and $m-\mathcal{M}-$ spaces are given in § 4.

The section 5 generalize the well - known definitions of probabilistic metric spaces. Finally §6 contains the $m-$ metrization theorems of a Hicks and Sharma type (compare [9] and [14]).

1. The $m-$ uniform structure. The definitions and lemmas

Let m be a positive integer. In the sequal M denotes the set $\{0, ..., m\}$ and $M_{k_1,...,k_l} = M \setminus \{k_1, ..., k_l\}$, $0 \le k_1 < k_2 < ...k_l \le m$, $0 < 1 < m$. In the sequal $\prod_M$ denotes the set of all permutations of M.

Let X be a nonempty set and if $\alpha = (a_i)_{i \in M} \in X^M$, then $p(\alpha)$ denotes the point $(a_{p(i)})_{i\in M}$, where $p \in \prod_M$. For $\alpha \in X^M$, $\alpha_{a_i \to a} = (a_0, ..., a_{i-1}, a, a_{i+1}, ..., a_m)$, $\alpha_{a_i \to a, a_j \to a'} = (a_0, ..., a_{i-1}, a, a_{i+1}, ..., a_{j-1}, a', a_{j+1}, ..., a_m)$, $i \ne j$, $a, a' \in X, ...$

For $p \in \prod_M$ we denote

$$\Delta_p = \{\alpha \in X^M : p(\alpha) = \alpha\}.$$

The diagonal set $\Delta \subset X^M$ is defined in the following

$$\Delta = \bigcup_{p \in \prod_M \setminus \{\mathrm{id}_M\}} \Delta_p,$$

where $\mathrm{id}_M(j) = j$, $j \in M$.

$V \subset X^M$ is said to be symmetric iff $V = p(V)$ for each $p \in \prod_M$.

We define

$$V^{-1} = \bigcup_{p \in \prod_M \setminus \{\mathrm{id}_M\}} p(V).$$

Remark 1.1. The following facts are obvious:

a) $V = V^{-1}$ iff $V = p(V)$ for each $p \in \prod_M$,

b) $V^{-1} = \bigcup_{p\in\prod_M \setminus\{\mathrm{id}_M\}} p^{-1}(V)$ for $V \subset X^M$.

For subsets $V_0, ..., V_m \subset X^M$ we define the set

$$V_0 \circ V_1 \circ ... \circ V_m = \circ_{k=0}^{m} V_k = \{\alpha \in X^M : \exists_{v\in X} \alpha_{a_i\to v} \in V_{m-i},\ i \in M\}.$$

Following [5], if $V_0 = V_1 = ... = V_m = V$ then we write $\circ V$ instead of $\circ_{k=0}^{m} V_k$.

Lemma 1.1.

a) If $U_i, V_i \subset X^M, U_i \subset V_i$, $0 \le i \le m$, *then*

$$\circ_{k=0}^{m} U_k \subset \circ_{k=0}^{0} V_k,$$

b) If $\Delta \subset V$ *then*

$$V \circ \underbrace{\Delta \circ ... \circ \Delta}_{m} = \Delta \circ V \circ \Delta \circ ... \circ \Delta = ... = \underbrace{\Delta \circ ... \circ \Delta}_{m} \circ V = V.$$

Proof. The property a) follows immediately from the definition of $\circ_{k=0}^{m} V_k$. For the proof of b) assume, for example, that $\alpha \in V \circ \Delta \circ ... \circ \Delta$. Thus for some $v \in X$ the following relations hold

(0) $$(v, a_1, ..., a_m) \in \Delta$$

(1) $$(a_0, v, ..., a_m) \in \Delta$$

...

$(m-1)$ $$(a_0, a_1, ..., v, a_m) \in \Delta$$

(m). $$(a_0, a_1, ..., a_{m-1}, v) \in V$$

Suppose that $a_i \neq a_j$ for each $i \neq j$, $i, j \in M$. From (0) - (m-1), $v = a_i$ for some $i \in M$. If $v \neq a_m$, then from (i), $a_i = a_j$ for some $j \in M$. If $v \neq a_m$, then from (i), $a_i = a_j$ for some $j \in M$ and $j \neq i$, $j \neq m$. This contradiction proves that if $a_i \neq a_j$ for each $i \neq j$, $i, j \in M$, then $v = m_m$, i.e. $\alpha_{a_m \to a_m} = \alpha \in V$. From the above argumentation, $V \circ \Delta \circ \ldots \circ \Delta \subset V$. If $\alpha = (a_i)_{i \in M} \in V$ then $\alpha_{a_i \to a_m} \in \Delta$ for $i \in \{0, \ldots, m-1\}$ and $\alpha_{a_m \to a_m} = \alpha \in V$. Thus $\alpha \in V \circ \Delta \circ \ldots \circ \Delta$ and $V \subset V \circ \Delta \circ .. \circ \Delta$.

Remark 1.2. From Lemma 1.1 we get simple conclusions:

a) If $\Delta \subset V$ then $V \subset \circ V$,

b) $\circ\Delta = \Delta$.

We may omitt the proofs of the following Lemmas 1.2 - 1.4:

Lemma 1.2. *For each $U, V \subset X^M$,*

$$(U \cap V)^{-1} = U^{-1} \cap V^{-1}.$$

Lemma 1.3. *For each $V_0, \ldots, V_m \subset X^M$,*

$$\circ_{k=0}^{m} V_k^{-1} \subset (\circ_{l=0}^{m} V_{m-1})^{-1}.$$

Lemma 1.4. *For each $n \in N$ and $V_0, \ldots, V_n \subset X^M$,*

$$\circ(\bigcap_{i=0}^{n} V_i) \subset \bigcap_{i=0}^{n} (\circ V_i).$$

Let $\mathcal{U}$ be a family of subsets U of X^M. Following S. Gähler [5] the family $\mathcal{U}$ is an $m-$ uniform structure if $\mathcal{U}$ is a filter and in addition the conditions hold:

U_1	$\Delta \subset U$ for each $U \in \mathcal{U}$,
U_2	for each $U \in \mathcal{U}$ and $p \in \prod_M$, $p(U) \in \mathcal{U}$,
U_3	for each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ that $\circ V \subset U$.

The ordered pair $(X, \mathcal{U})$ is then called an $m-$ uniform structure and members of $\mathcal{U}$ are called entourages. A subfamily $\mathcal{B}$ of $\mathcal{U}$ is said to be a basis for $\mathcal{U}$ iff every entourage contains some member of $\mathcal{B}$. It is easy to verify the following fact:

Lemma 1.5. *Let $\mathcal{B}$ be a family of subsets of X^M such that*

($\mathcal{B}$.1) $\Delta \in B$ *for each* $B \in \mathcal{B}$

($\mathcal{B}$.2) *for* $B_1, B_2 \in \mathcal{B}$ *there exists* $B_3 \in \mathcal{B}$, *such that* $B_3 \subset B_1 \cap B_2$,

($\mathcal{B}$.3) *for each* $B \in \mathcal{B}$ *there exists an* $A \in \mathcal{B}$ *such that* $\circ A \subset B$,

($\mathcal{B}$.4) *if* $B \in \mathcal{B}$ *and* $p \in \prod_M$, *then* (B) *contains some member of* $\mathcal{B}$.

Then there exists a unique $m-$ *uniformity* $\mathcal{U}$ *on* X *for which* $\mathcal{B}$ *is a basis.* $\mathcal{U}$ *is said to be generated by* $\mathcal{B}$ *and may be defined as the family*

$$\{U \in 2^{X^M} : \quad \text{there exists } B \in \mathcal{B}, \text{ such that } B \subset U\}.$$

Lemma 1.6. *In an* $m-$ *uniform structure* $(X,\mathcal{U})$ *for each* $U \in \mathcal{U}$ *there exists a symmetric set* $V \in \mathcal{U}$ *such that* $\circ V \subset U$.

Proof. If $U \in \mathcal{U}$, then from U_3 there exists $V^* \in \mathcal{U}$, that $\circ V^* \subset U$. By U_2, for each $p \in \prod_M$ we have $p(V^*) \in \mathcal{U}$. But $\mathcal{U}$ is a filter and therefore

$$V = \bigcap_{p \in \prod_M \setminus \{\mathrm{id}_M\}} p(V^*) \in \mathcal{U}.$$

From Lemma 1.2, the set V is symmetric.

Two basis are said to be equivalent iff they generate the same $m-$ uniform[illegible] following assertion is obvious:

Lemma 1.7. *Two basis* $\mathcal{B}$ *and* $\mathcal{B}_1$ *are equivalent iff each* $B \in \mathcal{B}$ *contains some* $B_1 \in \mathcal{B}_1$ *and vice - versa.*

Lemma 1.8. *If the family* $\mathcal{S}$ *of subsets* S *of* X^M *satisfies*

($\mathcal{S}$.1)) $\Delta \subset S$ *for each* $S \in \mathcal{S}$

($\mathcal{S}$.2) *for each* $S \in \mathcal{S}$, *there exists* $V \in \mathcal{S}$ *with* $\circ V \subset S$,

($\mathcal{S}$.3) *for each* $U \in \mathcal{S}$ *and* $p \in \prod_M$ *there exists* $V \in \mathcal{S}$ *thet* $V \subset p(U)$,

then $\mathcal{S}$ *is an* $m-$ *uniform subbasis.*

Proof. Let $\mathcal{B}$ be a family of all finite intersections of members of $\mathcal{S}$. Obviously, for each $U, V \in \mathcal{B}$, $U \cap V \in \mathcal{B}$. Also, if $B \in \mathcal{B}$, then

$$B = \bigcap_{i=0}^{n} S_i, \quad S_i \in \mathcal{S}, \quad i = 0, ..., n.$$

Thus for each $i \in \{0, ..., n\}$, there exists V_i such that $\circ V_i \subset S_i$. We have $A = \bigcap_{i=0}^{n} V_i \in \mathcal{B}$. Thus

$$\circ A = \circ(\bigcap_{i=0}^{n} V_i) \subset \bigcap_{i=0}^{n}(\circ V_i) \subset \bigcap_{i=0}^{n} S_i = B.$$

From Lemma 1.5, $\mathcal{B}$ is an $m-$ uniform basis.

2. The topology induced by the $m-$ uniformity and remarks on the generalized $m-$ metric space

Let $a, a' \in X$ and $\alpha = (a_i)_{i \in M_{k1}} \in X^{M_{k1}}$ for some $(k, 1) \in M^2$, $k < 1$. Then in the sequal $[a, a', \alpha]$ denote the point $\alpha' \in X^M$, such that $a'_k = a, a'_1 = a'$ and $a'_j = a_i$ for $i \in M_{k1}$, and

$$U_\alpha[a] = \{a' \in X : [a, a', \alpha] \in U\}$$

for any $U \subset X^M$.

Theorem 2.1. *Let* $(X, \mathcal{U})$ *be an* $m-$ *uniform structure for some* $m \geq 1$ *and*

$$U[a] = \bigcap_{s=1}^{n} U_{\alpha^s}[a]$$

where $\{\alpha^s \in X^{M_{01}} : s = 1, ..., n\}$, $n \in N$, $U \in \mathcal{U}$, *is an arbitrary finite system of points. Then the family*

$$\mathcal{U}[a] = \{U[a] : U \in \mathcal{U}\}, \ \ a \in X,$$

is a neighbourhood system on X.

Proof. We will prove here that the conditions (N.1) - (N.4) of Th. 3.3.2 of [19] (see also [18], p. 49, (A^*)) hold.

(N.1) Every point of X is contained in at least one neighbourhood, and is contained in each of its neighbourhoods. Indeed, if $U \in \mathcal{U}$ and $U[a] = \bigcap_{s=0}^{n} U_{\alpha^s}[a]$ then from the relations $a \in \Delta[a]$ and $\Delta \subset U$ we get $\Delta[a] \subset U[a]$ and consequently $a \in U[a]$;

(N.2) The intersection of any two neighbourhoods of a point is a neighbourhood of that point. Indeed, if $U, V \in \mathcal{U}$ then $U \cap V \in \mathcal{U}$. Thus from (N.1), $a \in (U \cap V)[a]$. If $a' \in (U \cap V)[a]$ then $a' \in \bigcap_{s=0}^{n}(U \cap V)_{\alpha^s}[a]$ i.e. $a' \in (U \cap V)_{\alpha^s}[a]$ for each $s = 0, ..., n$. Thus $[a, a', \alpha^s] \in U \cap V$ and $[a, a', \alpha^s] \in U$, $[a, a', \alpha^s] \in V$ for each $s = 0, ..., n$. Therefore $a' \in \bigcap_{s=0}^{n} U_{\alpha^s}[a]$ and $a' \in \bigcap_{s=0}^{n} V_{\alpha^s}[a]$ and $(U \cap V)[a] \subset U[a] \cap V[a]$.

(N.3) Any set which contains a neighbourhood of a point is itself a neighbourhood of that point. Indeed, if $U[a]$ is given and $A \supset U[a]$ then $V = U \cup A^M \in \mathcal{U}$, $V[a] = A$ and thus $A \in \mathcal{U}[a]$.

(N.4) Given $U[a]$, we will show that there exists $V[a]$ such that $U[a] \in \mathcal{U}[b]$ for each $b \in V[a]$. From Lemma 1.6, there exists $V \in \mathcal{U}$ that $\circ V \subset U$ and $V = V^{-1}$. Let $b \in V[a]$ and

$$V^* = \left(\bigcap_{i=0}^{m-2} V_{\alpha_{a_i \to a}}[b]\right) \cap V_\alpha[b], \ \alpha \in X^{M_{k1}}.$$

Then $V^* \in \mathcal{U}[b]$ and for $z \in V$ we have the relations $[a, z, \alpha_{a_i \to b}] \in V$, $i = 0, ..., m-2$, $[a, y, \alpha] \in V$ and $[y, z, \alpha] \in V$. Thus $[a, z, \alpha] \in V$ i.e. $z \in (\circ V)[a] \subset U[a]$. Therefore $V^* \subset U[a]$. The proof of Theorem 2.1 is complete.

If $\mathcal{U}$ is an $m-$ uniformity on X, then the topology $\mathcal{T}_{\mathcal{U}}$ defined by the neighbourhood system is called the topology induced by $\mathcal{U}$. An ordered triple $(X, \mathcal{U}, \mathcal{T}_{\mathcal{U}})$ is called an $m-$ uniform space.

The proofs of Lemmas 2.1 - 2.2 follow immediately from Theorem 2.1:

Lemma 2.1. *Let $(X, \mathcal{U}, \mathcal{T}_{\mathcal{U}})$ be an $m-$ uniform space. Then $G \subset X$ is open iff for each $a \in G$, there exists an entourage U such that $U[a] \subset G$.*

Lemma 2.2. *If $\mathcal{B}$ is a basis for $\mathcal{U}$ and $\beta = \{B :$ for each $a \in B$, there exists $U \in \mathcal{B}$, that $U[a] \subset B\}$, then β is a basis for $\mathcal{T}_{\mathcal{U}}$.*

Theorem 2.2. *Let $\mathcal{B}$ be a basis of the $m-$ uniformity $\mathcal{U}$ on X and $A \subset X$. Then*

$$A^- = \bigcap_{U \in \mathcal{B}} U^{-1}[A],$$

where $U[A] = \bigcup_{a \in A} U[a]$ for $U \in \mathcal{B}$.

Proof. By Lemma 2.2, $a \in A^-$ iff $U[a] \cap A \neq \emptyset$ for each $U \in \mathcal{B}$ iff there exists $x \in A$ such that $x \in U[a]$ for every $U \in \mathcal{B}$ iff $a \in U^{-1}[x]$ for $x \in A$ and each $U \in \mathcal{B}$ iff $a \in U^{-1}[A]$ for each $U \in \mathcal{B}$.

S.Gähler in [5] gives the following important for our investigations example of the $m-$ uniform space:

Example 2.1. Let $(E, \leq)$ be a partial ordered set and the order relation $\leq$ have additional properties:

$(\leq .1)$ there exists $O \in E$ that $(O, \epsilon) \in \leq$ for each $\epsilon \in E$,

$(\leq .2)$ for each $\epsilon, \epsilon' \in E_0 = E \setminus \{0\}$ there exists $\epsilon'' \in E_0$, that $(\epsilon'', \epsilon), (\epsilon'', \epsilon') \in \leq$ (i.e. E_0 is directed set).

For $\epsilon, \epsilon' \in E$ we write $\epsilon \leq \epsilon'$ iff $(\epsilon, \epsilon') \in \leq$, $\epsilon \not\leq \epsilon'$ if $(\epsilon.\epsilon') \notin \leq$ and $\epsilon < \epsilon'$ if $\epsilon \leq \epsilon'$ and $\epsilon \neq \epsilon'$.

Let X be a nonempty set and m be a positive integer. The function $\sigma : X^M \to E$ is a generalized $m-$ metric over X and E (see S.Gähler [5], p. 177) if

M'_{1a} $\sigma(\alpha) = 0$ for each $\alpha \in \Delta$,

M'_2 for each $\epsilon \in E_0$ and each $p \in \prod_M$ there exists $\epsilon' \in E_0$ such that $\sigma(p(\alpha)) \not\geq \epsilon$ whenever $\sigma(\alpha) \not\geq \epsilon$,

M'_3 for each $\epsilon \in E_0$ there exists $\epsilon' \in E_0$ such that $\sigma(\alpha) \not\geq \epsilon$ whenever $\sigma(\alpha_{a_i \to a}) \not\geq \epsilon'$ for each $i \in M$, $a \in X$.

The order pair (X, σ) is said to be a generalized $m-$ metric space if σ has properties M'_{1a}, M'_2 and M'_3.

Remark 2.1. a) S.Gähler in [5] defines the generalized $m-$ metric space in the case M is non - necessary finite set.

b) If (X, σ) is a generalized $m-$ metric spaces, then the family $\mathcal{B} = \{U_\epsilon \subset X^M : \epsilon \in E_0\}$, where $U_\epsilon = \{\alpha \in X^M : \sigma(\alpha) \not\geq \epsilon\}$, $\epsilon \in E_0$, is a basis of the $m-$ uniform structure (see [5], Th. 9_2) and if $\mathcal{U}$ is an $m-$ uniform structure on X, then there exists a generalized $m-$ metric σ which generates same $m-$ uniform structure (see [5], Th. 10).

Remark 2.2. a) Let (X, σ) be a generalized $m-$ metric space with card $M < \infty$. The topology τ_σ is defined in the following way: Let σ be a family of all subset of $X^{M_{k1}} \times E_0$. For $\Sigma_i \in \sigma$ also $\bigcup_i \Sigma_i \in \sigma$. $\Sigma \leq' \Sigma'$ iff $(\alpha, \epsilon) \in \Sigma$, $(\alpha, \epsilon') \in \Sigma'$ and $\epsilon \leq \epsilon'$. For $a \in X$ and $\Sigma \in \sigma$ in the sequal $W_\Sigma(a)$

denotes (see [5] , p. 183) the set $\{a' \in X : \sigma[a,a',\alpha] \not\geq \epsilon$ for all $(\alpha,\epsilon) \in \Sigma\}$ and in the sequal $W'_\Sigma(a)$ denotes the set $\{a' \in X : \sigma[a,a',\alpha] \not> \epsilon$ for all $(\alpha,\epsilon) \in \Sigma\}$. It is easy to verify, that $W_\Sigma(a) = \bigcup_{\Sigma' < \Sigma} W'_{\Sigma'}(a)$ for each $a \in X$ (see [5] , p. 184) and $\{W_\Sigma(a) : \Sigma \in \sigma\}$ is the system of neighbourhood at a, for each $a \in X$ (see [5], Theorems 12 - 17) .

b) In particular in a generalized $m-$ metric space (X,σ), $A^- = \bigcap_{\Sigma \in E} W_\Sigma(A)$ (see [5], Th. 18).

3. Remarks on separation axioms

S.Gähler ([6], Th. 25) proved, that if the generalized $m-$ metric space (X,σ) has the additional property

M'_{T_σ} For each two different points $x, y \in X$ there exists $\alpha \in X^{M_{01}}$ such that $\sigma[x,y,\alpha] \neq 0$, then X is a Hausdorff's space. Moreover, he proved in [6] (see Th. 27), that if the order relation $\leq$ is dense in E_0 and for each $a \in X$ and $\Sigma \in \sigma$ the set $W_\Sigma(a)$ is open and $W'_\Sigma(a)$ is closed then the generalized $m-$ metric space (X,σ) is a completely regular topological space.

The above Gähler's results on separation axioms for generalized $m-$ metric spaces give us the motivation for the below consideration

Theorem 3.1. *Let $\mathcal{B}$ be a family of subsets of X^M of the form $\mathcal{B} = \{U_\epsilon \subset X^M : \epsilon \in E_0\}$, where E_0 is a partially ordered set with properties $(\leq .1)$ - $(\leq .2)$ of Example 2.1. If*

(3.1) for each $\epsilon \in E_0$, $\Delta \subset U_\epsilon$,

(3.2) if $\delta, \epsilon \in E_0$, $\delta < \epsilon$, then $U_\delta \subset U_\epsilon$,

(3.3) for each $\epsilon \in E_0$ there exists $\epsilon' \in E_0$, that $\circ U_{\epsilon'} \subset U_\epsilon$

(3.4) for each $\epsilon \in E_0$ and each $p \in \prod_M$, $p(U) \in \mathcal{B}$,

then $\mathcal{B}$ is the base of an $m-$ uniform structure $\mathcal{U}$ and $(X,\mathcal{U},\mathcal{T}_\mathcal{U})$ is a completely regular topological space. If in addition the property holds

$$\Delta = \bigcap_{\epsilon \in E_0} U_\epsilon \tag{3.5}$$

then $(X,\mathcal{U},\mathcal{T}_\mathcal{U})$ is a Hausdorff's topological space.

Proof. Let us consider the family

$$\mathcal{B}' = \{U_{\alpha,\epsilon} \subset X^2 : \epsilon \in E_0 \quad \text{for some} \quad \alpha \in X^{M_{01}}\},$$

where

$$U_{\alpha,\epsilon} = \{(x,y) \in X^2 : [x,y,\alpha] \in U_\epsilon, \ \ \epsilon \in E_0\}.$$

Obviously $\Delta = \{(x,x) : x \in X\} \subset U_{\alpha,\epsilon} \subset X^2$ for each $\epsilon \in E_0$. If $W = V_{\epsilon_0} \circ V_{\epsilon_1} \circ ... \circ V_{\epsilon_m}$ and $[x,y,\alpha] \in W$ then $[v,y,\alpha] \in V_{\epsilon_m}$, $[x,v,\alpha] \in V_{\epsilon_{m-1}}$, $[x,y,\alpha_{a_i \to v}] \in V_{\epsilon_{m-i-2}}$, $i \in \{0,...,m-2\}$, for some $v \in X$.

In particular we get $(x,y) \in V_{\alpha,\epsilon_m}$ and $(x,v) \in V_{\alpha,\epsilon_{m-1}}$. Therefore

$$(x,y) \in V_{\alpha,\epsilon_{m-1}} \circ V_{\alpha,\epsilon_m}$$

and thus for each $U_{\alpha,\epsilon}$ there exists $V_{\alpha,\delta}$, that $\circ V_{\alpha,\delta} = V_{\alpha,\delta} \circ V_{\alpha,\delta} \subset U_{\alpha,\epsilon} \subset X^2$ (Obviously, $A \circ (B \circ C) = (A \circ B) \circ C$ for each $A, B, C \in \mathcal{B}'$). It is obvious, that if $U_{\alpha,\epsilon}$, $U_{\alpha,\epsilon'} \in \mathcal{B}'$ then there exists $\eta \in E_0$ that $U_{\alpha,\eta} \subset U_{\alpha,\epsilon} \cap U_{\alpha,\epsilon'}$. The family $\mathcal{B}'$ is the basis of the usual uniformity $\mathcal{U}'$ and the topology $\mathcal{T}_{\mathcal{U}'}$ is weaker then $\mathcal{T}_{\mathcal{U}}$. It is well - known (see for example [19], Th. 11.2.2) that $\mathcal{T}_{\mathcal{U}'}$ is completely regular and therefore $\mathcal{T}_{\mathcal{U}}$ is completely regular too.

If (3.5) holds then for each $a \in X$, $\{a\} = \Delta[a] = \bigcap_s \Delta_{\alpha^s}[a] = \bigcap_s (\bigcap_{U \in \mathcal{U}} U)_{\alpha^s}[a] = \bigcap_{U \in \mathcal{U}} U[a] = (\bigcap U)[a] \in \mathcal{U}[a]$ and $(X, \mathcal{U}, \mathcal{T}_{\mathcal{U}})$ is a Hausdorff's space.

4. Some examples of $m-$ uniform spaces

In this section of our paper we give definitions and examples of $m - \mathcal{H}-$ spaces and $m - \mathcal{M}-$ spaces. In such spaces we may prove some fixed - point theorems (see [14] and [15]).

Let $(J, \leq)$ be a directed and partially ordered set. Let X be a nonempty set, m be a positive integer and let $\mathcal{B}$ be a family of nonempty subsets of X^M of the form

$$\mathcal{B} = \{U_{j,\epsilon} \subset X^M : j \in J, \ \epsilon \in (0,r)\}, \ r \in R_+^\sharp, \ r > 0,$$

$R_+^\sharp = R_+ \cup \{+\infty\}$, such that

$$\Delta = \bigcap_{(j,\epsilon) \in J \times (0,r)} U_{j,\epsilon} \tag{4.1}$$

(4.2) $$U_{i,\delta} \subset U_{j,\epsilon} \quad \text{whenever} \quad (i,\delta) \leq (j,\epsilon)$$

($(i,\delta) \leq (j,\epsilon)$ iff $i \leq j$ and $\delta \leq \epsilon$),

(4.3) $$U_{j,\epsilon} = U_{j,\epsilon}^{-1} \quad \text{for each} \quad (j,\epsilon) \in J \times (0,r),$$

(4.4) for each $j \in J$ and each $\epsilon \in (0,r)$ there exists $\epsilon' \in (0,r)$,

that $\circ U_{j,\epsilon'} \subset U_{j,\epsilon}$,

(4.5) for each $j \in J$ each $\epsilon \in (0,r)$ and each $\alpha \in X^M$, if $\alpha \in U_{j,\epsilon}$

then there exists $\epsilon' < \epsilon$ that $\alpha \in U_{j,\epsilon'}$.

It is easy to verify that $\mathcal{B}$ is the base of an $m-$ uniform structure fulfilling assumptions of Theorem 3.1. We say that an $m-$ uniform structure is an $m-\mathcal{H}-$ structure if it is generated by the base $\mathcal{B}$ fulfilling (4.1) - (4.5).

If $(X,\mathcal{U})$ is an $m-\mathcal{H}-$ structure generated by $\mathcal{B}$, then the neighbourhood system $\{U[a] : a \in X,\ U \in \mathcal{U}\}$ (or equivalently $\{U_{j,\epsilon}[a] : j \in J,\ \epsilon \in (0,r),\ a \in X\}$ defines the topology $\mathcal{T}_\mathcal{U}$ and the ordered triple $(X,\mathcal{U},\mathcal{T}_\mathcal{U})$ is called an $m-\mathcal{H}-$ space.

A filter $\mathcal{F}$ in an $m-\mathcal{H}-$ space $(X,\mathcal{U},\mathcal{T}_\mathcal{U})$ is a Cauchy filter iff for each $(\alpha,j,\epsilon) \in X^{M_{01}} \times J \times (0,r)$, $F \times F \subset U_{j,\epsilon,\alpha}$ for some $F \in \mathcal{F}$. A $m-\mathcal{H}-$ space $(X,\mathcal{U},\mathcal{T}_\mathcal{U})$ is complete iff each Cauchy filter in X converges to a point of X.

Example 4.1. Let $(J,\leq)$ be a directed and partially ordered set and X be a nonempty set. Let $\sigma = (\sigma_j)_{j\in J}$ be a family of functions $\sigma_j : X^M \to R_+$ such that the conditions hold

$(M_{1a})_\mathcal{H}$ $\sigma_j(\alpha) = 0$ for each $\alpha \in \Delta$, $j \in J$,

$(M_{1b})_\mathcal{H}$ for each different $a, a' \in X$ there exists $\alpha \in X^{M_{01}}$ and $j \in J$, that $\sigma_j[a,a',\alpha] > 0$,

$(M_2)_\mathcal{H}$ $\sigma_j(\alpha) = \sigma_j(p(\alpha))$ for each $\alpha \in X^m$, $j \in J$ and $p \in \prod_M$,

$(M_3)_\mathcal{H}$ for each $j \in J$ and each $\epsilon \in (0,r)$ there exists $\delta > 0$ that for each $\alpha \in X^M$ and $v \in X$ we have $\sigma_j(\alpha) < \epsilon$ whenever $\sigma_j(\alpha_{a_i \to v}) < \delta$.

We say that the pair (X,σ) fulfilling $(M_{1a})_\mathcal{H}-(M_3)_\mathcal{H}$ (with the topology $\mathcal{T}_\sigma$ defined as usual) is a $\mathcal{H}-m-$ metric space. Obviously, (X,σ), $\sigma =$

$(\sigma_j)_{j\in J}$, is a generalized $m-$ metric space in the sense of S. Gähler (see Example 2.1).

A sequence (x_n) in a $\mathcal{H}-m-$ metric space (X,σ), $\sigma=(\sigma_j)_{j\in J}$, is a Cauchy sequence iff for every $\alpha\in X^{M_{01}}$ and $j\in J$, $\sigma_j[x_n,x_{n+p},\alpha]\to 0$ as $n,p\to\infty$ and $\mathcal{H}-m-$ metric space (X,σ) is sequentially complete iff each Cauchy sequence in X converges to a point of X.

Example 4.2. Let X be a nonempty set, m be a positive integer and let $\mathcal{B}$ be a family of nonempty subsets of X^M of the form $\mathcal{B}=\{U_\epsilon\subset X^M:\epsilon\in(0,r)\}$, $r\in R$, $r>0$, such that

$$\Delta=\bigcap_{0<\epsilon<r}U_\epsilon \tag{4.6}$$

$$\text{if}\quad 0<\delta<\epsilon<r\quad\text{then}\quad U_\delta\subset U_\epsilon, \tag{4.7}$$

$$U_\epsilon=U_\epsilon^{-1}\quad\text{for each}\quad\epsilon\in(0,r), \tag{4.8}$$

$$\text{for each}\quad\epsilon\in(0,r)\ \text{there exists}\ \epsilon'\in(0,r),\ \text{that}\quad\circ\,U_{\epsilon'}\subset U_\epsilon \tag{4.9}$$

$$\text{for each}\quad\epsilon\in(0,r)\quad\text{and each}\ \alpha\in X^M,\quad\text{if}\ \alpha\in U_\epsilon \tag{4.10}$$

then there exists $\epsilon'<\epsilon$ that $\alpha\in U_{\epsilon'}$.

We say (see [14]) that an $m-$ uniform structure is an $m-H-$ structure if it is generated by the base $\mathcal{B}$ fulfilling (4.6) - (4.10). Obviously, an $m-H-$ structure is an $m-\mathcal{H}-$ structure too.

Example 4.3. Let X be a nonempty set and $\sigma:X^M\to R_+$ we have properties

$(M_{1a})_H$ $\sigma(\alpha)=0$ for each $\alpha\in\Delta$,

$(M_{1b})_H$ for each different $a,a'\in X$ there exists $\alpha\in X^{M_{01}}$ that $\sigma[a,a',\alpha]>0$,

$(M_2)_H$ $\sigma(\alpha)=\sigma(p(\alpha))$ for each $\alpha\in X^M$ and $p\in\prod_M$,

$(M_3)_H$ for each $\epsilon>0$ there exists $\delta>0$ that for each $\alpha\in X^M$ and $v\in X$ we have $\sigma(\alpha)<\epsilon$ whenever $\sigma(\alpha_{a_i\to v})<\delta$, $i=0,...,m$.

We say (for $m=1$, compare [11]) that the pair (X,σ) fulfilling $(M_{1a})_H-(M_3)_H$ (with the topology $\mathcal{T}_\sigma$ defined as usual) is a $H-m-$ metric space.

Remark 4.1. Each $\mathcal{H}-m-$ metric space ($H-m-$ metric space) is an $m-\mathcal{H}-$ space ($m-H-$ space, respectively). For the proof of this assertion let (X,σ), $\sigma=(\sigma_j)_{j\in J}$, be an $\mathcal{H}-m-$ metric space and $\mathcal{H}_{j,\alpha}(\epsilon)=h(\epsilon-\sigma_j(\alpha))$, $\alpha\in X^M$, $\epsilon\in(0,r)$, $r\in R_+$, where $h(t)=0$ for $t\leq 0$ and $h(t)=r$ for $t>0$, $t\in R$. It is evident that $\mathcal{H}_{j,\alpha}(\cdot):R\to[0,r]$ is a non - decreasing, left continuous function with $\inf\{\mathcal{H}_{j,\epsilon}(t):t\in R\}=0$ and $\sup\{\mathcal{H}_{j,\alpha}(t):t\in R\}=r$, $j\in J$. Let $U_{j,\epsilon}=\{\alpha\in X^M:\mathcal{H}_{j,\alpha}(\epsilon)>r-\epsilon\}$, $\epsilon\in(0,r)$. It is easy to see that $\mathcal{B}=\{U_{j,\epsilon}:j\in J,\ \epsilon\in(0,r)\}$ is the base of the $m-$ uniformity $\mathcal{U}$ such that all properties (4.1) - (4.5) are fulfilled for $\mathcal{B}$ and $(X,\mathcal{U},\tau_{\mathcal{U}})$ is an $m-\mathcal{H}-$ space. We also have for $\epsilon\in(0,r)$, $\alpha\in U_{j,\epsilon}$ iff $\mathcal{H}_{j,\alpha}(\epsilon)>r-\epsilon$ iff $\sigma_j(\alpha)<\epsilon$. Thus $\tau_{\mathcal{U}}=\tau_\sigma$, $\sigma=(\sigma_j)_{j\in J}$.

We say that the function $T:[0,r)^M\to[0,r)$, $r\in R_+$, is a triangle function on $[0,r)$ if

$(T.1)$ $T(a,a,...,a)\leq a$ for each $a\in(0,r)$,

$(T.2)$ $T(\alpha)=T(p(\alpha))$ for each $\alpha\in[0,r)^M$ and $p\in\prod_M$,

$(T.3)$ $T(\alpha)\leq T(\alpha^1)$ for each $\alpha,\alpha^1\in[0,r)^M,\alpha\leq\alpha^1$ (i.e. $a_i\leq a_i^1$ for $i\in M$).

Let $(J,\leq)$ be a partially ordered and directed set. Let X be a non - void set, $\mathcal{B}$ be a family of nonempty subset of X^M of the form $\mathcal{B}=\{U_{j,\epsilon}\subset X^M: j\in J,\ \epsilon\in(0,r)\}$, $r\in R_+^\sharp$, such that (4.1) - (4.3) and (4.5) are fulfilled and in addition the following conditions hold

(4.11) for each $j\in J$ and $\epsilon=(\epsilon_0,...,\epsilon_m)\in(0,r)^M$

the relation holds $\circ_{k=0}^m U_{j,\epsilon_k}\subset U_{j,T(\epsilon)}$, where T is a triangle function on $[0,r)$.

We say that an $m-$ uniform structure $(X,\mathcal{U})$ is an $\mathcal{M}-m-$ structure if it is generated by $\mathcal{B}$ fulfilling (4.1) - (4.3) and (4.5) and (4.11). The ordered triple $(X,\mathcal{U},\mathcal{T}_{\mathcal{U}})$ is called then an $m-\mathcal{M}-$ space.

Example 4.4. Let $(J,\leq)$ be as above and m be a positive integer. Let X be a nonempty set and $\sigma=(\sigma_j)_{j\in J}$ be the family of functions $\sigma_j:X^M\to R_+$, $j\in J$, such that σ has properties

$(M_{1a})_{\mathcal{H}}-(M_2)_{\mathcal{H}}$ and in addition the following condition holds $(M_3)_{\mathcal{H}}$ for each $j\in J$, each $\alpha\in X^M$ and each $v\in X$,

$$\sigma_j(\alpha)\leq\sum_{i=1}^m\sigma_j(\alpha_{a_i\to v}).$$

The pair (X, σ) is called a space generated by $\sigma = (\sigma_j)_{j \in J}$.

Remark 4.2. Each generalized $m-$ metric space of Example 4.4 is a $\mathcal{M} - m-$ space. Indeed, if $\mathcal{B} = \{U_{j,\epsilon} : j \in J, \ \epsilon \in (0,r)\}$, where $U_{j,\epsilon} = \{\alpha \in X^M : \sigma_j(\alpha) < \epsilon\}$, $j \in J$, $\epsilon \in (0,r)$, then $\mathcal{B}$ is a basis of some $m-$ uniformity $\mathcal{U}$. All properties (4.1) - (4.3), (4.5) and (4.11) are fulfilled for the family $\mathcal{B}$ with $T(\alpha) = \min\{a_i : i \in M\}$, $\alpha \in [0,r]^M$.

Let X be a non - void set and let $\mathcal{B}$ be a family of nonempty subsets of X^M of the form $\mathcal{B} = \{U_\epsilon \subset X^M : \epsilon \in (0,r)\}$ such that (4.6) - (4.8) and (4.10) are fulfilled and in addition the following condition holds

(4.12) for each $\epsilon = (\epsilon_0, ..., \epsilon_m) \in (0,r)^M$, $\circ_{k=0}^m U_{\epsilon_k} \subset U_{T(\epsilon)}$, where T is a triangle function on $[0,r]$.

We say that an $m-$ uniform structure $(X,\mathcal{U})$ is an $m - M-$ structure if it is generated by $\mathcal{B}$ fulfilling (4.6) - (4.8), (4.10) and (4.12). The ordered triple $(X,\mathcal{U},\mathcal{T}_\mathcal{U})$ is called then a $m - M-$ space (for $m = 1$, see [14]).

Example 4.5. Following S.Gähler [3], let X be a nonempty set and m be a positive integer. The function $\sigma : X^M \to R_+$ is an $m-$ metric (see [3], [4] and [5]) if σ has properties $(M_{1a})_H - (M_2)_H$ and

$(M_3)_H$ for each $\alpha \in X^M$ and each $v \in X$,

$$\sigma(\alpha) \leq \sum_{i=0}^{m} \sigma(\alpha_{a_i \to v}).$$

The pair (X, σ) is called then an $m-$ metric space. It is easy to verify that an $m-$ metric space is an $m - M-$ space.

5. The definitions of $m-$ probabilistic spaces

In this paragraph we generalize the will - known definitions of $PPM-$ spaces and $PM-$ space (see for example [2], [8], [9], [13], [14], [17] and [20]).

A function $f : R \to [0,1]$ is a distribution function if it is a non - decreasing, left continuous function with $\inf f = 0$ and $\sup f = 1$. The set of all distribution functions we denote D (see also Schweizer and Sklar [20]).

Let $F : X^M \times R \to D$, $F_\alpha(t) =_{\text{not}} F(\alpha, t)$, $t \in R$, $\alpha \in X^M$. We say that F is a pre - probabilistic $m-$ metric structure ($PPM - m-$ structure) on X if the following conditions hold

(F.1) $F_\alpha(0) = 0$ for all $\alpha \in X^M$

(F.2) $F_\alpha(\epsilon) = 1$ for all $\epsilon > 0$ iff $\alpha \in \Delta$.

In this case the pair (X, F) is called a preprobabilistic $m-$ metric space ($PPM - m-$ space).

The $PPM-m-$ structure F on X is a probabilistic $m-$ metric structure ($PM-m-$ structure) and the pair (X, F) is a probabilistic $m-$ metric space ($PM - m-$ space) if F is symmetric, i.e. the following additional condition holds

(F.3) $F_\alpha = F_{p(\alpha)}$ for each $\alpha \in X^M$ and $p \in \prod_M$.

A $PM - m-$ space (X, F) is an $H - m-$ space (for $m = 1$ compare Hicks and Sharma [9]) that satisfies the following condition

$(F.4)_H$ for each $\epsilon > 0$ there exists $\delta > 0$ that $F_\alpha(\epsilon) > 1 - \epsilon$ whenever $F_{\alpha_{a_i \to v}}(\delta) > 1 - \delta$, for each $i \in M$ and $v \in X$.

Remark 5.1.

a) Hicks and Sharma prove in [9] that a topological space $(X, \mathcal{T})$ is metrizable iff there exists an $H - 1-$ structure on X which induces $\mathcal{T}$.

b) It is evident that if (X, F) is a $H - m-$ space then the family $\mathcal{B} = \{U_\epsilon \subset X^M : 0 < \epsilon < 1\}$, where $U_\epsilon = \{\alpha \in X^m : F_\alpha(\epsilon) > 1 - \epsilon\}$, $0 < \epsilon < 1$, is the base of a $m - H-$ structure $\mathcal{U}$.

c) A.Menger space is a $PM - 1-$ space that satisfies the condition

$$F_{(x,z)}(\epsilon + \delta) \geq T(F_{(x,y)}(\epsilon),\ F_{(y,z)}(\delta)), \tag{5.1}$$

where T is a $t-$ norm i.e. $T : [0,1]^2 \to [0,1]$ and

$$T(0,0) = 0,\ \ T(x,1) = x \tag{5.2}$$

$$T(x,y) = T(y,x) \tag{5.3}$$

$$T(x,y) \leq T(x^1, y^1) \ \text{ if } \ x \leq x^1,\ y \leq y^1, \tag{5.4}$$

$$T(T(x,y), z) = T(x, T(y,z)),\ \ x, y, z \in [0,1]. \tag{5.5}$$

d) The fixed - point theory in $PM-1-$ spaces was begin from 1972 with the paper [21] of Sehgal and Bharuda - Reid. They say in [21] that a mapping f of $PM-1-$ space (X,F) into itself is a contraction if there exists $k \in [0,1)$, such that for each $x,y \in X$, $F_{(fx,fy)}(kt) \geq F_{(x,y)}(t)$ for all $t>0$. Sehgal and Bharuda - Reid proved that if (X,F) is a complete Menger space with $T(x,y)=\min\{x,y\}$, $x,y \in [0,1]$, and $f: X \to X$ is a contraction, then there exists a unique fixed point $\bar{x}$ of f in X and $\lim f^n x = \bar{x}$ for each $x \in X$. Many authors generalize the result from [21] for selfmappings in a sequentially complete Menger space in the case if $m=1$ (see for example references of [15]). The associative law (5.5) of the definition of a Menger space is very strong and we can not translate (5.5) from the case $m=1$ into the case $m \geq 1$. Therefore we will consider the generalization of a Menger space in which the condition (5.5) is replaced by less restrictive $(F.4)_M$.

We say that a $PM-m-$ space (X,F) is a Menger $m-$ space if it has properties (F.1) - (F.3) and $(F.4)_M$ for $\epsilon=(\epsilon_0,...,\epsilon_m) \in (0,1)^M$, $F_\alpha(T(\epsilon)) > 1-T(\epsilon)$, whenever $F_{\alpha_{a_i \to v}}(\epsilon_i) > 1-\epsilon_i$ for each $i \in M$, $\alpha \in X^M$, $v \in X$, where T is a triangle function fulfilling (T.1) - (T.3).

Following Nguyen Xuan Tan [17] let D_1 be the set of all non - decreasing left - continuous functions from $R^\sharp$ into R_+, where $R^\sharp = R \cup \{-\infty\} \cup \{+\infty\}$.

We say that (X,F) is a generalized probabilistic $m-$ metric space, shortly $GPM-m-$ space if $F: X^M \times R^\sharp \to D_1$ and the following conditions are fulfilled:

(5.6) $F_\alpha(0)=0$ for all $\alpha \in X^M$,

(5.7) if $\alpha \in \Delta$ then $F_\alpha(\epsilon)=1$ for all $\epsilon>0$,

(5.8) $F_\alpha(\epsilon)=F_{p(\alpha)}(\epsilon)$ for each $p \in \prod_M$ and $\epsilon>0$,

(5.9) $F_\alpha(\min\{\epsilon_0,...,\epsilon_m\}) \geq \min\{F_{\alpha_{a_i \to v}}(\epsilon_0),...,F_{\alpha_{a_m \to v}}(\epsilon_m)\}$

for each $\epsilon=(\epsilon_0,...,\epsilon_m) \in (0,\infty)^M$, $\alpha \in X^M$ and $v \in X$.

Remark 5.2.

a) It is easy to see that $\sup\{F_\alpha(t) : t \in R_+^\sharp\} = \sup\{F_\alpha(t) : t>0\} \leq 1$ for each $\alpha \in X^M$. Indeed, if $\alpha \in \Delta$, then from (5.7) , $F_\alpha(t)=1$ for each $t>0$. Suppose that $\alpha \notin \Delta$. Then $\alpha_{a_0 \to a_1} \in \Delta$ and obviously $F_{\alpha_{a_0 \to a_1}}(t)=1$ for any $t>0$.

Thus from (5.9) we get

$$1 = F_{\alpha_{a_0 \to a_1}}(t) = F_{\alpha_{a_0 \to a_1}}(\min\{t, ..., t\}) \leq$$

$$\min\{F_{\alpha_{a_0 \to a_0}}(t),\ F_{\alpha_{a_1 \to a_0}}(t), ..., F_{\alpha_{a_m \to a_0}}(t)\} =$$

$$Min\{F_\alpha(t), 1, ..., 1\} = F_\alpha(t).$$

b) A $GPM - 1$ space was introduced by Nguyen Xuan Tan [17]. In fact, Nguyen Xuan Tan assumes that a triangle function T (i.e. $T(x,y) = \min\{x,y\},\ x,y \in (0,\infty)$) fulfils an associative law and he de facto use this property of T in his argumentations.

Let D_2 be the set of all non - decreasing left - continuous functions from $R^\sharp$ into $[0,1]$. We say that (X,F) is a generalized probabilistic $m-\mathcal{H}-$ space if (5.6) - (5.8) and $(F.4)_H$ are fulfilled for $F : X^M \times R^\sharp \to D_2$. The pair $(X,F),\ F : X^M \times R^\sharp \to D_2$ is said to be a generalized probabilistic $m-\mathcal{M}-$ space if (5.6) - (5.8) and $(F.4)_M$ are fulfilled.

6. Some $m-$ metrization theorem

Theorems of this section are generalizations of results of Hicks and Sharma, [8] and [9] (see also [14]).

Theorem 6.1. *Let $(X,\mathcal{U},\mathcal{T}_\mathcal{U})$ be an $m-\mathcal{H}-$ space with the base $\mathcal{B} = \{U_{j,\epsilon} \subset X^M : j \in J,\ \epsilon \in (0,r)\},\ r \in R^\sharp_+$, of its $m-$ uniformity $\mathcal{U}$. Then the family of functions $\sigma = (\sigma_j)_{j\in J},\ \sigma_j : X^M \to R^\sharp_+$,*

$$\sigma_j(\alpha) = \begin{cases} \sup\{\epsilon \in (0,r) : \alpha \in U_{j,\epsilon} & \text{if } \alpha \in \bigcup_{0<\epsilon<r} U'_{j,\epsilon} \\ 0, & \text{if } \alpha \in \bigcap_{0<\epsilon<r} U_{j,\epsilon} \end{cases}$$

$\alpha \in X^M$, where $U'_{j,\epsilon} = X^M \setminus U_{j,\epsilon},\ \epsilon \in (0,r)$, has properties:

(6.1) *the family $\sigma = (\sigma_j)_{j\in J}$ is a $\mathcal{H} - m-$ metric on X,*

(6.2) $\mathcal{T}_\sigma = \mathcal{T}_\mathcal{U}$.

(6.3) *(X,σ) is sequentially complete iff $(X,\mathcal{U},\mathcal{T}_\mathcal{U})$ is complete.*

Proof. At first (compare [9] and [11]) we prove the following property of σ :

(6.1) $\alpha \in U_{j,\epsilon}$ iff $\sigma_j(\alpha) < \epsilon$.

Indeed, if $\alpha \in U_{j,\epsilon}$ then from (4.5), $\alpha \in U_{j,\epsilon'}$ for some $\epsilon' < \epsilon$ iff $\alpha \notin U'_{j,\epsilon'}$. Thus for $\sigma_j(\alpha) > 0$, $\sigma_j(\alpha) = \sup\{\delta \in (0,r) : \alpha \notin U_{j,\delta}\} \leq \epsilon' < \epsilon$. If $0 < \sigma_j(\alpha) < \epsilon$ then $\alpha \in \bigcup_{0<\delta<r} U'_{j,\delta}$ and $\delta' = \sup\{\delta \in (0,r) : \alpha \in U'_\delta\} < \epsilon$. Thus there exists $\eta > 0$ that $\delta' + \eta < \epsilon$ and $\alpha \notin U'_{j,\delta'+\eta}$ iff $\alpha \in U_{j,\delta'+\eta} \subset U_{j,\epsilon}$.

Now, from (4.1) and (4.3), $\sigma_j(\alpha) = 0$ for each $\alpha \in \Delta$ and if a and a' are different points of X, then there exists $\alpha \in X^{M_{01}}$ that $\sigma_j[a, a', \alpha] > 0$ for some $j \in J$. From (4.3), $\sigma_j(\alpha) = \sigma_j(p(\alpha))$ for each $p \in \prod_M$. For the proof of the property $(M_3)_{\mathcal{H}}$ assume that $\epsilon \in (0,r)$ and $j \in J$. From (4.4), there exists $\delta \in (0,r)$, that $\circ U_{j,\delta} \subset U_{j,\epsilon}$ and thus if $\alpha_{a_i \to v} \in U_{j,\delta}$, $i \in M$, then $\alpha \in U_{j,\epsilon}$. From (6.4), we get $\sigma_j(\alpha) < \epsilon$ whenever $\sigma_j(\alpha_{a_i \to v}) < \delta$, $i \in M$. Thus for each $\epsilon \in (0,r)$ there exists $\delta \in (0,r)$, that $\sigma_j(\alpha) < \epsilon$ whenever $\sigma_j(\alpha_{a_i \to v}) < \delta$, $i \in M$, $v \in X$, $\alpha \in X^M$. The assertions (6.2) - (6.3) follow immediately from (6.4).

Remark 6.1. Let $(X, \mathcal{U}, \mathcal{T}_\mathcal{U})$ be an $m - H-$ space with the base $\mathcal{B} = \{U_\epsilon : 0 < \epsilon < r\}$ of its $m-$ uniformity $\mathcal{U}$. Then the function $\sigma : X^M \to R_+$,

$$\sigma(\alpha) = \begin{cases} \sup\{\epsilon \in (0,r) : \alpha \in U'_\epsilon\} & \text{if } \alpha \in \bigcup_{0<\epsilon<r} U'_\epsilon \\ 0, & \text{if } \alpha \in \bigcap_{0<\epsilon<r} U_\epsilon, \end{cases}$$

$\alpha \in X^M$, where $U'_\epsilon = X^M \setminus U_\epsilon$, $\epsilon \in (0,r)$, has properties:

(6.5) the function σ is an $H - m-$ metric on X,

(6.6) $\mathcal{T}_\sigma = \mathcal{T}_\mathcal{U}$

(6.7) (X, σ) is sequentially complete iff $(X, \mathcal{U}, \mathcal{T}_\mathcal{U})$ is complete.

The above assertion follows immediately from Theorem 6.1 (for $m = 1$, see [14]).

Example 6.1. Let (X, F) be a probabilistic $m - \mathcal{H}-$ space. Let $\{U_{j,\epsilon} = \alpha \in X^M : F_\alpha(j) > 1 - \epsilon\}$, $j \in R^\sharp$, $\epsilon \in (0,1)$. Then the family $\mathcal{B} = \{U_{j,\epsilon} : j \in R^\sharp_+, \epsilon \in (0,1)\}$ is the base of an $m - \mathcal{H}-$ structure $\mathcal{U}$. From Theorem 6.1 there exists the family $\sigma = (\sigma_j)_{j \in R_+}$ such that (6.1) - (6.3) are fulfilled.

Remark 6.2. Let us consider two "contraction conditions" with a contraction constant $k \in (0,1)$ for a selfmapping f in a probabilistic $\mathcal{H} - m-$ space (X, F) :

(6.8) for each $t \in R$, $\epsilon \in (0,1)$ and $\alpha \in X^M$

$$F_{\alpha_{a_0 \to fa_0, a_1 \to fa_1}}(t) > 1 - k\epsilon \text{ whenever } F_\alpha(t) > 1 - \epsilon,$$

(6.9) for each $t \in R$, $\epsilon \in (0,1)$ and $\alpha \in X^M$, $F_{\alpha_{a_0 \to fa_0, a_1 \to fa_1}}(kt) > 1 - k\epsilon$ whenever $F_\alpha(t) > 1 - \epsilon$.

It is easy to see that conditions (6.8) - (6.9) are equivalent to the following conditions:

(6.10) if $\alpha \in U_{t,\epsilon}$ then $\alpha_{a_0 \to fa_0, a_1 \to fa_1} \in U_{t,k\epsilon}$ for each $t \in R_+$, $\epsilon \in (0,1)$,

(6.11) if $\alpha \in U_{t,\epsilon}$ then $\alpha_{a_0 \to fa_0, a_1 \to fa_1} \in U_{kt,k\epsilon}$ for each $t \in R_+$, $\epsilon \in (0,1)$, respectively.

From Theorem 6.1 the conditions (6.10) - (6.11) imply the following contraction conditions:

(6.12) $\sigma_t(\alpha_{a_0 \to fa_0, a_1 \to fa_1}) \leq k\sigma_t(\alpha)$, $t \in R_+$,

(6.13) $\sigma_{kt}(\alpha_{a_0 \to fa_0, a_1 \to a_1}) \leq k\sigma_t(\alpha)$, $t \in R_+$, respectively.

But it is evident that (6.13) implies (6.12) because $\sigma_{t_1} \leq \sigma_{t_2}$ if $t_1 \geq t_2$ for each $t_1, t_2 \in R_+$.

Example 6.2. Let (X, F) be a probabilistic $H - m-$ space. Let $U_\epsilon = \{\alpha \in X^M : F_\alpha(\epsilon) > 1 - \epsilon\}$, $\epsilon \in (0,1)$. Then the family $\mathcal{B} = \{U_\epsilon \subset X^M : \epsilon \in (0,1)\}$ is the base of an $H - m-$ structure $\mathcal{U}$. On the base of Remark 6.1 (see also [14]) there exists $\sigma : X^M \to R_+$, such that (6.5) - (6.7) are fulfilled.

Remark 6.3. Let f be a selfmapping in a probabilistic $H - m-$ space (X, F). Suppose that the condition holds

(6.14) for each $\epsilon \in (0,1)$ and $\alpha \in X^M$,

$F_{\alpha_{a_0 \to fa_0, a_1 \to fa_1}}(k\epsilon) > 1 - k\epsilon$ whenever $F_\alpha(\epsilon) > 1 - \epsilon$, where $k \in (0,1)$.

The condition (6.14) is equivalent to the following

(6.15) $\alpha_{a_0 \to fa_0, a_1 \to fa_1} \in U_{k\epsilon}$ whenever $\alpha \in U_\epsilon$ for each $\epsilon \in (0,1)$, $\alpha \in X^M$,

and this condition implies the contraction condition:

(6.16) $\sigma(\alpha_{a_0 \to fa_0, a_1 \to fa_1}) \leq k\sigma(\alpha)$

for each $\alpha \in X^M$, where σ is defined in Remark 6.1.

Theorem 6.2. *Let $(X, \mathcal{U}, \mathcal{T}_\mathcal{U})$ be an $\mathcal{M} - m-$ space with the base $\mathcal{B} = \{U_{j,\epsilon} : j \in J, \epsilon \in (0,1)\}$ of its $\mathcal{M} - m-$ structure $\mathcal{U}$ and $\sigma_j : X^M \to R_+^{\natural}$, $j \in J$, are defined as in Theorem 6.1. Then*

(6.17) (X,σ) is a space generated by the family $\sigma = (\sigma_j)_{J\in J}$ (as in Example 4.4),

(6.18) $\mathcal{T}_{\mathcal{U}} = \mathcal{T}_\sigma$

(6.19) *as* (6.16).

Proof. It is enough to prove "the triangle inequality" for $\sigma = (\sigma_j)_{j\in J}$. Let $\sigma_j(\alpha_{a_i\to v}) = \epsilon_i$, $\epsilon_i \in (0,r)$, $i \in M$. Then $\alpha_{a_i\to v} \in U_{\epsilon'_i}$ for each $\epsilon'_i > \epsilon_i$, $i \in M$. Thus from (4.11),

$$\alpha \in U_{T(\epsilon_0,\ldots;\epsilon_m)} \subset U_{T(\lambda,\ldots,\lambda)} \subset U_\lambda,\ \lambda = \max\{\epsilon'_i : i \in M\}.$$

If for example $\lambda = \epsilon'_0$, then $\sigma(\alpha) < \epsilon'_0 = \epsilon_0 + \delta = \sigma_j(\alpha_{a_i\to v}) + \delta$ and since δ is an arbitrary number of $(0,r)$, then

$$\sigma_j(\alpha) \le \sigma_j(\alpha_{a_i\to v}) \le \sum_{i=0}^{m} \sigma_j(\alpha_{a_i\to v}).$$

Remark 6.4. Let $(X,\mathcal{U},\mathcal{T}_{\mathcal{U}})$ be an $m - M-$ space with the base $\mathcal{B} = \{U_\epsilon : 0 < \epsilon < r\}$, $r \in R_+$, of its $m-$ uniformity $\mathcal{U}$. Then the function $\sigma : X^M \to R_+$ defined an in Remark 6.1 has properties

(6.20) σ is an $m-$ metric on X

(6.21) $\mathcal{T}_{\mathcal{U}} = \mathcal{T}_\sigma$

(6.22) as (6.19).

The above assertion follows immediately from Theorem 6.2 (for $m = 1$, see [11]).

Remark 6.5. The results of this paragraph give a possibility to prove some fixed - point theorems in PM - m - spaces or in GPM - m - spaces for the large class of contraction mappings. For this purpose it is enough to use fixed - point results for contractions in m - metric spaces or in genralized m - metric spaces, respectively.

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REZIME

NEKA UOPŠTENJA VEROVATNOSNIH METRIČKIH PROSTORA

U ovom radu su ispitivana neka m - metrička tvrdjenja na m - uniformnim prostorima. Uvedeni su uopšteni m - verovatnosni prostori i date neke napomene o kontrakciji u m - uniformnim prostorima.

Received by the editors December 24, 1989