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# A REMARK ON KANEKO REPORT ON GENERAL CONTRACTIVE TYPE CONDITIONS FOR MULTIVALUED MAPPINGS IN TAKAHASHI CONVEX METRIC SPACES

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**Abstract**

The purpose of this paper is to generalize the fixed point theorems for multivalued mappings proved in [2] for the class of Takahashi convex metric spaces.

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## 1. Introduction

In 1970 Takahashi [3] introduced the definition of convexity in metric space and generalized some important fixed point theorems previously proved for Banach spaces. Subsequently many others authors have obtained additional results in this setting. This paper is a continuation of this investigation.

## 2. Preliminaries

**Definition 1.** *Let $X$ be a metric space and $I$ be the closed unit interval. A mapping $W: X \times X \times I \to X$ is said to be a* **convex structure** *on $X$ if for all $x, y, u \in X$, $\lambda \in I$,*

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda) d(u, y).$$

*$X$ together with a convex structure is called a (Takahashi)* **convex metric space.**

Clearly any convex subset of normed space is a convex metric space with $W(x, y, \lambda) = \lambda x + (1 - \lambda) y$.

**Definition 2.** *Let $X$ be a convex metric space. A nonempty subset $K$ of $X$ is said to be* **convex** *if $W(x, y, \lambda) \in K$ whenever $x, y \in K$ and $\lambda \in I$.*

**Remark.** Takahashi has shown ([3]) that the open and closed balls are convex, and that the arbitrary intersection of convex sets is convex too.

For arbitrary $A \subset X$ let

$$\tilde{W}(A) := \{W(x, y, \lambda) : x, y, \in A, \lambda \in I\}.$$

Using this notation we can say that $K$ is convex if and only if $\tilde{W}(K) \subset K$.

A few additional definitions will be needed too.

**Definition 3.** *A convex metric space $(X, d)$ will be said to have* **Property (C)** *iff every bounded decreasing net of nonempty closed convex subsets of $X$ has a nonempty intersection.*

**Remark.** Every weakly compact convex subset of a Banach space has *Property (C).*

**Definition 4.** *A subset $A$ of a metric space $(X, d)$ is called* **proximal** *if for each $x \in X$, there exists an element $a \in A$ such that $d(x, a) = d(x, A)$ where $d(x, a) = \inf\{d(x, y) : y \in A\}$.*

We denote the family of all nonempty bounded proximal subsets of $X$ by $2^X_{bp}$ and the Hausdorff metric defined on $2^X_{bp}$ induced by $d$ by $H$, i.e., for $A, B \in 2^X_{bp}$,

$$H(A,B) = \max\{\sup_{x\in A} d(x,B),\ \sup_{x\in B} d(x,A)\}.$$

Let $T : X \to 2^X_{bp}$. Then, for $x \in X$, by an orbit of $x$ under $T$, $\sigma(x)$, we mean the sequence $\{x_n : x_0 = x,\ x_n \in Tx_{n-1}\}$. The orbit $\sigma(x)$ is called **strongly regular** if

$$\sigma(x) = \{x_n : x_n \in Tx_{n-1},\ d(x_n, x_{n-1}) = d(x_{n-1}, Tx_{n-1})\}.$$

**Definition 5.** *The convex hull of a set $A$ $(A \subset X)$ is the intersection of all convex sets in $X$ containing $A$ and it is denoted by conv $A$.*

**Proposition 1.** [1]. *Let $X$ be a convex metric space and let, for $n \in N$, $A_n := \tilde{W}^n(A)$, $(A \subset X)$. Then*

$$conv\ A = \lim A_n = \bigcup_{n=1}^{\infty} A_n.$$

**Proposition 2.** [1]. *For any subset $A$ of convex metric space $X$*

$$diam\ (conv\ A) = diam\ A.$$

**Definition 6.** *A convex metric space with a convex structure $W$ is a* **P-convex metric space** *if for all $x, y, z, u \in X$ and $\lambda \in I$*

$$d(W(x,y,\lambda),\ W(u,z,\lambda)) \le \lambda d(x,u) + (1-\lambda)\, d(y,z).$$

## 3. Results

**Theorem 1.** *Let $(X,d)$ be a complete $P$-convex metric space with the continuous convex structure $W$ and $K$ a nonempty closed bounded convex subset of $X$ with Property (C). Let $T$ be a mapping of $K$ into the family of proximal subsets of $K$ that satisfies.*

$$H(Tx,Ty) \le \phi(\max\{d(x,Tx),\ d(y,Ty)\})$$

*for each* $x, y \in K$ *and* $\phi : [0, +\infty) \to [0, +\infty)$ *nondecreasing right continuous function such that* $\phi(t) < t$ *for* $t > 0$. *Then there exists a nonempty subset* $M$ *of* $K$ *such that* $Tx = M$ *for all* $x \in M$.

*Proof.* For any $x_0 \in K$ we may construct a strongly regular orbit at $x_0$ for $T$. First we claim that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Observe that

$$C_n = d(x_n, Tx_n) \le H(Tx_{n-1}, Tx_n) \le \phi(\max\{d(x_{n-1}, Tx_{n-1}),\ d(x_n, Tx_n)\})$$

so that if $C_{n-1} < C_n$, then $C_n \le \phi(C_n) < C_n$. This is a contradiction. Thus $C_n \le \phi(C_{n-1}) < C_{n-1}$. Since $\{C_n\}_{n\in N}$ is a monotone decreasing sequence of non-negative real numbers, $\lim_{n\to\infty} C_n = C$ exists. If $C > 0$, then using the right continuity of $\phi$ we obtain

$$C \le \lim_{C_n \to C+0} \phi(C_n) = \phi(C) < C$$

This contradiction shows that $C = 0$.

Now we let $H_\varepsilon = \{x | d(x, Tx) \le \varepsilon\}$ for each $\varepsilon > 0$. From the argument above we have that $H_\varepsilon \ne \phi$ for each $\varepsilon > 0$.

Our second claim is that $\overline{\text{conv}}\, T(H\varepsilon) \subseteq H_\varepsilon$ for each $\varepsilon > 0$.

Let $y \in \overline{\text{conv}}\ T(H_\varepsilon)$ and let $\delta > 0$ be given. Then there exists $y^* \in conv\ T(H_\varepsilon)$ such that

$$d(y, y^*) \le \delta$$

Since $y^* \in \text{conv}\, T(H_\varepsilon)$ there exitst $n_0 \in N$ such that $y^* \in \tilde{W}^{n_0}(T(H\varepsilon))$. Then $y^* = W(y_1, y_2, \lambda_1)$, for $y_1, y_2 \in \tilde{W}^{n_0-1}(T(H_\varepsilon))$ and $\lambda_1 \in I$. But $y_1 = W(y_{11}, y_{12}, \lambda_{11})$, $y_2 = W(y_{21}, y_{22}, \lambda_{12})$, $\lambda_{11}, \lambda_{12} \in I$ $y_{11}, y_{12}, y_{21}, y_{22} \in \tilde{W}^{n_0-2}(T(H_\varepsilon))$ and so on.
After not more than $n_0$ steps we shall have that elements belong to $T(H_\varepsilon)$. Let us denote them by $\{y_i^*\}_{i\in J_1}$. ($J_1$ is a finit set). Since $y_i^* \in T(H_\varepsilon), i \in J_1$, there exists $y_i \in H_\varepsilon$ such that $y_i^* \in T(y_i)$. But $Ty$ is proximal so, for every $i \in J_1$, for some $z_i \in Ty$,

$$d(y_i^*, z_i) = d(y_i^*, Ty).$$

Now let $z$ be defined by $\{z_i\}_{i\in J_1}$ in the same way as $y*$ by $\{y_i^*\}_{i\in J_1}$. Of course $z \in Ty$. Now we have that

$$d(y, Ty) \le d(y, y^*) + d(y^*, Ty) \le \delta + d(W(y_1, y_2, \lambda_1), W(z_1, z_2, \lambda_1))$$

$$\leq \delta + \lambda_1 d(y_1, z_1) + (1-\lambda_1) d(y_2, z_2) \leq \delta + \lambda_1 d(W(y_{11}, y_{12}, \lambda_{12}),$$

$$W(z_{11}, z_{12}, \lambda_{12})) + (1-\lambda_1) d(W(y_{21}, y_{22}, \lambda_{21}), W(z_{21}, z_{22}, \lambda_{12}), \leq$$

$$\leq \delta + \lambda_1 \cdot \lambda_{12}\, d(y_{11}, z_{11}) + \lambda_1(1-\lambda_{12}) d(y_{12}, z_{12}) + (1-\lambda_1)\lambda_{21} \cdot$$

$$\cdot d(y_{21}, z_{21}) + (1-\lambda_1)(1-\lambda_2)\, d(y_{22}, z_{22}) \leq \ldots \leq$$

$$\leq \delta + \sum_{i \in J} \alpha_i\, d(y_i^*, z_i) = \delta + \sum_{i \in J} \alpha_i\, d(y_i^*, Ty) \leq \delta + \sum_{i \in J} \alpha_i H(Ty_i, Ty) \leq$$

$$\delta + \sum_{i \in J} \alpha_i \phi(\max\{d(y_i, Ty_i), d(y, Ty)\}) \leq \delta + \sum_{i \in J} \alpha_i \phi(\max\{\varepsilon, d(y, Ty)\})$$

$\alpha_i \geq 0, \ i \in J$ and $\sum_{i \in J} \alpha_i = 1$ ($J$ is a finit set).

If $d(y, Ty) > \varepsilon$, than $d(y, Ty) \leq \delta + \phi(d(y, Ty))$. since $\delta > 0$ is arbitrary, this leads to an obvious contradiction that $d(y, Ty) \leq \phi(d(y, Ty)) < d(y, Ty)$. Hence we must have $d(y, Ty) \leq \varepsilon$ and $y \in H_\varepsilon$. This proves our second claim.

Let $\mu = \{\overline{conv}\ T(H_\varepsilon) | \varepsilon > 0\}$. Then $\mu$ is a bounded decreasing net of nonempty closed convex subsets so by Property (C) it has nonempty intersection. Hence $\phi \neq \cap \ \mu \subseteq \cap\{H_\varepsilon | \varepsilon > 0\}$. This shows that the function $x \to d(x, Tx)$ attains its infinum over $K$ and because of the first claim this infinum must be zero. Let $M = \cap\{H_\varepsilon | \varepsilon > 0\}$ and the proof is complete.

Using the proof of Theorem 1 and Theorem 2 [2] one can prove :

**Theorem 2.** *Let $(X, d)$ be a complete $P$-convex metric space with continuous convex structure and $K$ a nonempty convex closed bounded subset of $X$ with Property (C). Let $T$ be a mapping of $K$ into the family of nonempty proximal subsets of $K$ that satisfies:*

*For given $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in K$*

$$\varepsilon \leq \max\{d(x, Tx),\ d(y, Ty)\} < \varepsilon + \delta \Rightarrow H(Tx, Ty) < \varepsilon.$$

*Then there exists a nonempty subset $M$ of $K$ such that $Tx = M$ for all $x \in M$.*

## References

[1] Gajić, Lj., On convexity in convex metric spaces with application, J. Natur. Phys. Sci. 3(1989), 39-48.

[2] Kaneko, H., A report on general contractive type conditions for multi-valued mappings, Math. Japonica, 33,4(1988), 543-550.

[3] Takahashi, W., A convexity in metric space and nonexpansive mappings I, Kodai Math. Sem. Rep., 22 (1970), 142-149.

[4] Tallman, L., Fixed points for condensing multifunctions in metric space with convex structure, Kodai. Math. Sem. Rep., 29 (1977), 62-70.

**REZIME**

NAPOMENA O KANEKOVOM IZVEŠTAJU O OPŠTIM USLOVIMA KONTRAKTIVNOG TIPA ZA VIŠEZNAČNA PRESLIKAVANJA U TAKAHASHIJEVIM KONVEKSNIM METRIČKIM PROSTORIMA

U radu se uopštavaju teoreme o nepokretnoj tački za višeznačna preslikavanja dokazana u [2] za klasu Takahashijevih konveksnih metričkih prostora.

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