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ABELIAN THEOREM FOR THE GENERAL INTEGRAL TRANSFORMATION OF DISTRIBUTIONS

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Abstract

Two stuctural propositions for distributions with S-asymptotic are proved. This makes possible the proof of an Abelian type theorem for a general integral transformation of distributions.

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1. Notations

For $a = (a_1, \ldots, a_n)$, $x = (x_1, \ldots, x_n)$ belonging to R^n and $w = (w_1, \ldots, w_n)$ belonging to $C^n : w^a = w_1^{a_1} \ldots w_n^{a_n}; x > 0$ means $x_i > 0, i = 1, \ldots, n; \prec x, a \succ = \sum_{i=1}^n x_i a_i$.
$|x| = \sum_{i=1}^n |x_i|, \|x\|^2 = \sum_{i=1}^n x_i^2$. By P we shall denote the set of functions $c : R^n \to R_+ \cdot A = \{A_i\}, i \in N$, is a strictly monotone basis of neighbourhoods of "∞" in $R^n \cup \{\infty\}$; with $\cap_{i=1}^n A_i = \{\infty\}$. By Ω we shall denote an unbounded set in R^n such that $\Omega + \Omega \subset \Omega$ and for which "∞" is an adherent point. With this notation, for a complex valued function G, $\lim_{h \in \Omega, h \to \infty} G(h) = \gamma \in C$ means that for every $\varepsilon > 0$ there exists an $A_j \in A$ such that $|G(h) - \gamma| < \varepsilon$ for $h \in \Omega \cap A_j$. For a distribution T, by $F[T]$ we shall denote its Fourier transform.

2. S-asymptotic

We shall use the S-asymptotic (shift asymptotic) [3], [5] to make precise the asymptotic behaviour of distributions. S-asymptotic has many applications especially in the quantum field theory [1].

Definition 1. *A distribution T has the S-asymptotic in D' related to Ω and $c \in P$ with the limit $U \in D'$, if the following limit exsists*

$$\lim_{h\in\Omega, h\to\infty} < T(x+h)/c(h), \varphi(x) >=< U, \varphi > \tag{2.1}$$

for every $\varphi \in D$. Relation (2.1) can be written

$$\lim_{h\in\Omega, h\to\infty} (T * \hat{\varphi})(h)/c(h) = (U * \hat{\varphi})(0) \tag{2.1}$$

where $\hat{\varphi}(x) = \varphi(-x)$. This property of T can be written in short as

$$T(x+h)^s \sim c(h) \cdot U(x), h \in \Omega, \text{ in } D'. \tag{2.2}$$

If T belongs to a subspace M' of D' and if the limit in relation (2.1) exists for every $\varphi \in M, M$ being the basis space for M', we shall say that T has the S-asymptotic in M'.

Proposition 1. *If $T(x+h)^s \sim c(h).U(x), h \in \Omega$ and $U \neq 0$, then the limit*

$$\lim_{h\in\Omega, h\to\infty} c(h+h_0)/c(h) = d(h_0), h_0 \in \Omega \tag{2.3}$$

exists and U satisfies the equation $U(x+h_0) = d(h_0)U(x)$.

Proof. There exists $\varphi \in D$ such that $< U, \varphi > \neq 0$. For this φ

$$\lim_{h\in\Omega, h\to\infty} \frac{c(h+h_0)}{c(h)} < \frac{T(x+(h+h_0))}{c(h+h_0)}, \varphi(x) >=$$

$$\lim_{h\in\Omega, h\to\infty} < \frac{T((x+h_0)+h)}{c(h)}, \varphi(x) >, h_0 \in \Omega.$$

Hence the assertion of Proposition 1. -

We can obtain more precise results about c and U if we take for Ω a special set. A cone is often used as the set Ω.

Proposition 2. *Suppose: 1. Ω is a convex cone Γ, int $\Gamma \neq \emptyset$; 2. The set A consists of $A_i = \{x \in R^n, \|x\| > i\}, i \in N$; 3. $T(x+h)^s \sim c(h).U(x), h \in \Gamma, U \neq 0$. Then,*

$$\lim_{h \in \Omega, h \to \infty} c(h+h_0)/c(h) = \exp \prec p, h_0 \succ, h_0 \in R^n, p \in R^n$$

and $U = C \exp \prec p, x \succ$, where C is a constant.

For the proof see [3, p. 84].

The next two propositions will be used to prove an Abelian type theorem for the integral transform of distributions.

Proposition 3. *Suppose $T_0 \in B'$ and $T_0(x+h)^s \sim 1 \cdot U(x), h \in \Omega$ in D', then:*

1. $T_0 = \sum_{i=0}^{2} \Delta^{ik} F_i, k \in N \cup \{0\}$, where F_i are continuous functions belonging to L^∞ and Δ is the Laplace operator.
2. For every $0 \leq i \leq 2$ functions $F_i(x+h)$ converege uniformly to bounded functions $f_i, i = 0, 1, 2$, when x belongs to a compact set K and $h \in \Omega, h \to \infty$.
3. T_0 has the S-asymptotic in B', as well, related to Ω and $c = 1$, with the limit U.

In [5] we proved the same Proposition in the case $\Omega \equiv \Gamma$. Since the proof of this Proposition in the case of a general Ω does not require any substantial change, we can omit it.

Now, we shall make precise a class of distributions which will be used in the subsequent pages.

Definition 2. *A distribution T belongs to the class $\mathcal{J}(a, b)$, where $a, b \in R^n$ if the distribution $T_0 = T(x) \cdot \exp \prec -a, x \succ \cdot (z+x)^{-b}$ belongs to B'; $Im z_i \neq 0, i = 1, \ldots, n$.*

Proposition 4. *Suppose that $T \in D'$ and $T(x+h)^s \sim c(h).U(x), h \in \Omega$ in D'. If:*

1. $T \in \mathcal{J}(a,b)$*;*

2. $\exp \prec -a, h \succ c(h)/(z+x+h)^b$ *converges to* $V(x) \neq 0$ *in* E*,*

then the distribution $T_0(x) = T(x) exp \prec -a, x \succ (z+x)^{-b}$ *has the property* $T_0(x+h)^s \sim 1 \cdot U(x)V(x) \exp \prec -a, x \succ, h \in \Omega$ *in* D'*.*

Proof. For the distribution T_0 we have:

$$< T_0(x+h), \varphi(x) >=$$

$$< \frac{T(x+h)}{c(h)}, \frac{c(h) \exp \prec -a, h \succ}{(z+x+h)^b} \cdot \exp \prec -a, x \succ \varphi(x) >, \varphi \in D.$$

Since $T(x+h)/c(h)$ converges weakly in D', the set $\{T(x+h)/c(h), h \in \Omega\}$ is weakly bounded, and thus it is a bounded set. By Theorem X in [4], T.I, it follows that

$$\lim_{h \in \Omega, h \to \infty} < T_0(x+h), \varphi(x) >=< U(x)V(x) \exp \prec -a, x \succ, \varphi(x) >$$

for every $\varphi \in D$.

3. A general integral transform of distributions

In the following we shall use the well known function $\eta_\omega \in C^\infty, \omega > 0$ [6]:

$$\eta_\omega(x) = \int_{B(0,2\omega)} q_\omega(x-t)dt, \ x \in R^n,$$

where

$$q_\omega(x) = \begin{cases} D\omega^{-n} \exp(-\frac{\omega^2}{\omega^2 \|x\|^2}), & \|x\| < \omega \\ 0, & \|x\| \geq \omega \end{cases} : D \int_{R^n} q_1(t)dt = 1.$$

The function η_ω, has the properties: $0 \leq \eta_\omega(x) \leq 1, x \in R^n; \eta_\omega(x) = 1, x \in B(0,\omega); \eta_\omega(x) = 0, \|x\| > 3\omega; |D^k \eta_\omega(x)| \leq C_k \omega^{-k}, x \in R^n$. The constants C_k do not depend on ω.

Definition 3. *Suppose that $k(s+x)$ is a smooth function for every fixed $s \in S_0 \subset C^n$. The k-transformation of a distribution $T \in D', K(T)(s)$, is defined by the limit*

$$K(T)(s) = \lim_{\omega \to \infty} < T(x), k(s+x)\eta_\omega(x) >,$$

if this limit exists when $s \in S_0$.

Proposition 5. *If for fixed $a, b \in R^n$ and $s \in S_0$ the function $k(s+x)(z+x)^b \exp \prec a, x \succ \in D_{L^1}$, then the k-transformation exists for $T \in \mathcal{J}(a,b), s \in S_0$ and we have:*

$$K(T)(s) = < \frac{T(x)}{(z+x)^b \exp \prec a, x \succ}, k(s+x)(z+x)^b \exp \prec a, x \succ > .$$

Proof. We have only to prove that $\eta_\omega(x)k(s+x)(s_0+x)^b \cdot \exp \prec a, x \succ$ converges in D_{L^1} when $\omega \to \infty$. That is a direct consequence of the property of the function $\eta_\omega : |D^k \eta_\omega(x)| \leq C_k \omega^{-|k|}$. -

Remark. If $T \in D'(R)$ is a regular distribution defined by the function f and if the classical transform

$$\int_{-\infty}^{\infty} f(x)k(s+x)dx, s \in S_0$$

exists, then the k-transform of $T = f$ exists and

$$\lim_{\omega \to \infty} < f(x), k(s+x)\eta_\omega(x) > = \int_{-\infty}^{\infty} f(x)k(s+x)dx, s \in S_0.$$

We shall show this.

$$K(f)(s) = \lim_{\omega \to \infty} \int_{-3\omega}^{-\omega} + \int_{-\omega}^{\omega} + \int_{\omega}^{3\omega} f(x)k(s+x)\eta_\omega(x)dx, s \in S_0.$$

We have to show only that the first and the third integral converge to zero when $\omega \to \infty$. Since for $\omega \leq x \leq 3\omega$, the function η_ω is positive and monotonely decreasing, by the mean value theorem there exists a $\xi, 0 < \xi < 2$, such that

$$\int_{\omega}^{3\omega} \eta_\omega(x) Re[f(x)k(s+x)]dx = \eta_\omega(\omega) \int_{\omega}^{\omega+\xi\omega} Re[f(x)k(s+x)]dx$$

for $s \in S_0$. The last integral tends to zero when $\omega \to \infty$, because we supposed that the classical k-transformation exists. We have the same situation with the imaginary part of this integral and with the first integral, as well.

In such a way, the k-tranformation generalises the clasical one.

Proposition 6. *We suppose:*

1. *For fixed $a \in R^n$ and $s \in S_0$ the function $k(s+x) \cdot \exp \prec a, x \succ \in D_{L^2}$.*
2. *For an $s' \in S_0$, with the property $s' + R^n \subset S_0$, we have: $F[k(s' - x) \exp \prec a, -x \succ](y) \neq 0, y \in R^n$.*

Then, the k-transformation exists for $T \in \mathcal{J}(a, -c), c_i > 1/2, i = 1, \ldots, n$, it is a one-to-one mapping defined in the set $\mathcal{J}(a, -c)$ and the set $\{k(s' + x + h) \exp \prec a, x + h \succ, h \in R^n\}$ is dense in D_{L^2}.

Proof. If for $s \in S_0, k(s + x) \exp \prec a, x \succ \in D_{L^2}$, then for $Im z_i \neq 0, i = 1, \ldots, n, k(s + x) \exp \prec a, x \succ (z + x)^{-c}$ belongs to D_{L^1}. By Proposition 5 there exists the k-transformation of T, $K(T)(s)$, and

$$K(T)(s) =< T(x)(z + x)^c \exp \prec -a, x \succ, k(s + x)(z + x)^{-c} \exp \prec a, x \succ>$$

$$=< T(x) \exp \prec -a, x \succ, k(s + x) \exp \prec a, x \succ>;$$

because $T \exp \prec -a, x \succ \in D'_{L^2}$ (see Theorem XXVI in [4], T.II,p. 59). Now, for $h \in^n$ we have:

$$K(T)(s' - h) = \exp \prec a, h \succ ([T(x) \exp \prec -a, x \succ]*$$

$$*[k(s' - x) \exp \prec a, -x \succ])(h).$$

If $K(T)(s'-x) = 0$ for all $h \in R^n$, then $\exp \prec a, -h \succ K(T)(s'-h) = 0$, as well. Using the Fourier transform and its properties (see [4], T.II, p, 126) we have:

$$F[T(x)e^{\prec -a,x \succ}](y)F[k(s' - x)e^{\prec a,-x \succ}](y) = 0, y \in R^n$$

If $F[k(s' - x) \exp \prec a, -x \succ](y) \neq 0, y \in R^n$ then $F[T(x) \exp \prec -a, x \succ]$ $(y) = 0, y \in R^n$ and therefore $T = 0$.

We know (see [2]p.239) that the linear envelope of the set $\{k(s' - h + x)\exp \prec a, x - h \succ, h \in R^n\}$ is dense in D_{L^2} if and only if for every $W \in D'_{L^2}$ and $h \in R^n$ from $< W(x), k(s' - h + x)\exp \prec a, x - h \succ >= 0$ it follows that $W = 0$. To show this, we shall use the same properties of the Fourier transform as we did above:

$$U =< W(x), k(s' - h + x)\exp \prec a, x - h \succ >=$$

$$= (W(x) * [k(s' - x)\exp \prec a, -x \succ])(h), h \in R^n.$$

By the Fourier transform these relations give:

$$0 = F[W](y)F[k(s' - x)\exp \prec a, -x \succ](y), y \in R^n.$$

We have only to use supposition 2 of Proposition 6 to obtain that $W = 0$.

4. Abelian type theorem

Theorem 1. *We suppose:*

1. $T \in \mathcal{J}(a, b)$, *where* $a \in R^n$ *and* $b \in (R_+ \cup \{0\})^n$;
2. $T(x + h)^s \sim c(h).U(x), h \in \Omega$, *in* D';
3. $k(s + x)\exp \prec a, x \succ (z + x)^b \in D_{L^1}, Imz_i \neq 0, i = 1, \ldots, n$, *where* $s \in S_0$ *and* $S_0 - \Omega \subset S_0$;
4. $\lim_{h\in\Omega, h\to\infty} \frac{c(h)\exp\prec -a,h\succ}{(z+x+h)^b} = V(x) \neq 0$ *in* E.

Then the k-transformation of T *exists and*

$$\text{(4.1)} \qquad \lim_{h\in\Omega, h\to\infty} \frac{K(T)(s-h)}{c(h)} =< U(x)V(x)\exp \prec -a, x \succ,$$

$$V^{-1}(x)\exp \prec a, x \succ k(s + x) > .$$

Proof. If we denote by $T_0(x) = T(x)\exp \prec -a, x \succ (z + x)^{-b}$, by proposition 5 the k-transformation exists for $T \in \mathcal{J}(a, b), s \in S_0$ and

$$K(T)(s) = T_0(x), k(s + x)\exp \prec a, x \succ)(z + x)^b.$$

Hence,

$$\frac{K(T)(s-h)}{c(h)} = \langle T_0(x+h), \frac{(z+x+h)^b}{c(h)\exp \prec -a, x \succ} k(s+x)\exp \prec a, x \succ \rangle$$

By propositions 3 and 4, $T_0(x+h)^s \sim 1 \cdot U(x)V(x)\exp \prec -a, x \succ, h \in \Omega$, in B' and

$$\frac{K(T)(s-h)}{c(h)} = \sum_{i=0}^{2}(-1)^{ik}\int_{R^n} F_i(x+h)\Delta^{ik}\frac{(z+x+h)^b}{c(h)\exp \prec -a, h \succ}\cdot$$
$$\cdot k(s+x)\exp \prec a, x \succ dx.$$

The expression $\Delta^{ik}(z+x+h)^b k(s+x)\exp \prec a, x \succ$ consists of the finite sum of elements which have the following form:

$$H_{j,p,q}(x,h) = C_{j,p}(z+x+h)^{b-(j-p)} D_k^p[k(s+x)\exp \prec a, x \succ],$$

where $0 \le p \le j \le b$.

We shall prove that $|H_{j,p,q}(x,h)|/c(h)\exp \prec -a, h \succ$ is bounded by a function belonging to L^1 for all $h \in R^n$.

First, if $(z+x)^b k(s+x)\exp \prec a, x \succ$ belongs to D_{L^1}, then $(z+x)^b D_x^p[k(s+x)\exp \prec a, x \succ]$ belons to L^1, as well.

Secondly, we shall prove two inequalities:

$$|(z+x+h)^q| = \prod_{i=1}^{n}|z_i+x_i+h_i|^{q_i} \prec \prod_{i=1}^{n}(|z_i+x_i|+1)^{q_i}(|z_i+h_i|+$$
$$+|x_i|)^{q_i} \prec \prod_{i=1}^{n}[|z_i+x_i|(1+\frac{1}{|Imz_i|})]^{q_i}[|z_i+h_i|(1+\frac{|x_i|}{|Imz_i|})]^{q_i}$$
$$\le C_q|(z+x)^q||(z+h)^q|, q_i \ge 0, i = 1, \dots, n.$$
$$|(z+x+h)^{-(j-p)}| \le C'_{j,p}, j \ge p; x \in R^n, h \in \Omega.$$

Now,

$$|H_{j,p,q}(x,h)|/c(h) \le C_{j,p}C'_{j,p}C_b|\frac{(z+h)^b}{c(h)}||(z+x)^b|\cdot$$
$$\cdot|D_x^q k(s+x)e^{\prec a,x \succ}|.$$

This inequality shows that $|H_{j,p,q}(x,h)|/c(h)\exp \prec -a, h \succ$ is bounded by a function belonging to L^1 when $h \in \Omega$. When we exchange the limit and the integral in relation (7), we just obtain the assertion of Theorem 1, given by relation (3.1).

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REZIME

ABELOVA TEOREMA ZA OPŠTU INTEGRALNU TRANSFORMACIJU DISTRIBUCIJA

Opšta integralna transformacija distribucije $T \in D$ uvedena je na sledeći način:

Funkcija η_ω [6] je glatka funkcija sa sledećim osobinama: $0 \leq \eta_\omega(x) \leq 1, x \in R^n; \eta_\omega(x) = 1, \|x\| < \omega; \eta_\omega(x) = 0, \|x\| > 3\omega$.

$k(s+x)$ je glatka funkcija za $x \in R^n$ u $s \in S_0 \subset C^m$. k-transformacija distribucije $T \in D', K(T)(s)$, definisana je graničnim procesom

$$K(T)(s) = \lim_{\omega \to \infty} < T(x), k(s+x)\eta_\omega(x) >,$$

ako ova granica postoji za $s \in S_0$.

Za ovu integralnu transformaciju posebno je značajna klasa distribucija $\mathcal{J}(a,b)$. Distribucija T pripada klasi $\mathcal{J}(a,b)$ ako distribucija $T_0(x) = T(x) \exp \prec -a, x \succ (z+x)^{-b}$ pripada skupu ograničenih distribucija, B' za $Imz_i \neq 0, i = 1, \ldots, n$.

Pored uslova egzistencije k-transformacije i njenih osobina, dokazana je i sledeća

TEOREMA 1. Pretpostavimo:

1. $T \in \mathcal{J}(a,b)$, gde $a \in R^n$ i $b \in (R_+ \cup \{0\})^n$;
2. T ima S-asimptotiku u odnosu na c sa granicom u dok $h \in \Omega$.
3. $k(s+x)\exp \prec a,x \succ (z+x)^b \in D_{L^1}, Imz_i \neq 0, i=1,\dots,n$, gde $s \in S_0$ i $S_0 - \Omega \subset S_0$;
4. $\lim_{h\in\Omega,h\to\infty} \frac{c(h)\exp\prec -a,h\succ}{(z+x+h)^b} = V(x) \neq 0$ u E.

Tada k-transformacija od T postoji i

$$\lim_{h\in\Omega,h\to\infty} \frac{K(T)(s-h)}{c(h)} =< U(x)V(x)\exp \prec -a,x \succ,$$

$$V^{-1}(x)\exp \prec a,x \succ k(s+x) > .$$

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