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THE SPACES OF WEIGHTED AND TEMPERED ULTRADISTRIBUTIONS
Part I

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Abstract

The spaces $\mathcal{D}_{L,\mu}{}^{(M_\alpha)}(\mathbf{R})$ and $\dot{\mathcal{B}}_\mu^{(M_\alpha)}$, $s \in [1,\infty)$, $\mu \in \mathbf{R}$, of weighted ultradifferentiable functions and the space $S^{(M_\alpha)}(\mathbf{R})$ of rapidly decreasing ultradifferentiable functions on the real line $\mathbf{R}$ are defined. Their topological structure and their relations with $\mathcal{D}_{L^s}(\mathbf{R})$, $\dot{\mathcal{B}}(\mathbf{R})$ and $\mathcal{S}(\mathbf{R})$ are investigated. The basic properties of the (ultra)differentiation, the multiplication and the convolution in the spaces are obtained. The space $O_M^{(M_\alpha)}(\mathbf{R})$ of multipliers of the space $S^{(M_\alpha)}(\mathbf{R})$ is determined.

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0. Introduction

Following the approach of Komatsu ([5]) to the theory of Beurling ultradistributions we define and investigate the spaces $\mathcal{D}_{L,\mu}^{(M_\alpha)}(\mathbf{R})$ and $\dot{\mathcal{B}}_\mu^{(M_\alpha)}$, $(s \in [1,\infty]$, $\mu \in \mathbf{R})$ of weighted ultradifferentiable functions, the space $S^{(M_\alpha)}(\mathbf{R})$ of rapidly decreasing ultradifferentiable functions and $O_M^{(M_\alpha)}(\mathbf{R})$ as the natural generalizations of the spaces $\mathcal{D}_{L^s}(\mathbf{R})$, $\dot{\mathcal{B}}(\mathbf{R})$, $\mathcal{S}(\mathbf{R})$, $\mathcal{O}_M(\mathbf{R})$ ([11]), $\mathcal{D}_{L^s}^{(M_\alpha)}(\mathbf{R})$, $\dot{\mathcal{B}}^{(M_\alpha)}(\mathbf{R})$ ([8] and [9]), $\mathcal{D}_{L^s,\mu}(\mathbf{R})$, $\dot{\mathcal{B}}_\mu(\mathbf{R})$ ([6]) and $S_{(M_\alpha)}$

([10]). The spaces $\mathcal{D}^{(M_\alpha)}_{L^s,\mu}(\mathbf{R})$ are defined as projective limits, when $h \to \infty$, of the spaces $\mathcal{D}^{M_\alpha,h}_{L^s,\mu}(\mathbf{R})$, or equivalently of the spaces $\mathcal{D}^{M_\alpha,h}_{\mu,L^s}(\mathbf{R})$, the properties of which will be investigated in the Section 1.. In Section 2. are obtained the relations between the spaces $\mathcal{D}^{(M_\alpha)}_{L^s,\mu}(\mathbf{R})$, for various $s \in [1,\infty]$, the space $S^{(M_\alpha)}(\mathbf{R})$ and the spaces $\mathcal{D}(\mathbf{R})$, $\mathcal{E}(\mathbf{R})$ ([11]), $\mathcal{D}^{(M_\alpha)}(\mathbf{R})$, $\mathcal{E}^{(M_\alpha)}(\mathbf{R})$ ([5]). In Section 3. we investigate the elementary operations for the weighted and tempered ultradifferentiable functions. It is proved that the spaces $\mathcal{D}^{(M_\alpha)}_{L^2,\mu}(\mathbf{R})$ and $S^{(M_\alpha)}(\mathbf{R})$ are stable under differential resp. ultradifferential operators, if condition (M.2)' resp. (M.2) (see below) is fulfilled. The basic properties of the pointwiese multiplication in these spaces are obtained and the space of multipliers of the space $S^{(M_\alpha)}(\mathbf{R})$ is defined and investigated. Let us give a survey of definitions needed in the paper. By $\mathbf{N}$ we denote the set of non-negative integers.

Let $(M_\alpha)_{\alpha\in\mathbf{N}}$ be a sequence of positive numbers, which satisfy the following conditions:

(M.1) $M_\alpha^2 \le M_{\alpha-1}M_{\alpha+1}$, $\alpha = 1,2,\ldots$.

(M.2)' There are constants A and H such that

$$M_{\alpha+1} \le AH^\alpha M_\alpha, \quad \alpha = 1,2,\ldots$$

(M.3)' $\sum_{\alpha=1}^{\infty} \frac{M_{\alpha-1}}{M_\alpha} < \infty$, $\alpha = 1,2,\ldots$

Condition (M.2)' will be in some assertions replaced by the following stronger condition

(M.2) There are constants A and H such that

$$M_\alpha \le AH_0^\alpha \min_{0\le\beta\le\alpha} M_\beta M_{\alpha-\beta}, \quad \alpha \in \mathbf{N}.$$

Put $h > 0$, $\rho > 0$. Let us recall ([5])

$$\mathcal{E}^{M_\alpha,h}([-\rho,\rho]) = \{\varphi \in C^\infty(\mathbf{R}), \|\varphi\|_{\rho,h} = \sum_{\alpha\in\mathbf{N}} \frac{h^\alpha \|\varphi^{(\alpha)}\|_{\infty,[-\rho,\rho]}}{M_\alpha} < \infty\},$$

$$\mathcal{E}^{(M_\alpha)}(\mathbf{R}) = \text{proj }\lim_{\rho\to\infty} \text{proj }\lim_{h\to\infty} \mathcal{E}^{M_\alpha,h}([-\rho,\rho]),$$

$$\mathcal{D}^{M_\alpha,h}_{[-\rho,\rho]} = \{\varphi \in C^\infty(\mathbf{R}), \text{supp }\varphi \subset [-\rho,\rho], \sum_{\alpha\in\mathbf{N}} \frac{h^\alpha \|\varphi^{(\alpha)}\|_{\infty,[-\rho,\rho]}}{M_\alpha < \infty}$$

$$\mathcal{D}^{(M_\alpha)}_{[-\rho,\rho]} = \text{proj }\lim_{h\to\infty} \mathcal{D}^{M_\alpha,h}_{[-\rho,\rho]}, \ \mathcal{D}^{(M_\alpha)}(\mathbf{R}) = \text{ind }\lim_{\rho\to 0} \mathcal{D}^{(M_\alpha)}_{[-\rho,\rho]}.$$

A differential operator

$$P(D) = \sum_{\alpha \in \mathbf{N}} a_\alpha D^\alpha,$$

where $D = \frac{1}{i}\frac{d}{dx}$, $i = (-1)^{1/2}$ and $a_\alpha \in \mathbf{C}$, is called an ultradifferential operator of the class (M_α), whenever the coefficients satisfy the estimate $| a_\alpha | \leq CL^\alpha / M_\alpha, \alpha \in \mathbf{N}$, for some constants C and L.

We always assume that s is a constant, such that $s \in [1, \infty]$. The usual norm in $L^s(\mathbf{R})$ is denoted by $\| \ \|_s$. The spaces $\mathcal{D}_{L^s}^{(M_\alpha)}(\mathbf{R})$ and $\dot{\mathcal{B}}^{(M_\alpha)}(\mathbf{R})$ were defined in [9] and [7], by

$$\mathcal{D}_{L^s}^{M_\alpha,h}(\mathbf{R}) = \{\varphi \in C^\infty(\mathbf{R}), \gamma_{s,h}(\varphi) = \sum_{\alpha \in \mathbf{N}} \frac{h^\alpha \| \varphi^{(\alpha)} \|_s}{M_\alpha} < \infty\}, h > 0,$$

$$\mathcal{D}_{L^s}^{(M_\alpha)}(\mathbf{R}) = \text{proj } \lim_{h \to \infty} \mathcal{D}_{L^s}^{M_\alpha,h}(\mathbf{R}),$$

$\dot{\mathcal{B}}^{(M_\alpha)}(\mathbf{R})$ is a subspace of $\dot{\mathcal{B}}^{(M_\alpha)}(\mathbf{R}) = \mathcal{D}_{L^\infty}^{(M_\alpha)}(\mathbf{R})$, which is the completion of $\mathcal{D}^{(M_\alpha)}(\mathbf{R})$ under the family of norms $\gamma_{s,h}$, where $h > 0$.

Let $\mu \in \mathbf{R}$. Recall [6],

$$\mathcal{D}_{L^s,\mu}(\mathbf{R}) = \{\varphi \in \mathcal{D}(\mathbf{R}), < x >^\mu \varphi \in \mathcal{D}_{L^s}(\mathbf{R})\},$$

where $< x > := (1+ | x |^2)^{1/2}$, equipped with the topology, which is induced by the bijection

$$\mathcal{D}_{L^s}(\mathbf{R}) \to \mathcal{D}_{L^s,\mu}(\mathbf{R}), \qquad \varphi \mapsto < x >^{-\mu} \varphi.$$

$\dot{\mathcal{B}}_\mu(\mathbf{R})$ stands for the space $\mathcal{D}_{L^\infty,\mu}(\mathbf{R})$ and the space $\dot{\mathcal{B}}_\mu(\mathbf{R})$ is its subspace, which is the completion of $\mathcal{D}(\mathbf{R})$ in $\dot{\mathcal{B}}_\mu(\mathbf{R})$.

The notation „$A \hookrightarrow B$", means that the space A is dense in the space B and that the inclusion mapping $i : A \to B$ is continuous.

1. Basic spaces

Let $\mu \in \mathbf{R}$ and $h > 0$.

Definition 1. *Denote by* $\mathcal{D}^{M_\alpha,h}_{L^s,\mu}(\mathbf{R})$ *the space of all the smooth functions* φ*, such that* $(< x >^\mu \varphi)^{(\alpha)} \in L^s(\mathbf{R})$*, for each* $\alpha \in \mathbf{N}$*, and that*

$$\| \varphi \|_{\mathcal{D}^{M_\alpha,h}_{L^s,\mu}(\mathbf{R})} = \gamma_{s,h}(< x >^\mu \varphi) = \sum_{\alpha\in\mathbf{N}} \frac{h^\alpha \| (< x >^\mu \varphi)^{(\alpha)} \|_s}{M_\alpha} < \infty,$$

equipped with the topology induced by the norm $\| \; \|_{\mathcal{D}^{M_\alpha,h}_{L^s,\mu}(\mathbf{R})}$. *By* $\mathcal{D}^{M_\alpha,h}_{\mu,L^s}(\mathbf{R})$ *we denote the space of all the functions* $\varphi \in C^\infty(\mathbf{R})$*, which satisfay that* $< x >^\mu \varphi^{(\alpha)} \in L^s(\mathbf{R})$*, for all* $\alpha \in \mathbf{N}$*, and*

$$\| \varphi \|_{\mathcal{D}^{M_\alpha,h}_{\mu,L^s}(\mathbf{R})} = \sum_{\alpha\in\mathbf{N}} \frac{h^\alpha \| < x >^\mu \varphi^{(\alpha)} \|_s}{M_\alpha} < \infty.$$

equipeed with the topology induced by the norm $\| \; \|_{\mathcal{D}^{M_\alpha,h}_{\mu,L^s}(\mathbf{R})}$.

Proposition 1. *The spaces* $\mathcal{D}^{M_\alpha,h}_{\mu,L^s}(\mathbf{R})$ *and* $\mathcal{D}^{M_\alpha,h}_{L^s,\mu}(\mathbf{R})$ *are Banach spaces.*

Proof. The space $\mathcal{D}^{M_\alpha,h}_{L^s,\mu}(\mathbf{R})$ is a Banach space, since it is isomorphic to the space $\mathcal{D}^{M_\alpha,h}_{L^s}(\mathbf{R})$, which is a Banach space ([7]). The isomorphism is given by

$$\mathcal{D}^{M_\alpha,h}_{L^s,\mu}(\mathbf{R}) \to \mathcal{D}^{M_\alpha,h}_{L^s}(\mathbf{R}), \qquad \varphi \mapsto < x >^\mu \varphi.$$

Let (φ_n) be a Cauchy sequence in $\mathcal{D}^{M_\alpha,h}_{\mu,L^s}(\mathbf{R})$ and $\epsilon > 0$, then there exists $n_0(\epsilon) \in \mathbf{N}$ such that, for all $n, m > n_0(\epsilon)$,

(1.2;1) $\quad \sum_{\alpha\in\mathbf{N}} \frac{h^\alpha \| < x >^\mu (\varphi_n - \varphi_m)^{(\alpha)} \|_s}{M_\alpha} < \epsilon.$

This implies that, for all $\alpha \in \mathbf{N}, (< x >^\mu \varphi_n^{(\alpha)})_n$ is a Cauchy sequence in the complete space $L^s(\mathbf{R})$. So, there exists $< x >^\mu \phi_\alpha \in L^s(\mathbf{R})$, the limit of the sequence $(< x >^\mu \varphi_n^{(\alpha)})_n$ in $L^s(\mathbf{R}), \alpha \in \mathbf{N}$. It follows that $< x >^\mu \varphi_n^{(\alpha)} \to < x >^\mu \phi_\alpha$ in $\mathcal{D}'(\mathbf{R})$, i.e. $\varphi_n^{(\alpha)} \to \phi_\alpha$ in $\mathcal{D}'(\mathbf{R})$. Because of the continuity of the differentiation in $\mathcal{D}'(\mathbf{R})$ we have, $\phi_0^{(\alpha)} = \phi_\alpha \in L^s(\mathbf{R})$ in $\mathcal{D}'(\mathbf{R})$, for $\alpha \in \mathbf{N}$. This implies that $\phi_0^{(\alpha)} = \phi_\alpha$ holds, also, in the classical sense. By letting $m \to \infty$ in (1.2;1) we get

(1.2;2) $\quad \sum_{\alpha\in\mathbf{N}} \frac{h^\alpha \| < x >^\mu (\varphi_n - \phi_0)^{(\alpha)} \|_s}{M_\alpha} < \epsilon$, for $n > n_0$.

Let $n > n_0$, from (1.2;2) we have

$$\sum_{\alpha\in\mathbf{N}} \frac{h^\alpha \|< x >^\mu \phi_0^{(\alpha)}\|_s}{M_\alpha} \le \sum_{\alpha\in\mathbf{N}} \frac{h^\alpha \|< x >^\mu (\varphi_n - \phi_0)^{(\alpha)}\|_s}{M_\alpha} +$$

$$+ \sum_{\alpha\in\mathbf{N}} \frac{h^\alpha \|< x >^\mu (\varphi_n)^{(\alpha)}\|_s}{M_\alpha} < \infty,$$

which implies that $\phi_0 \in \mathcal{D}^{M_\alpha,h}_{\mu,L^s}(\mathbf{R})$.

$\varphi_n \to \phi_0$ in $\mathcal{D}^{M_\alpha,h}_{\mu,L^s}(\mathbf{R})$ because (1.2;2) holds for each $\epsilon > 0$.□

In the proof of Proposition 2 we will use the inequalities stated by the next Lemma.

Lemma 1. *Let $\mu \in \mathbf{R}$ and $r, n \in \mathbf{N}$. Then*

(1.3;1) $\quad |(< x >^\mu)^{(\alpha)}| \le (2+|\mu|)^\alpha \alpha! < x >^\mu,$

(1.3;2) $\quad \left| \left(\frac{(<x>^\mu)^{(\alpha)}}{<x>^\mu} \right)^{(\beta)} \right| \le 2^\beta (2+|\mu|)^{\alpha+\beta}(\alpha+\beta)!.$

Proof: 1° Let $\alpha \in \mathbf{N}$. We have

$$(< x >^\mu)^{(\alpha)} = ((1+ix)^{\mu/2}(1-ix)^{\mu/2})^{(\alpha)}$$

$$= \sum_{\gamma=0}^{\alpha} \binom{\alpha}{\gamma} ((1+ix)^{\mu/2})^{(j)}((1-ix)^{\mu/2})^{(\alpha-\gamma)}$$

$$= \sum_{\gamma=0}^{\alpha} \binom{\alpha}{\gamma} (-1)^{\alpha-\gamma}(i)^\alpha < x >^\mu (\frac{1-xi}{1+xi})^\gamma (\frac{1}{1-xi})^\alpha.$$

$$\cdot \frac{\mu}{2} \cdot (\frac{\mu}{2} - 1) \cdot \ldots \cdot (\frac{\mu}{2} - (\gamma - 1)) \cdot \frac{\mu}{2} \cdot (\frac{\mu}{2} - 1) \cdot \ldots \cdot (\frac{\mu}{2} - (\alpha - \gamma - 1)).$$

For $\alpha > \gamma > 0$ the product

$$\frac{\mu}{2} \cdot (\frac{\mu}{2} - 1) \cdot \ldots \cdot (\frac{\mu}{2} - (\gamma - 1)) \cdot \frac{\mu}{2} \cdot (\frac{\mu}{2} - 1) \cdot \cdot (\frac{\mu}{2} - (\alpha - \gamma - 1)),$$

may be written in the form

$$\sum_{\omega=2}^{\alpha} s_\omega (\frac{\mu}{2})^\omega,$$

where s_ω stands for the sum of $\binom{\alpha-2}{\omega-2}$ real numbers, such that the absolute value of each of them is not greater than $(\gamma-1)!(\alpha-\gamma-1)!$. It follows that

$$\left| \frac{\mu}{2}\cdot\left(\frac{\mu}{2}-1\right)\cdot\ldots\cdot\left(\frac{\mu}{2}-(\gamma-1)\right)\cdot\frac{\mu}{2}\cdot\left(\frac{\mu}{2}-1\right)\cdot\cdot\left(\frac{\mu}{2}-(\alpha-\gamma-1)\right)\right|$$

$$\leq (\gamma-1)!(\alpha-\gamma-1)!\sum_{\omega=2}^{\alpha}\binom{\alpha-2}{\omega-2}\left|\frac{\mu}{2}\right|^{\alpha}\leq(\gamma-1)!(\alpha-\gamma-1)!\left(1+\left|\frac{\mu}{2}\right|\right)^{\alpha}$$

$$\leq \left(1+\left|\frac{\mu}{2}\right|\right)^{\alpha}(\alpha-2)!\leq\left(1+\left|\frac{\mu}{2}\right|\right)^{\alpha}\alpha!.$$

Using the same idea we get

(1.3;3) $\quad \left|\frac{\mu}{2}\cdot\left(\frac{\mu}{2}-1\right)\cdot\ldots\cdot\left(\frac{\mu}{2}-(\alpha-1)\right)\right|\leq\left(1+\left|\frac{\mu}{2}\right|\right)^{\alpha}\alpha!.$

Notice, $\left|(-1)^{\gamma}(i)^{\alpha-\gamma}\left(\frac{1-xi}{1+xi}\right)^{\gamma}\left(\frac{1}{1-xi}\right)^{\alpha}\right|\leq 1$. Hence, we have the estimation

$$\left|(<x>^{\mu})^{(\alpha)}\right|\leq\sum_{\gamma=0}^{\alpha}\binom{\alpha}{\gamma}<x>^{\mu}\left(1+\left|\frac{\mu}{2}\right|\right)^{\alpha}\alpha!\leq 2^{\alpha}\left(1+\left|\frac{\mu}{2}\right|\right)^{\alpha}\alpha!<x>^{\mu}$$

$$=(2+|\mu|)^{\alpha}\alpha!<x>^{\mu}.$$

2^{o} By (1.3;1)

$$\left|\left(\frac{(<x>^{\mu})^{(\alpha)}}{<x>^{\mu}}\right)^{(\beta)}\right|\leq\sum_{\gamma=0}^{\beta}\binom{\beta}{\gamma}\left|(<x>^{\mu})^{(\alpha+\gamma)}(<x>^{-\mu})^{(\beta-\gamma)}\right|$$

$$\leq\sum_{\gamma=0}^{\beta}\binom{\beta}{\gamma}\cdot\left((2+|\mu|)^{\alpha+\gamma}(\alpha+\gamma)!<x>^{\mu}\right)\cdot\left((2+|\mu|)^{\beta-\gamma}(\beta-\gamma)!<x>^{-\mu}\right)$$

$$\leq\sum_{\gamma=0}^{\beta}\binom{\beta}{\gamma}\cdot(2+|\mu|)^{\alpha+\beta}\cdot(\alpha+\beta)!\leq 2^{\beta}\cdot(2+|\mu|)^{\alpha+\beta}\cdot(\alpha+\beta)!\Box$$

Proposition 2. *Suppose (M.1). Then for every* $h>0$ *resp.* $k>0$ *there is* $\tilde{h}>0$ *resp.* $\tilde{k}>0$, *such that the inclusion mapping*

(1.4;1) $\quad \mathcal{D}^{M_\alpha,\tilde{h}}_{\mu,L^s}(\mathbf{R})\to\mathcal{D}^{M_\alpha,h}_{L^s,\mu}(\mathbf{R})$

resp.

(1.4;2) $\quad \mathcal{D}^{M_\alpha,\tilde{k}}_{L^s,\mu}(\mathbf{R})\to\mathcal{D}^{M_\alpha,k}_{\mu,L^s}(\mathbf{R})$

is continuous.

Proof. Let us show that for given $h > 0$ and $\tilde{h} = 4h$, the mapping (1.4;1) is contiuous.

As M_α fulfils (M.1), it holds

(1.4;3) $\frac{1}{M_{\alpha-\beta}}\frac{M}{M_\beta} \geq \frac{1}{M_\alpha}, \alpha, \beta \in \mathbf{N}, \beta \leq \alpha.$

According to [5], Lemma 4.1 there exists $c > 0$, which depends on h and μ, such that for all $\beta \in \mathbf{N}$

$$\frac{(4h)^\beta (2+|\mu|)^\beta \beta! M_0}{M_\beta} \leq c.$$

From Lemma 1 it follows that

$$\| (< x >^\mu \varphi)^{(\alpha)} \|_s \leq \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \| (< x >^\mu)^{(\beta)} \varphi^{(\alpha-\beta)} \|_s$$

$$\leq \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} (2+|\mu|)^\beta \beta! \, \|< x >^\mu \varphi^{(\alpha-\beta)} \|_s \, .$$

This implies

$$\sum_{\alpha \in \mathbf{N}} \frac{h^\alpha \| (< x >^\mu \varphi)^{(\alpha)} \|_s}{M_\alpha} \leq$$

$$\leq \sum_{\alpha \in \mathbf{N}} \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \frac{(4h)^{\alpha-\beta}(4h)^\beta (2+|\mu|)^\beta \beta! M_0 \, \|< x >^\mu \varphi^{(\alpha-\beta)} \|_s}{4^\alpha M_{\alpha-\beta} M_\beta} \leq$$

$$\leq c \sum_{\alpha \in \mathbf{N}} \frac{2^\alpha}{4^\alpha} \sum_{\gamma \in \mathbf{N}} \frac{(4h)^\gamma \, \|< x >^\mu \varphi^{(\gamma)} \|_s}{M_\gamma} \leq$$

$$\leq 2c \sum_{\gamma \in \mathbf{N}} \frac{(4h)^\gamma \, \|< x >^\mu \varphi^{(\gamma)} \|_s}{M_\gamma},$$

Let us prove the continuity of the mapping (1.4;2), where $\tilde{k} = 8\text{k}$. By (M.1) for given k there exists a constant $c > 0$ such that for all $\beta, \gamma \in \mathbf{N}$

$$\frac{(8k)^{\beta+\gamma} 2^\gamma (2+|\mu|)^{\beta+\gamma} (\beta+\gamma)! M_0}{M_{\beta+\gamma}} \leq c,$$

As, for $f, g \in C^\infty(\mathbf{R})$ and $\alpha \in \mathbf{N}$,

$$f \cdot g^{(\alpha)} = \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} (-1)^\beta (f^{(\beta)} g)^{(\alpha-\beta)},$$

by Lemma 1 it follows

$$\|< x >^{\mu} \varphi^{(\alpha)} \|_s \leq \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \| ((< x >^{\mu})^{(\beta)} \varphi)^{(\alpha-\beta)} \|_s \leq$$

$$\sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \| (\frac{(< x >^{\mu})^{(\beta)}}{< x >^{\mu}} < x >^{\mu} \varphi)^{(\alpha-\beta)} \|_s \leq$$

$$\sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \sum_{\gamma=0}^{\alpha-\beta} \binom{\alpha-\beta}{\gamma} \| (< x >^{\mu} \varphi)^{(\alpha-\beta-\gamma)} \left(\frac{(< x >^{\mu})^{(\beta)}}{< x >^{\mu}} \right)^{(\gamma)} \|_s$$

$$\sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \sum_{\gamma=0}^{\alpha-\beta} \binom{\alpha-\beta}{\gamma} 2^{\gamma} (2+ | \mu |)^{\beta+\gamma} (\gamma+\beta)! \| (< x >^{\mu} \varphi)^{(\alpha-\beta-\gamma)} \|_s .$$

We conclude

$$\sum_{\alpha \in \mathbf{N}} \frac{k^{\alpha} \| < x >^{\mu} \varphi^{(\alpha)} \|_s}{M_{\alpha}} \leq \sum_{\alpha \in \mathbf{N}} \frac{(8k)^{\alpha} \| < x >^{\mu} \varphi^{(\alpha)} \|_s}{8^{\alpha} M_{\alpha}} \leq$$

$$\leq \sum_{\alpha \in \mathbf{N}} \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \sum_{\gamma=0}^{\alpha-r} \binom{\alpha-\beta}{\gamma} \frac{(8k)^{\beta+\gamma} 4^{\beta+\gamma} (2+ | \mu |)^{\beta+\gamma} (\gamma+\beta)! M_0}{M_{\beta+\gamma}} \cdot$$

$$\cdot \frac{(8k)^{\alpha-(\beta+\gamma)}}{8^{\alpha} M_{\alpha-(\beta+\gamma)}} \| (< x >^{\mu} \varphi)^{(\alpha-(\beta+\gamma))} \|_s \leq$$

$$\leq c \sum_{\alpha \in \mathbf{N}} \frac{1}{8^{\alpha}} 2^{\alpha+\alpha} \sum_{\beta \in \mathbf{N}} \frac{(8k)^{\beta} \| (< x >^{\mu} \varphi)^{(\beta)} \|_s}{M_{\beta}} \leq$$

$$\leq 2c \sum_{\beta \in \mathbf{N}} \frac{8k)^{\beta} \| (< x >^{\mu} \varphi)^{(\beta)} \|_s}{M_{\beta}} . \square$$

2. Spaces of weighted ultradifferentiable functions

Let $\mu \in \mathbf{R}$ and $h > 0$.

Definition 2.

$$\mathcal{D}_{L^s,\mu}^{(M_\alpha)}(\mathbf{R}) = \lim_{h \to \infty} \mathcal{D}_{L^s,\mu}^{M_\alpha,h}(\mathbf{R}).$$

$$\mathcal{B}_{\mu}^{(M_\alpha)}(\mathbf{R}) = \mathcal{D}_{L^\infty,\mu}^{(M_\alpha)}(\mathbf{R}).$$

$\mathcal{B}_{\mu}^{(M_\alpha)}(\mathbf{R})$ *is a subspace of* $\dot{\mathcal{B}}_{\mu}^{(M_\alpha)}(\mathbf{R})$ *which is the completion of* $\mathcal{D}^{(M_\alpha)}(\mathbf{R})$ *under the family of norms* $\{\|\ \|_{\mathcal{D}_{L^\infty,\mu}^{M_\alpha,h}(\mathbf{R})}, h > 0\}$.

$$S^{(M_\alpha)}(\mathbf{R}) = proj\ \lim_{\mu\to\infty} \mathcal{D}_{L^s,\mu}^{(M_\alpha)}(\mathbf{R}).$$

Remark 2.2. Proposition 2, implies that families of norms $\{\|\ \|_{\mathcal{D}_{L^s,\mu}^{M_\alpha,h}(\mathbf{R})}$, $h > 0\}$ and $\{\|\ \|_{\mathcal{D}_{L^s,\mu}^{M_\alpha,h}(\mathbf{R})}, h > 0\}$ are equivalent on the spaces $\mathcal{D}_{L^s,\mu}^{(M_\alpha)}(\mathbf{R})$ and $\mathcal{D}^{(M_\alpha)}(\mathbf{R})$. Hence, $\mathcal{D}_{L^s,\mu}^{(M_\alpha)}(\mathbf{R}) = \text{proj}\lim_{h\to\infty} \mathcal{D}_{\mu,L^s}^{M_\alpha,h}(\mathbf{R})$, and $\dot{\mathcal{B}}_{\mu}^{(M_\alpha)}(\mathbf{R})$ is the completion of $\mathcal{D}^{(M_\alpha)}(\mathbf{R})$ under the family of norms $\|\ \|_{\mathcal{D}_{\mu,L^\infty}^{M_\alpha,h}(\mathbf{R})}, h > 0$.

Theorem 1. 1° *There exists* $c > 0$ *such that for each* $\varphi \in \mathcal{D}_{\mu+2/s,L^\infty}^{M_\alpha,h}(\mathbf{R})$

(2.3;1) $\quad \|\varphi\|_{\mathcal{D}_{\mu,L^s}^{M_\alpha,h}(\mathbf{R})} \le c\,\|\varphi\|_{\mathcal{D}_{\mu+2/s,L^\infty}^{M_\alpha,h}(\mathbf{R})}$.

2° *If (M.2)' is fulfiled, there exists* $c > 0$ *such that for each* $\varphi \in \mathcal{D}_{\mu+2/s}^{M_\alpha,h}(\mathbf{R})$

(2.3;2) $\quad \|\varphi\|_{\mathcal{D}_{L^\infty,\mu}^{M_\alpha,h}(\mathbf{R})} \le c\,\|\varphi\|_{\mathcal{D}_{\mu+2/s}^{M_\alpha,h}(\mathbf{R})}$.

In this case,

(2.3;3) $\quad S^{(M_\alpha)}(\mathbf{R}) = proj\ \lim_{\mu\to} \mathcal{D}_{L^s,\mu}^{(M_\alpha)}(\mathbf{R}).$

Proof: 1° It holds

$$\|\varphi\|_{\mathcal{D}_{\mu,L^s}^{M_\alpha,h}(\mathbf{R})}$$

$$= \sum_\alpha \frac{h^\alpha}{M_\alpha} \|<x>^\mu \varphi^{(\alpha)}\|_s$$

$$\le \sum_\alpha \frac{h^\alpha}{M_\alpha} \|<x>^{\mu+2/s} \varphi^{(\alpha)}\|_\infty \|<x>^{-2/s}\|_s$$

$$\le c\sum_\alpha \frac{h^\alpha}{M_\alpha} \|<x>^{\mu+2/s} \varphi^{(\alpha)}\|_\infty = c\,\|\varphi\|_{\mathcal{D}_{\mu+2/s,L^\infty}^{M_\alpha,h}(\mathbf{R})},$$

which implies the continuity of the mapping (2.3;1).

2° Suppose that (M.2)′ is fullfiled. It holds that

$$\|\varphi\|_{\mathcal{D}_{L^\infty,\mu}^{M_\alpha,h}(\mathbf{R})} = \sum_\alpha \frac{h^\alpha}{M_\alpha} \|(<x>^\mu \varphi)^{(\alpha)}\|_\infty = \sum_\alpha \frac{h^\alpha}{M_\alpha} \left\| \int_{-\infty}^{x} (<x>^\mu \varphi)^{(\alpha+1)} dx \right\|_\infty$$

$$\leq \sum_\alpha \frac{h^\alpha}{M_\alpha} \mid \int_{\mathbf{R}} (< x >^\mu \varphi)^{(\alpha+1)} dx \mid \leq \sum_\alpha \frac{h^\alpha}{M_\alpha} \| (< x >^\mu \varphi)^{(\alpha+1)} \|_1$$

$$\leq \frac{A}{hH} \sum_\alpha \frac{(hH)^{\alpha+1}}{M_\alpha} \| (< x >^\mu \varphi)^{(\alpha+1)} \|_1 \leq \frac{A}{hH} \| 4\varphi \|_{\mathcal{D}^{M_\alpha, hH}_{L^1,\mu}(\mathbf{R})} .$$

Let $s > 1$. From the proof of Proposition 2 and Hölder's inequality it follows that there exists $c' > 0$ and $c > 0$ such that, for $s' = s/(s-1)$,

$$\| \varphi \|_{\mathcal{D}^{M_p,h}_{\mu,L^1}(\mathbf{R})} = \sum_\alpha \frac{h^\alpha}{M_\alpha} \| < x >^\mu \varphi^{(\alpha)} \|_1 \leq$$

$$\leq \sum_\alpha \frac{h^\alpha}{M_\alpha} \| < x >^{\mu+2/s} \varphi^{(\alpha+1)} \|_s \| < x >^{-2/s} \|_{s'} \leq c \| \varphi \|_{\mathcal{D}^{M_p,h}_{\mu+2/s}(\mathbf{R})} .$$

Since $\bigcap_\mu \mathcal{D}^{M_p,h}_{L^s,\mu}(\mathbf{R}) = \bigcap_\mu \mathcal{D}^{M_p,h}_{L^\infty,\mu}(\mathbf{R})$. (2.3;1) and (2.3;2) imply the last assertion. □

From now on, we shall allways assume (M.1) and (M.3)$'$. These conditions are sufficient that the space $\mathcal{D}^{(M_\alpha)}(\mathbf{R})$ is non-trivial (see [5],Theorem 4.2.). Obviously, $\mathcal{D}^{(M_\alpha)}(\mathbf{R}) \subset S^{(M_\alpha)}(\mathbf{R})$ but the space $S^{(M_\alpha)}(\mathbf{R})$ is larger than the space $\mathcal{D}^{(M_\alpha)}(\mathbf{R})$. For example the function

(2;1) $\quad \varphi(x) = \sum_{j=1}^\infty \frac{\rho(x-x_j)}{<x_j>^j}, \quad x \in \mathbf{R},$

where $(x_j)_j$ is a sequence of real numbers, such that $\mid x_j \mid +2 \leq \mid x_{j+1} \mid$, the function $\rho \in \mathcal{D}^{(M_\alpha)}(\mathbf{R})$ is such that $\rho(x) = 1$, for $x \in [-1/2, 1/2]$, and that supp $\rho \subset [-1,1]$ and $\rho(x) \geq 0$ for all $x \in \mathbf{R}$ (the existence of ρ follows from [5], p.61) belongs to $S^{(M_\alpha)}(\mathbf{R})$ but does not belong to $\mathcal{D}^{(M_\alpha)}(\mathbf{R})$. Let us prove that. Since the supports of the terms are disjoint, the function φ is infinitely differentiable. Furthermore, if $\alpha \in \mathbf{N}$ and $\mu \in \mathbf{R}$, then for

(2;1.1) $\quad \mid x_j \mid -1 \leq \mid x \mid \leq \mid x_j \mid +1$

it holds

$$< x >^\mu \varphi^{(\alpha)} = \frac{< x >^\mu}{< x_j >^\mu} \frac{\rho^{(\alpha)}(x - x_j)}{< x_j >^{j-\mu}},$$

$$< x >< x_j >^{-1} \leq 6 \quad \text{and} \quad \mid \rho^{(\alpha)}(x - x_j) \mid = \max_{x\in\mathbf{R}} \mid \rho^{(\alpha)}(x) \mid .$$

We see, that if x satisfies (2;1.1), $j > \mu$ and $h > 0$, it holds that

$$\sum_\alpha \frac{h^\alpha}{M_\alpha} \mid < x >^\mu \varphi^{(\alpha)}(x) \mid \leq 6^\mu \sum_\alpha \frac{h^\alpha}{M_\alpha} \| \rho^{(\alpha)} \|_\infty < \infty.$$

This implies $\varphi \in S^{(M_\alpha)}(\mathbf{R})$.

Proposition 3. *The mapping*

(2.4;1) $\mathcal{D}_{L^s}^{(M_\alpha)}(\mathbf{R}) \to \mathcal{D}_{L^s,\mu}^{(M_\alpha)}(\mathbf{R}), \varphi \mapsto < x >^{-\mu} \varphi$

resp.

(2.4;2) $\dot{\mathcal{B}}^{(M_\alpha)}(\mathbf{R}) \to \dot{\mathcal{B}}_\mu^{(M_\alpha)}(\mathbf{R}), \varphi \mapsto < x >^{-\mu} \varphi.$

is an isomorphism.

Proof. From the definitions of the spaces $\mathcal{D}_{L^s}^{(M_\alpha)}(\mathbf{R})$ and $\mathcal{D}_{L^s,\mu}^{(M_\alpha)}(\mathbf{R})$ it follows immediately that mapping (2.3;1) is an isomorphism.

Let $\varphi \in \dot{\mathcal{B}}^{(M_\alpha)}(\mathbf{R})$. There exists a sequence (φ_n) from $\mathcal{D}^{(M_\alpha)}(\mathbf{R})$ such that $\varphi_n \to \varphi$ in $\mathcal{D}_{L^\infty}^{(M_\alpha)}(\mathbf{R})$. From the definition of the space $\mathcal{D}_{L^\infty,\mu}^{(M_\alpha)}(\mathbf{R})$, it follows that $< x >^{-\mu} \varphi_n$ is a sequence from $\mathcal{D}^{(M_\alpha)}(\mathbf{R})$ which converges to $< x >^{-\mu} \varphi$ in $\mathcal{D}_{L^\infty,\mu}^{(M_\alpha)}(\mathbf{R})$, i.e. $< x >^{-\mu} \varphi \in \dot{\mathcal{B}}_\mu^{(M_\alpha)}$. So, (2.4;2) is well-defined. Analogously, we obtain that the mapping

$$\dot{\mathcal{B}}_\mu^{(M_\alpha)}(\mathbf{R}) \to \dot{\mathcal{B}}^{(M_\alpha)}(\mathbf{R}), \quad \varphi \mapsto < x >^{-\mu} \varphi,$$

is well-defined and that it is the inverse mapping for (2.4;1).

The continuity of mapping (2.4;2) and its inverse follows immediately from the definitions of spaces $\dot{\mathcal{B}}^{(M_\alpha)}(\mathbf{R})$ and $\dot{\mathcal{B}}_\mu^{(M_\alpha)}(\mathbf{R})$.□

Theorem 2. *The space* $\mathcal{D}_{L^s,\mu}^{(M_\alpha)}(\mathbf{R})$ *is a (FG)-space (Gelfand space). (For the definition of (FG)-spaces see [1], p.46)*

Proof. Since $\mathcal{D}_{L^s,\mu}^{M_\alpha,n}(\mathbf{R})$, $n \in \mathbf{N}$, are Banach spaces (Proposition 1),

$$\ldots \subset \mathcal{D}_{L^s,\mu}^{M_\alpha,n}(\mathbf{R}) \subset \ldots \mathcal{D}_{L^s,\mu}^{M_\alpha,2}(\mathbf{R}) \subset \mathcal{D}_{L^s,\mu}^{M_\alpha,1}(\mathbf{R})$$

and, for each $n \in \mathbf{N}$ the inclusion mapping

$$\mathcal{D}_{L^s,\mu}^{M_\alpha,n}(\mathbf{R}) \to \mathcal{D}_{L^s,\mu}^{M_\alpha,n+1}(\mathbf{R})$$

is continuous, it follows that $\mathcal{D}_{L^s,\mu}^{(M_\alpha)}(\mathbf{R})$ is a (FG)-space.□

The previous theorem and the proof of [1],p.47,2.2.Satz imply that $\mathcal{D}_{L^s,\mu}^{(M_\alpha)}(\mathbf{R})$ can be represented as a strict (FG)-space.

Theorem 3. $S^{(M_\alpha)}(\mathbf{R})$ *is a strict (F) space. (See [1],p.108 and 47.)*

The proof of the theorem is based on the next two lemmas.

Let $S_\mu^{M_\alpha,h}$ be a closure of $S^{(M_\alpha)}(\mathbf{R})$ in the space $\mathcal{D}_{L^\infty,\mu}^{M_\alpha,h}$.

Lemma 2. *Let $\mu, h > 0$. A subset K of $S_\mu^{M_\alpha,h}$ is a relatively compact set if and only if*
(i) K is bounded
and
(ii) for each $\epsilon > 0$ there exist $\beta \in \mathbf{N}$ such that for all $\varphi \in K$

$$\sum_{\alpha \geq \beta} \frac{h^\alpha}{M_\alpha} \| < x >^\mu \varphi^{(\alpha)} \|_\infty < \epsilon.$$

Proof. 1^o Suppose that $K \subset S_\mu^{M_\alpha,h}$ is a relatively compact set. It follows immediately that K is bounded. Let us prove that (ii) holds. If $\varphi_1, \varphi_2, \dots \varphi_n$ is an ϵ-net for $K, \epsilon > 0$, then there exists $\gamma \in \mathbf{N}$ such that

$$\sum_{\alpha \geq \gamma} \frac{h^\alpha}{M_\alpha} \| < x >^\mu \varphi_i^{(\alpha)} \|_\infty < \epsilon, \quad i \in 1, 2, \dots, n.$$

For each $\varphi \in K$, we have

$$\sum_{\alpha \geq \gamma} \frac{h^\alpha}{M_\alpha} \| < x >^\mu \varphi^{(\alpha)} \|_\infty$$

$$\leq \sum_{\alpha \geq \gamma} \frac{h^\alpha}{M_\alpha} \| < x >^\mu (\varphi - \varphi_i)^{(\alpha)} \|_\infty + \sum_{\alpha \geq \gamma} \frac{h^\alpha}{M_\alpha} \| < x >^\mu \varphi_i^{(\alpha)} \|_\infty < 2\epsilon,$$

which implies (ii).

2^o Suppose that K fulfiles (i) and (ii). Proposition 1 implies that $S_\mu^{M_\alpha,h}$ is a Banach space. Let us prove that a sequence $(\varphi_n)_n$ of elements of K has a Cauchy subsequence, this imples (see [1], p.76, Korollar 1) that K is relatively compact set.

It follows from (i) that there exists $c > 0$ such that for each $\alpha \in N$, $x \in \mathbf{R}$, $\ell > 0$ and $\varphi \in K$ there exists $\xi \in (x, x + \ell)$ such that

$$| < x >^\mu \varphi^{(\alpha)}(x) | \leq c \frac{M_\alpha}{h^\alpha}$$

and

$$| < x >^\mu \varphi^{(\alpha)}(x + \ell) - \varphi^{(\alpha)}(x)) | \leq | < \xi >^\mu \varphi^{(\alpha+1)}(\xi) | \leq c \frac{M_\alpha}{h^\alpha} \ell.$$

The definition of $S_\mu^{M_\alpha,h}$ and the fact that if $\varphi \in S^{(M_\alpha)}(\mathbf{R})$ then for each $\alpha \in \mathbf{N}$ the sequence $(\varphi_{n,\alpha+1}^{(\alpha)})$ converges to ϕ_α in $\mathcal{D}'(\mathbf{R})$. Because of the continuity of differentiation in $\mathcal{D}'(\mathbf{R})$ we have $\phi_\alpha = \phi_0^{(\alpha)}$ in the distributional sence. Since $\phi_\alpha \in L^\infty(\mathbf{R})$, this holds in the clasical sence, too. Therefore for all $\alpha \in \mathbf{N}$

$$(< x >^\mu \varphi_{n,n}^{(\alpha)})_n \xrightarrow{L^\infty(\mathbf{R})} < x >^\mu \phi_0^{(\alpha)} \text{ as } n \to \infty,$$

which implies that, for large enough $n, m \in \mathbf{N}$ and given $\epsilon > 0$,

(2.7;1) $\quad \sum_{\alpha<\beta} \frac{h^\alpha}{M_\alpha} \|< x >^\mu (\varphi_{n,n} - \varphi_{m,m})^{(\alpha)} \|_\infty \leq \epsilon/3.$

Let us prove that the subsequence $(\varphi_{n,n})_n$ of the sequence $(\varphi_n)_n$ is a Chauchy sequence in $S_\mu^{M_\alpha,h}$. If $\epsilon > 0, n, m \in \mathbf{N}$,

(ii) implies that there exists $\beta \in \mathbf{N}$ such that

$$\sum_\alpha \frac{h^\alpha}{M_\alpha} \|< x >^\mu (\varphi_{n,n} - \varphi_{m,m})^{(\alpha)} \|_\infty \leq \sum_{\alpha<\beta} \frac{h^\alpha}{M_\alpha} \|< x >^\mu (\varphi_{n,n} - \varphi_{m,m})^{(\alpha)} \|_\infty$$

$$+ \sum_{\alpha\geq\beta} \frac{h^\alpha}{M_\alpha} \|< x >^\mu \varphi_{n,n}^{(\alpha)} \|_\infty + \sum_{\alpha\geq\beta} \frac{h^\alpha}{M_\alpha} \|< x >^\mu \varphi_{m,m}^{(\alpha)} \|_\infty$$

$$\leq \sum_{\alpha<\beta} \frac{h^\alpha}{M_\alpha} \|< x >^\mu (\varphi_{n,n} - \varphi_{m,m})^{(\alpha)} \|_\infty \leq 2\epsilon/3.$$

It follows from above and (2.7;1) that for each $\epsilon > 0$ and large enough $n, m \in \mathbf{N}$

$$\sum_\alpha \frac{h^\alpha}{M_\alpha} \|< x >^\mu (\varphi_{n,n} - \varphi_{m,m})^{(\alpha)} \|_\infty \leq \epsilon. \square$$

Lemma 3. *If $\nu, \mu, k, h \in \mathbf{N}, \mu \leq \nu$ and $h < k$, the inclusion mapping*
(2.8;1) $\quad i : S_\nu^{M_\alpha,k} \to S_\mu^{M_\alpha,h}$
is a compact operator.

Proof. We have to prove that the ball B of radius one from $S_\nu^{M_\alpha,k}$ is a relatively compact set in $S_\mu^{M_\alpha,k}$. The continuity of (2.8;1), which follows immediately from the definition of $S_\nu^{M_\alpha,k}$ and $S_\mu^{M_\alpha,h}$, implies that that B is bounded in $S_\mu^{M_\alpha,k}$. Let us prove that condition (ii) of Lemma 2 is fulliled.

The assumption gives $\lim_{\alpha\to\infty}(h/k)^\alpha = 0$. Hence, for each $\epsilon > 0$ there exists $\beta \in \mathbf{N}$ such that for all $\alpha \geq \beta$ it holds $h^\alpha \leq \epsilon k^\alpha$. Therefore, for each $\varphi \in B$

$$\sum_{\alpha\geq\beta} \frac{h^\alpha}{M_\alpha} \| < x >^\mu \varphi^{(\alpha)} \|_\infty \leq \epsilon \sum_{\alpha\geq\beta} \frac{k^\alpha}{M_\alpha} \| < x >^\nu \varphi^{(\alpha)} \|_\infty \leq \epsilon.$$

The assertion of the lemma follows from above and the previous lemma.□

Proof of Theorem 3: As the space $S^{(M_\alpha)}(\mathbf{R})$ may be equivalently defined by $S^{(M_\alpha)}(\mathbf{R}) = \operatorname{proj}\lim_{h\to\infty,\, h\in\mathbf{N}} S_h^{M_\alpha,h}$, Lemma 2 and Lemma 3 imply that $S^{(M_\alpha)}(\mathbf{R})$ is an (F)-space. Since $S^{(M_\alpha)}(\mathbf{R})$ is dense in $S_h^{M_\alpha,h}, h > 0, S^{(M_\alpha)}(\mathbf{R})$ is a strict (F)-space.□

As an immediate consequence of the previous theorem (see [1]) we have the next assertion.

Corollary 1. *(i)$S^{(M_\alpha)}(\mathbf{R})$ is compleate, bornological, Montel and Schwartz space.*

(ii) the strong dual $S'^{(M_\alpha)}(\mathbf{R})$ of $S^{(M_\alpha)}(\mathbf{R})$ is an (LS)-space, moreover

$$S'^{(M_\alpha)}(\mathbf{R}) = ind\lim_{h\to\infty,\, h\in\mathbf{N}}(S_h^{M_\alpha,h})$$

where "b" denotes the strong dual.

Theorem 4. *If $1 \leq s \leq r < \infty$ and $\mu \in \mathbf{R}$ then*

$$\mathcal{D}^{(M_\alpha)}(\mathbf{R}) \hookrightarrow S^{(M_\alpha)}(\mathbf{R}) \hookrightarrow \mathcal{D}^{(M_\alpha)}_{L^s,\mu}(\mathbf{R}) \hookrightarrow \mathcal{D}^{(M_\alpha)}_{L^r,\mu}(\mathbf{R}) \hookrightarrow \dot{\mathcal{B}}^{(M_\alpha)}_\mu(\mathbf{R})$$

$$\mathcal{D}^{(M_\alpha)}_{L^r,\mu}(\mathbf{R}) \hookrightarrow \dot{\mathcal{B}}^{(M_\alpha)}_\mu(\mathbf{R}) \hookrightarrow \mathcal{E}^{(M_\alpha)}(\mathbf{R})$$

$$\mathcal{D}^{(M_\alpha)}(\mathbf{R}) \hookrightarrow \mathcal{D}(\mathbf{R}), \quad S^{(M_\alpha)}(\mathbf{R}) \hookrightarrow \mathcal{S} = \bigcap_\mu \mathcal{D}_{L^s,\mu}(\mathbf{R}),$$

$$\mathcal{D}^{(M_\alpha)}_{L^s,\mu}(\mathbf{R}) \hookrightarrow \mathcal{D}_{L^s,\mu}(\mathbf{R}), \quad \mathcal{D}^{(M_\alpha)}_{L^r,\mu}(\mathbf{R}) \hookrightarrow \mathcal{D}_{L^r,\mu}(\mathbf{R})$$

$$\dot{\mathcal{B}}^{(M_\alpha)}_\mu(\mathbf{R}) \hookrightarrow \dot{\mathcal{B}}_\mu(\mathbf{R})$$

$$\mathcal{D}^{(M_\alpha)}_{L^r,\mu}(\mathbf{R}) \hookrightarrow \mathcal{D}_{L^r,\mu}(\mathbf{R}), \quad \dot{\mathcal{B}}^{(M_\alpha)}_\mu(\mathbf{R}) \hookrightarrow \dot{\mathcal{B}}_\mu(\mathbf{R})$$

$$\mathcal{E}^{(M_\alpha)}(\mathbf{R}) \hookrightarrow \mathcal{E}(\mathbf{R}).$$

Proof. Obviously,

$$\mathcal{D}^{(M_\alpha)}(\mathbf{R}) \subset \mathcal{D}^{(M_\alpha)}_{L^s,\mu}(\mathbf{R}) \subset \mathcal{D}^{(M_\alpha)}_{L^r,\mu}(\mathbf{R}) \subset \dot{\mathcal{B}}^{(M_\alpha)}_{\mu}(\mathbf{R}) \subset \mathcal{E}^{(M_\alpha)}(\mathbf{R}),$$

$$\mathcal{D}^{(M_\alpha)}(\mathbf{R}) =< x >^{\mu} \mathcal{D}^{(M_\alpha)}(\mathbf{R}) \text{ and } \mathcal{E}^{(M_\alpha)}(\mathbf{R}) =< x >^{\mu} \mathcal{E}^{(M_\alpha)}(\mathbf{R}).$$

We have ([8],Proposition 1)

$$\mathcal{D}^{(M_\alpha)}(\mathbf{R}) \hookrightarrow \mathcal{D}^{(M_\alpha)}_{L^s}(\mathbf{R}) \hookrightarrow \mathcal{D}^{(M_\alpha)}_{L^r}(\mathbf{R}) \hookrightarrow \dot{\mathcal{B}}^{(M_\alpha)}_{\mu}(\mathbf{R}),$$

$$\mathcal{D}^{(M_\alpha)}_{L^r}(\mathbf{R}) \hookrightarrow \mathcal{E}^{(M_\alpha)}(\mathbf{R}),$$

(from [7] Theorem 3.)

$$\mathcal{D}^{(M_\alpha)}_{L^s}(\mathbf{R}) \hookrightarrow \mathcal{D}_{L^s}(\mathbf{R}),$$

and ([6])

$$\mathcal{D}(\mathbf{R}) \hookrightarrow \mathcal{S}(\mathbf{R}) = \bigcap_{\mu} \mathcal{D}_{L^s,\mu}(\mathbf{R}) \hookrightarrow \mathcal{D}_{L^s,\mu}(\mathbf{R}) \hookrightarrow \mathcal{D}_{L^r,\mu}(\mathbf{R}) \hookrightarrow \dot{\mathcal{B}}_{\mu}(\mathbf{R}).$$

Let us show that $\mathcal{D}^{(M_\alpha)}_{L^s,\mu}(\mathbf{R})$ is a dense subspace of $\mathcal{D}^{(M_\alpha)}_{L^r,\mu}(\mathbf{R})$. If $\phi \in \mathcal{D}^{(M_\alpha)}_{L^r,\mu}(\mathbf{R})$ then there exists $\varphi \in \mathcal{D}^{(M_\alpha)}_{L^r}(\mathbf{R})$ such that $\phi =< x >^{-\mu} \varphi$ (Proposition 3). As $\mathcal{D}^{(M_\alpha)}_{L^s}(\mathbf{R})$ is a dense subspace of $\mathcal{D}^{(M_\alpha)}_{L^r}(\mathbf{R})$, there exists a sequence $(\varphi_n) \in \mathcal{D}^{(M_\alpha)}_{L^s}(\mathbf{R})$, such that $\varphi_n \to \varphi$ in $\mathcal{D}^{(M_\alpha)}_{L^r}(\mathbf{R})$. From Proposition 3 it follows that the sequence, $< x >^{-\mu} \varphi_n \in \mathcal{D}^{(M_\alpha)}_{L^s,\mu}(\mathbf{R})$ converges to $< x >^{-\mu} \varphi = \phi \in \mathcal{D}^{(M_\alpha)}_{L^r,\mu}(\mathbf{R})$ in $\mathcal{D}^{(Mp)}_{L^r,\mu}(\mathbf{R})$, when $n \to \infty$.

Since, the inclusion mapping $i : \mathcal{D}^{(M_\alpha)}_{L^s,\mu}(\mathbf{R}) \to \mathcal{D}^{(M_\alpha)}_{L^r,\mu}(\mathbf{R})$ can be represented as a composition of three continuous mappings in the following way

$$\mathcal{D}^{(M_\alpha)}_{L^s,\mu}(\mathbf{R}) \overset{\varphi \mapsto <x>^{\mu}\varphi}{\longrightarrow} \mathcal{D}^{(M_\alpha)}_{L^s}(\mathbf{R}) \overset{\varphi \mapsto \varphi}{\longrightarrow} \mathcal{D}^{(M_\alpha)}_{L^r}(\mathbf{R}) \overset{\varphi \mapsto <x>^{-\mu}\varphi}{\longrightarrow} \mathcal{D}^{(M_\alpha)}_{L^r,\mu}(\mathbf{R})$$

(see Proposition 3 and [8] Proposition 1), it is continuous.

Using the same idea as above, one can prove that

$$\mathcal{D}^{(M_\alpha)}_{L^s,\mu}(\mathbf{R}) \hookrightarrow \mathcal{D}^{(M_\alpha)}_{L^r,\mu}(\mathbf{R}) \hookrightarrow \dot{\mathcal{B}}^{(M_\alpha)}_{\mu}(\mathbf{R}),$$

$$\mathcal{D}^{(M_\alpha)}_{L^s,\mu}(\mathbf{R}) \hookrightarrow \mathcal{D}_{L^s,\mu}(\mathbf{R}).$$

$$\mathcal{D}^{(M_\alpha)}_{L^r,\mu}(\mathbf{R}) \hookrightarrow \mathcal{D}_{L^r,\mu}(\mathbf{R}), \quad \dot{\mathcal{B}}^{(M_\alpha)}_{\mu}(\mathbf{R}) \hookrightarrow \dot{\mathcal{B}}_{\mu}(\mathbf{R}),$$

and that $\mathcal{D}^{(M_\alpha)}(\mathbf{R})$ resp. $\mathcal{D}^{(M_\alpha)}_{L^r,\mu}(\mathbf{R})$ is dense in $\mathcal{D}^{(M_\alpha)}_{L^s,\mu}(\mathbf{R})$ resp. $\mathcal{E}^{(M_\alpha)}(\mathbf{R})$. The continuity of the inclusion mappings $i : \mathcal{D}^{(M_\alpha)}(\mathbf{R}) \to \mathcal{D}^{(M_\alpha)}_{L^s,\mu}(\mathbf{R})$ and $i : \mathcal{D}^{(M_\alpha)}_{L^r,\mu}(\mathbf{R}) \to \mathcal{E}^{(M_\alpha)}(\mathbf{R})$ follows immediately from the definitions of the these spaces.

Let us show that $\mathcal{D}^{(M_\alpha)}(\mathbf{R}) \subset \mathcal{S}^{(M_\alpha)}(\mathbf{R})$. The inclusion mapping $\mathcal{D}^{(M_\alpha)}(\mathbf{R}) \to \mathcal{S}^{(M_\alpha)}(\mathbf{R})$ is continuous, since for each $\mu \in \mathbf{R}$, the inclusion $\mathcal{D}^{(M_\alpha)}(\mathbf{R}) \to \mathcal{D}^{(M_\alpha)}_{L^\infty,\mu}(\mathbf{R})$ is continuous. To prove that $\mathcal{D}^{(M_\alpha)}(\mathbf{R})$ is dense in $\mathcal{S}^{(M_\alpha)}(\mathbf{R})$, let $\varphi \in \mathcal{S}^{(M_\alpha)}(\mathbf{R})$ be given; choose some $\rho \in \mathcal{D}^{(M_\alpha)}(\mathbf{R})$ such that $\rho \leq 1$ and $\rho = 1$ for $|x| \leq 1$(the existence of ρ follows from [5], Lemma 5.1) and set $\varphi_j(x) = \rho(x/j)\varphi(x) \in \mathcal{D}^{(M_\alpha)}(\mathbf{R}), j = 1,2,\ldots$. Let $h > 0$ and $\mu \geq 0$ be given. We have

$$\| \varphi - \varphi_j \|_{\mathcal{D}^{M_\alpha,h}_{\mu,L^\infty}}$$

$$= \sum_\alpha \frac{h^\alpha \| < x >^\mu (\varphi - \varphi_j)^{(\alpha)} \|_\infty}{M_\alpha} = \sum_\alpha \frac{(2h)^\alpha \| < x >^\mu \varphi^{(\alpha)} \chi_j \|_\infty}{2^\alpha M_\alpha}$$

where $\chi_j(x) = 1$ for $|x| > j$ and $\chi_j(x) = 0$ else. From the definition of the space $\mathcal{S}^{(M_\alpha)}(\mathbf{R})$, it follows that there exists a constant $c(\mu, h)$, which does not depend on α and, such that

$$(2h)^\alpha | < x >^\mu \varphi^{(\alpha)}(x) | / M_\alpha \leq < x >^{-1} c(\mu, h), \quad x \in \mathbf{R}.$$

This imples that, for $\alpha \in \mathbf{N}, (2h)^\alpha \| < x >^\mu \varphi^{(\alpha)} \chi_j \|_s / M_\alpha$ converges uniformly to zero as $j \to \infty$. Hence, $\| \varphi - \varphi_j \|_{\mathcal{D}^{M_\alpha,h}_{L^s,\mu}} \to 0$ as $j \to \infty$.

$\mathcal{E}^{(M_\alpha)}(\mathbf{R}) \hookrightarrow \mathcal{E}(\mathbf{R})$ and $\mathcal{D}^{(M_\alpha)}(\mathbf{R}) \hookrightarrow \mathcal{D}(\mathbf{R})$ follow from [3],Theorem 7.3. and the definitions of these spaces.

Since $\mathcal{D}^{(M_\alpha)}(\mathbf{R}) \hookrightarrow \mathcal{D}(\mathbf{R}) \hookrightarrow \mathcal{S}(\mathbf{R})$, and $\mathcal{D}^{(M_\alpha)}(\mathbf{R}) \subset \mathcal{S}^{(M_\alpha)}(\mathbf{R})$, it follows that $\mathcal{S}^{(M_\alpha)}(\mathbf{R})$ is dense in $\mathcal{S}(\mathbf{R})$. The continuity of the inclusion mapping follows immediately from the definitions of the spaces.□

3. Elementary operations

3.1. (Ultra)differentiation

Theorem 5. *Suppose (M.2)'. Then the mappings*

(3.1;1) $\quad (-1)^\beta D^\beta : \mathcal{D}^{(M_\alpha)}_{L^s,\mu}(\mathbf{R}) \to \mathcal{D}^{(M_\alpha)}_{L^s,\mu}(\mathbf{R}),$

(3.1;2) $(-1)^\beta D^\beta : \dot{\mathcal{B}}_\mu^{(M_\alpha)}(\mathbf{R}) \to \dot{\mathcal{B}}_\mu^{(M_\alpha)}(\mathbf{R})$,
(3.1;3) $(-1)^\beta D^\beta : \mathcal{S}^{(M_\alpha)}(\mathbf{R}) \to \mathcal{S}^{(M_\alpha)}(\mathbf{R})$,
where $\beta \in \mathbf{N}$, *are continuous.*

Proof. Since

$$\frac{1}{M_\alpha} \leq \frac{A^\beta H^{(\alpha+\beta-1)\beta/2}}{M_{\alpha+\beta}}, \qquad \alpha, \beta \in \mathbf{N}.$$

We have, for each $\varphi \in \mathcal{D}_{\mu,L^s}^{(M_\alpha)}(\mathbf{R})$

$$\| D^\beta \varphi \|_{\mathcal{D}_{\mu,L^s}^{M_\alpha,h}} \leq \sum_\alpha \frac{h^\alpha \| < x >^\mu \varphi^{(\alpha+\beta)} \|}{M_\alpha}{}_s$$

$$\leq \sum_\alpha \frac{h^\alpha H^{(\alpha+\beta-1)\beta/2} A^\beta \| < x >^\mu \varphi^{(\alpha+\beta} |_s}{M_{\alpha+\beta}}$$

$$\leq A^\alpha H^{\alpha(\alpha-1)/2} (H^{\alpha/2} h)^{-\alpha} \sum_\beta \frac{(H^{\alpha/2} h)^{\beta+\alpha} \| < x >^\mu \varphi^{(\alpha+\beta)} \|_s}{M_{\alpha+\beta}}$$

$$\leq A^\beta H^{-\beta/2} h^{-\beta} \| \varphi \|_{\mathcal{D}_{\mu,L^s}^{M_\alpha, hH^{\beta/2}}} .$$

From above and the definition of the spaces $\mathcal{D}_{L^s,\mu}^{(M_\alpha)}(\mathbf{R})$ and $\mathcal{S}^{(M_\alpha)}(\mathbf{R})$, it folows the continuity of (3.1;1) and (3.1;3). Let us prove that the mapping $(-1)^\beta D^\beta : \dot{\mathcal{B}}_\mu^{(M_\alpha)}(\mathbf{R}) \to \dot{\mathcal{B}}_\mu^{(M_\alpha)}(\mathbf{R})$ is continuous. As (3.1;1) is continuous, it follows, from the definition of the space $\dot{\mathcal{B}}_\mu^{(M_\alpha)}(\mathbf{R})$, that the mapping $(-1)^\beta D^\beta : \dot{\mathcal{B}}_\mu^{(M_\alpha)}(\mathbf{R}) \to \mathcal{D}_{L^\infty,\mu}^{(M_\alpha)}(\mathbf{R})$ is also continuous. To prove the continuity of (3.1;2) it is enough to show that $(-1)^\beta D^\beta$ maps $\dot{\mathcal{B}}_\mu^{(M_\alpha)}(\mathbf{R})$ into $\dot{\mathcal{B}}_\mu^{(M_\alpha)}(\mathbf{R})$. If $\varphi \in \dot{\mathcal{B}}_\mu^{(M_\alpha)}(\mathbf{R})$ there exists a sequence $(\varphi_n)_n$ from $\mathcal{D}^{(M_\alpha)}(\mathbf{R})$, such that $\varphi_n \to \varphi$ in $\dot{\mathcal{B}}_\mu^{(M_\alpha)}(\mathbf{R})$, when $n \to \infty$. Since mapping $(-1)^\beta D^\beta$: $\mathcal{D}^{(M_\alpha)}(\mathbf{R}) \to \mathcal{D}(\mathbf{R})$ ([5]), we have $(-1)^\beta D^\beta \varphi_n \in \mathcal{D}(\mathbf{R})$, the mapping $(-1)^\beta D^\beta$: $\dot{\mathcal{B}}_\mu^{(M_\alpha)}(\mathbf{R}) \to \mathcal{D}_{L^\infty,\mu}^{(M_\alpha)}(\mathbf{R})$ is continuous and $(-1)^\beta D^\beta \varphi_n \to (-1)^\beta D^\beta \varphi$ as $n \to \infty$ in $\mathcal{D}_{L^\infty,\mu}^{(M_\alpha)}(\mathbf{R})$. Thus, $(-1)^\beta D^\beta \varphi \in \dot{\mathcal{B}}_\mu(\mathbf{R})$.

Theorem 6. *Let* $P^*(D) = \sum_{\alpha=0}^\infty a_\alpha (-1)^\alpha D^\alpha$ *be an ultradifferential operator of the class* (M_α).
If (M.2)', then
(3.2;1) $P^*(D) : \mathcal{D}_{L^s,\mu}^{(M_\alpha)}(\mathbf{R}) \to \mathcal{D}_{L^s,\mu}(\mathbf{R})$,

(3.2;2) $P^*(D)$: $\dot{\mathcal{B}}_\mu^{(M_\alpha)}(\mathbf{R}) \to {}_\mu(\mathbf{R})$,
(3.2;3) $P^*(D) : \mathcal{S}^{(M_\alpha)}(\mathbf{R}) \to \mathcal{S}(\mathbf{R})$,
are continuous linear mappings.
If (M.2), then
(3.2;4) $P(D) : \mathcal{D}_{L^s,\mu}^{(M_\alpha)}(\mathbf{R}) \to \mathcal{D}_{L^s,\mu}^{(M_\alpha)}(\mathbf{R})$,
(3.2;5) $P(D) : \dot{\mathcal{B}}_\mu^{(M_\alpha)}(\mathbf{R}) \to \dot{\mathcal{B}}_\mu^{(M_\alpha)}(\mathbf{R})$,
(3.2;6) $P(D) : \mathcal{S}^{(M_\alpha)}(\mathbf{R}) \to \mathcal{S}^{(M_\alpha)}(\mathbf{R})$,
are continuous linear mappings.

Proof: 1^oLet (M.2)$'$. For fixed $m \in \mathbf{N}$ and each $\beta \le m$, we have

$$||< x >^\mu (P^*(D)\varphi)^{(\beta)} ||_s \le \sum_\alpha C \frac{L^\alpha}{M_\alpha} ||< x >^\mu \varphi^{(\alpha+\beta)} ||_s \le$$

$$\le \sum_\alpha C \frac{L^\alpha A^\alpha H^{\alpha\beta+\beta(\beta-1)/2}}{M_{\alpha+\beta}} ||< x >^\mu \varphi^{(\alpha+\beta)} ||_s \le$$

$$\le C H^{\beta(\beta-1)} \sum_\alpha \frac{(1+L)^{\alpha+\beta}(1+A)^{\alpha+\beta}(1+H)^{m(\alpha+\beta)}}{M_{\alpha+\beta}} ||< x >^\mu \varphi^{(\alpha+\beta)} ||_s \le$$

$$\le C(1+H)^{m(m-1)/2} || \varphi ||_{\mathcal{D}_{\mu,L^s}^{M_\alpha,k}(\mathbf{R})},$$

where $k = (1+L)(1+A)(1+H)^m$. Hence, (3.2;1) is a linear continuous mapping.

Since the mapping $P^*(D) : \mathcal{D}^{(M_\alpha)}(\mathbf{R}) \to \mathcal{D}(\mathbf{R})$ is continuous ([5], Theorem 2.12, p.47), from above and the definition of the space $\dot{\mathcal{B}}_\mu^{(M_\alpha)}(\mathbf{R})$ we get, analogously as in the proof of Theorem 3, the continuity of mapping (3.2;5).

The continuity of (3.2;3) follows immediately from the continuity of (3.2;1).

2^o Assume (M.2).

The continuity of (3.2;4) follows from the fact that the mapping

$$P^*(D) : \mathcal{D}_{\mu,L^s}^{M_\alpha,k}(\mathbf{R}) \to \mathcal{D}_{\mu,L^s}^{M_\alpha,h}(\mathbf{R}),$$

where $k = (1+h)(1+2L)H$, is continuous. Indeed,

$$|| P^*(D)\varphi ||_{\mathcal{D}_{\mu,L^s}^{M_\alpha,h}(\mathbf{R})} \le \sum_\alpha \sum_\beta \frac{h^\beta \mid a_\alpha \mid ||< x >^\mu \varphi^{(\alpha+\beta)} ||_s}{M_\beta}$$

$$\leq C\sum_{\alpha} 2^{-\alpha}\sum_{\beta}\frac{h^{\beta}(2L)^{\alpha}\,\|< x >^{\mu}\varphi^{(\alpha+\beta)}\|_s}{M_{\alpha}M_{\beta}}$$

$$\leq AC\sum_{\alpha} 2^{-\alpha}\sum_{\beta}\frac{(1+h)^{\alpha+\beta}(1+2L)^{\alpha+\beta}H^{\alpha+\beta}\,\|< x >^{\mu}\varphi^{(\alpha+\beta)}\|_s}{M_{\alpha+\beta}}$$

$$\leq 2AC\,\|\varphi\|_{\mathcal{D}^{M_\alpha,k}_{\mu,L^s}(\mathbf{R})}\,.$$

Using the same idea as in the first part of the proof and the result that $P^*(D):\mathcal{D}^{(M_\alpha)}(\mathbf{R})\to\mathcal{D}^{(M_\alpha)}(\mathbf{R})$(5,Theorem 2.12), one can prove the continuity of the mapping (3.2;5).

The continuity of (3.2;6) follows immediately from the continuity of (3.2;4).□

3.2. Multiplication

Theorem 7. *Let $\mu,\nu\in\mathbf{R}$ and $1\leq s,q,r\leq\infty$.*

1° If $\frac{1}{s}+\frac{1}{q}\geq\frac{1}{r}$; then the mappings (pointwise multiplications)

(3.3;1) $\mathcal{D}^{(M_\alpha)}_{L^s,\mu}(\mathbf{R})\times\mathcal{D}^{(M_\alpha)}_{L^q,\nu}(\mathbf{R})\to\mathcal{D}^{(M_\alpha)}_{L^r,\mu}\mathbf{R}),(\varphi,\phi)\mapsto\varphi\phi,$

(3.3;2) $\mathcal{D}^{(M_\alpha)}_{L^s,\mu}(\mathbf{R})\times\dot{\mathcal{B}}^{(M_\alpha)}_{\nu}(\mathbf{R})\to\mathcal{D}^{(M_\alpha)}_{L^s,\mu}(\mathbf{R}),(\varphi,\phi)\mapsto\varphi\phi,$

(3.3;3) $\mathcal{D}^{(M_\alpha)}_{L^\infty,\mu}(\mathbf{R})\times\dot{\mathcal{B}}^{(M_\alpha)}_{\nu}(\mathbf{R})\to\dot{\mathcal{B}}^{(M_\alpha)}_{\mu+\nu}(\mathbf{R}),(\varphi,\phi)\mapsto\varphi\phi,$

(3.3;4) $\dot{\mathcal{B}}^{(M_\alpha)}_{\mu}(\mathbf{R})\times\dot{\mathcal{B}}^{(M_\alpha)}_{\nu}(\mathbf{R})\to\dot{\mathcal{B}}^{(M_\alpha)}_{\mu+\nu}(\mathbf{R}),(\varphi,\phi)\mapsto\varphi\phi,$ *are continuous.*

Proof. 1° Let $\varphi\in\mathcal{D}^{(M_\alpha)}_{L^s,\mu}(\mathbf{R})$ and $\phi\in\mathcal{D}^{(M_\alpha)}_{L^q,\nu}(\mathbf{R})$ and $(1/s)+(1/q)=1/t$. The generalized Hölder inequaluty ([2],p.189) and (M.1) implay

$$\|\varphi\phi\|_{\mathcal{D}^{M_\alpha,h}_{L^r,\mu}}=\sum_{\alpha}\frac{h^{\alpha}}{M_\alpha}\,\|(< x >^{\mu}\varphi< x >^{\nu}\phi)^{(\alpha)}\|_r$$

$$=\sum_{\alpha}\sum_{k=0}^{\alpha}\binom{\alpha}{k}\frac{h^{\alpha}\,\|(< x >^{\mu}\varphi)^{(k)}(< x >^{\nu}\phi)^{(\alpha-k)}\|_r}{M_\alpha}$$

$$\leq\sum_{\alpha}\sum_{k=0}^{\alpha}\binom{\alpha}{k}\frac{h^{\alpha}\,\|(< x >^{\mu}\varphi)^{(k)}\|_s\|(< x >^{\nu}\phi)^{(\alpha-k)}\|_q}{M_\alpha}$$

$$\leq\sum_{\alpha}\frac{M_0}{4^{\alpha}}\sum_{k=0}^{\alpha}\frac{(4h)^k}{M_k}\,\|(< x >^{\mu}\varphi)^{(k)}\|_s\,\frac{(4h)^{\alpha-k}}{M_{\alpha-k}}\,\|(< x >^{\nu}\phi)^{(\alpha-k)}\|_q\leq$$

$$\leq 2M_0 \parallel \varphi \parallel_{\mathcal{D}^{M_\alpha,4h}_{L^q,\mu}(\mathbf{R})} \parallel \phi \parallel_{\mathcal{D}^{M_\alpha,4h}_{L^s,\mu}(\mathbf{R})} .$$

This proves the continuity of the mapping

$$\mathcal{D}^{(M_\alpha)}_{L^s,\mu}(\mathbf{R}) \times \mathcal{D}^{(M_\alpha)}_{L^q,\nu}(\mathbf{R}) \to \mathcal{D}^{(M_\alpha)}_{L^t,\mu}(\mathbf{R}), (\varphi,\phi) \mapsto \varphi\phi.$$

Since $\mathcal{D}^{(M_\alpha)}_{L^s,\mu}(\mathbf{R}) \subset \mathcal{D}^{(M_\alpha)}_{L^t,\mu}(\mathbf{R})$, mapping (3.3;1) is continuous. The continuity of (3.3;2),(3.3;3) and (3.3;4) follows from the continuity of the mapping (3.3;1), the definition of space $\dot{\mathcal{B}}^{(M_\alpha)}_{\mu}(\mathbf{R})$ and the fact that the pointwise multiplication maps $\mathcal{E}^{(M_\alpha)}(\mathbf{R}) \times \mathcal{D}^{(M_\alpha)}(\mathbf{R})$ into $\mathcal{D}^{(M_\alpha)}(\mathbf{R})$ continuously ([5], p.69).

Corollary 2. *If $p, q \in [1,\infty]$, $q < p$ and $\mu < \nu - ((1/q) - (1/p))$ then*

$$\mathcal{D}^{(M_\alpha)}_{L^p,\mu}(\mathbf{R}) \subset \mathcal{D}^{(M_\alpha)}_{L^q,\nu}(\mathbf{R}).$$

Proof. Let $(1/r) = (1/q) - (1/p)$. Then $r \in [1,\infty)$. The assumption $\mu < \nu - ((1/q) - (1/p))$ implies $r(\mu - \nu) < -1$ and hence $< x >^{\mu-\nu} \in L^r(\mathbf{R})$. Therefore from (M.3)$'$ and Lemma 1 it follows that for each $h > 0$

$$\sum_\alpha \frac{h^\alpha}{M_\alpha} \parallel (< x >^{\mu} < x >^{-\nu})^{(\alpha)} \parallel_r \leq \sum_\alpha \frac{h^\alpha}{M_\alpha} (2+ \mid \mu - \nu \mid)^\alpha \alpha! \parallel < x >^{\mu-\nu} \parallel_r$$

$$\leq \parallel < x >^{\mu-\nu} \parallel_r \sum_\alpha \frac{1}{2^\alpha} ((2h(2+ \mid \mu - \nu \mid))^\alpha \alpha!)/M_\alpha < \infty,$$

which implies that $< x >^{-\nu} \in \mathcal{D}^{(M_\alpha)}_{L^r,\mu}(\mathbf{R})$. Therefore, the inclusion mapping

$$i : \mathcal{D}^{(M_\alpha)}_{L^p,\mu}(\mathbf{R}) \to \mathcal{D}^{(M_\alpha)}_{L^q,\nu}(\mathbf{R})$$

can be represented as a composition of the following continuous (see Proposition 3 and Theorem 7) mappings

$$\mathcal{D}^{(M_\alpha)}_{L^p,\nu}(\mathbf{R}) \to \mathcal{D}^{(M_\alpha)}_{L^p}(\mathbf{R}), \varphi \mapsto < x >^{\nu} \varphi$$

and

$$\mathcal{D}^{(M_\alpha)}_{L^p}(\mathbf{R}) \to \mathcal{D}^{(M_\alpha)}_{L^r,\mu}(\mathbf{R}) \cdot \mathcal{D}^{(M_\alpha)}_{L^p}(\mathbf{R}) = \mathcal{D}^{(M_\alpha)}_{L^q,\mu}(\mathbf{R}), \varphi \mapsto < x >^{-\nu} \varphi,$$

and so it is continuous.

$\mathcal{D}^{(M_\alpha)}_{L^p,\nu}(\mathbf{R})$ is dense in $\mathcal{D}^{(M_\alpha)}_{L^q,\mu}(\mathbf{R})$ since $\mathcal{D}^{(M_\alpha)}(\mathbf{R}) \subset \mathcal{D}^{(M_\alpha)}_{L^p,\nu}(\mathbf{R})$ and $\mathcal{D}^{(M_\alpha)}(\mathbf{R})$ is dense in $\mathcal{D}^{(M_\alpha)}_{L^q,\mu}(\mathbf{R})$.□

3.3. The space of multipliers

Theorem 8. *Let $\varphi \in \mathcal{E}^{(M_\alpha)}(\mathbf{R})$. The following conditions are equivalent (i) for all $\psi \in S^{(M_\alpha)}(\mathbf{R}), \varphi\psi \in S^{(M_\alpha)}(\mathbf{R})$, (ii) for every $h > 0$ there exists $\mu \leq 0$, such that*

$$\sum_\alpha \frac{h^\alpha}{M_\alpha} \| < x >^\mu \varphi^{(\alpha)} \|_\infty < \infty.$$

Proof. 1^o Let us prove, by contradiction, that (i) implies (ii). Suppose that condition (i) is and (ii) is not fulfilled. Then there is $h > 0$ and a sequence $x_j, j \in \mathbf{N}$, which tends to infinity, such that

$$\sum_\alpha \frac{h^\alpha}{M_\alpha} \mid \varphi^{(\alpha)}(x_j) \mid > j < x_j >^j .$$

Without loss of generality we may suppose that $\mid x_j \mid +2 \leq \mid x_{j+1} \mid$. Consider the function $\psi \in S^{(M_\alpha)}(\mathbf{R})$, defined by (2;1).

For each $j \in \mathbf{N}$,
(3.5;1)$\sum_\alpha \frac{h^\alpha}{M_\alpha} \mid (\psi\varphi)^{(\alpha)}(x_j) \mid = \sum_\alpha \frac{h^\alpha}{M_\alpha} \mid \sum_{k=0}^{\alpha} \binom{\alpha}{k} \psi^{(k)}(x) \varphi^{(\alpha-k)}(x_j) \mid$

$$= \sum_\alpha \frac{h^\alpha}{M_\alpha} \mid \sum_{k=0}^{\alpha} \binom{\alpha}{k} \frac{\rho^{(k)}(0)}{< x_j >^j} \varphi^{(\alpha-k)}(x_j) \mid$$

$$= \sum_\alpha \frac{h^\alpha}{M_\alpha} \mid \frac{1}{< x_j >^j} \varphi^{(\alpha)}(x_j) \mid > \text{ j}.$$

On the other hand, (i) implies that for each $\psi \in S^{(M_\alpha)}(\mathbf{R})$ there exist a constant $c = c(h, \psi, \varphi)$ such that

$$\sum_\alpha \frac{h^\alpha}{M_\alpha} \mid (\psi\varphi)^{(\alpha)}(x) \mid < \frac{c}{< x >}, \quad x \in \mathbf{R}.$$

It follows that

$$\sum_\alpha \frac{h^\alpha}{M_\alpha} \mid (\psi\varphi)^{(\alpha)}(x) \mid \to 0, \text{ as } x \to \infty,$$

which contradict (3.5;1).

2^o Assume that (ii) is fulfilled. If $\psi \in S^{(M_\alpha)}(\mathbf{R}), h > 0$ and $\nu \in \mathbf{R}$ then there exists μ such that

$$\sum_\alpha \frac{h^\alpha}{M_\alpha} \| < x >^\nu (\psi\varphi)^{(\alpha)} \|_\infty \leq \sum_\alpha \frac{1}{4^\alpha}\frac{(4h)^\alpha}{M_\alpha} \sum_{j=0}^{\alpha} \binom{\alpha}{j} \| < x >^\nu \psi^{(j)}\varphi^{(\alpha-j)} \|_\infty$$

$$\leq M_o \sum_\alpha \frac{1}{4^\alpha} \sum_{j=0}^{\alpha} \frac{(4h)^\alpha}{M_{\alpha-j}M_j} \binom{\alpha}{j} \| < x >^{\nu-\mu} \psi^{(j)} \|_\infty \| < x >^\mu \varphi^{(\alpha-j)} \|_\infty$$

$$\leq M_o \sum_\alpha \frac{1}{4^\alpha} \sum_{j=0}^{\alpha} \frac{(4h)^j}{M_j} \binom{\alpha}{j} \| < x >^{\nu-\mu} \psi^{(j)} \|_\infty \frac{(4h)^{\alpha-j}}{M_{\alpha-j}} \| < x >^\mu \varphi^{(\alpha-j)} \|_\infty$$

$$\leq 2M_o \sum_\alpha \frac{(4h)^\alpha}{M_\alpha} \| < x >^{\nu-\mu} \psi^{(\alpha)} \|_\infty \sum_\alpha \frac{(4h)^\alpha}{M_\alpha} \| < x >^\mu \varphi^{(\alpha)} \|_\infty < \infty,$$

which means that (i) is fulfilled.

Definition 3. $O_M^{(M_\alpha)}(\mathbf{R})$ *is the vector space of all* $\varphi \in \mathcal{E}^{(M_\alpha)}(\mathbf{R})$, *such that for all* $\psi \in S^{(M_\alpha)}(\mathbf{R})$ *the piontwise product* $\varphi\psi$ *belongs to* $S^{(M_\alpha)}(\mathbf{R})$. *The topology on* $O_M^{(M_\alpha)}(\mathbf{R})$ *is induiced by the family of seminorms*

$$p_{\psi,h,\nu}(\varphi) = \sum_\alpha \frac{h^\alpha}{M_\alpha} \| < x >^\nu (\psi\varphi)^{(\alpha)} \|_\infty, \quad \psi \in S^{(M_p)}(\mathbf{R}), h > 0, \nu \in \mathbf{R}.$$

Theorem 9. *The pointwise multiplication*

$$S^{(M_\alpha)}(\mathbf{R}) \times O_M^{(M_\alpha)}(\mathbf{R}) \to S^{(M_\alpha)}(\mathbf{R}), \quad (\psi, \varphi) \mapsto \psi\varphi$$

is a separately continuous mapping. Proof: *From the proof of Theorem 3.5. (2^o) follows the continuity of the mapping*

$$S^{(M_\alpha)}(\mathbf{R}) \to S^{(M_\alpha)}(\mathbf{R}), \psi \mapsto \psi\varphi, \varphi \in O_M^{(M_\alpha)}(\mathbf{R}).$$

The continuity of the mapping

$$O_M^{(M_\alpha)}(\mathbf{R}) \to S^{(M_\alpha)}(\mathbf{R}), \varphi \mapsto \psi\varphi, \psi \in S^{(M_\alpha)}(\mathbf{R})$$

follows immediately from the definition of the spaces $S^{(M_\alpha)}(\mathbf{R})$ *and* $O_M^{(M_\alpha)}(\mathbf{R})$.□

Remark 3.8. If $\varphi \in \mathcal{E}^{(M_\alpha)}(\mathbf{R})$ is such that $\varphi\psi \in S^{(M_\alpha)}(\mathbf{R})$ for all $\psi \in S^{(M_\alpha)}(\mathbf{R})$ for all $\psi \in S^{(M_\alpha)}(\mathbf{R})$ then by Theorem 5 $\varphi \in O_M^{(M_\alpha)}(\mathbf{R})$.

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REZIME

PROSTORI TEŽINSKIH I TEMPERIRANIH ULTRADISTRIBUCIJA DEO I

Definisani su prostori $\mathcal{D}_{L^s,\mu}^{(M_\alpha)}$, $\dot{\mathcal{B}}_\mu^{(M_\alpha)}(\mathbf{R})$ težinskih ultradiferencijabilnih funkcija i prostor $\mathcal{S}^{(M_\alpha)}(\mathbf{R})$ brzo opadajućih ultradiferencijabilnih funkcija, koji čine prirodnu generalizaciju Švarcovih prostora $\mathcal{D}_{L^s}(\mathbf{R})$, $\dot{\mathcal{B}}(\mathbf{R})$ i $\mathcal{S}(\mathbf{R})$. Ispitana je topološka struktura tih prostora i njihove relacije sa poznatim prostorima distribucija. Dokazana su osnovna svojstva operacija (ultra)diferenciranja i množenja u tim prostorima. Odredjen je prostor $O_M^{(M_\alpha)}(\mathbf{R})$ množioca prostora $\mathcal{S}^{(M_\alpha)}(\mathbf{R})$.

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