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SET-VALUED ISOMETRIES ON THE REAL LINE

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Abstract

Set-valued functions, defined on $\mathbb{R}$ and having closed intervals on $\mathbb{R}$ as values are considered. Due to the Hausdorff metric defined on the set of closed intervals, a set-valued isometry is defined. A complete characterisation of such real, set-valued isometries is given.

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1.

Let (X,d) be a metric space and Y be a metric vector space. Denote by $n(Y)$ the family of all nonempty subsets of Y and by $cc(Y)$ the family of all convex and compact elements of $n(Y)$. Any function mapping X into $n(Y)$ is called a *set-valued* (or *multivalued*) function. In this paper we restrict ourselves to considering set-valued functions having values in $cc(Y)$. The *Hausdorff metric* d_H in $cc(Y)$ is defined by:

$$d_H(A,B) := \inf\{r \geq 0: \ A \subset B + rS, \ B \subset A + rS\}$$

for $A, B \in cc(y)$, where S denotes a unit ball in Y.
Now, we define a *set-valued isometry* as a function $F : X \to cc(Y)$ satisfying the condition

$$(1) \qquad d_H(F(x), F(y)) = d(x,y) \quad \text{for } x, y \in X.$$

In the present paper we deal with set-valued isometries on the real line only; so, from now on, we assume that $X = Y = \mathbb{R}$. The family $cc(\mathbb{R})$ coincide with the set of all compact intervals on $\mathbb{R}$. Each set-valued function $F : \mathbb{R} \to cc(\mathbb{R})$ has the form:

$$F(x) = [f(x), g(x)] \quad \text{for } x \in \mathbb{R} \tag{2}$$
with functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(x) \leq g(x)$ for $x \in \mathbb{R}$.

Let us start with the following, trivial to obtain,

Lemma 1. *Let $A = [a_1, a_2]$, $B = [b_1, b_2]$ with $a_1, a_2, b_1, b_2 \in \mathbb{R}$ such that $a_1 \leq a_2$ and $b_1 \leq b_2$. Then*

$$d_H(A, B) = \max\{|a_1 - b_1|, |a_2 - b_2|\}.$$

According to the above lemma a function $F : \mathbb{R} \to cc(\mathbb{R})$ having form (2) is an isometry if and only if

$$\max\{|f(x) - f(y)|, |g(x) - g(y)|\} = |x - y| \quad \text{for } x, y \in \mathbb{R}. \tag{3}$$

Functional equation (3), with unknown functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f \leq g$, will be the subject matter of our considerations.

2.

It is obvious that (3) implies inequalities

$$|f(x) - f(y)| \leq |x - y| \quad \text{for } x, y \in \mathbb{R} \tag{4}$$

and

$$|g(x) - g(y)| \leq |x - y| \quad \text{for } x, y \in \mathbb{R}. \tag{5}$$

Therefore, both f and g are continuous. Functions f and g satisfy (3) if and only if any translations $f + c_1$ and $g + c_2$ ($c_1, c_2 \in \mathbb{R}$) do. Hence we can assume, without loss of generality, that $f(0) = 0$.

Proposition 1. *Let functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(x) \leq g(x)$ for $x \in \mathbb{R}$ satisfy (3) and, additionally, $f(0) = 0$, $g(0) = a \geq 0$. Then at least one of the following conditions holds:*

1) $f(x) = x$ *for all* $x \in \mathbb{R}$,

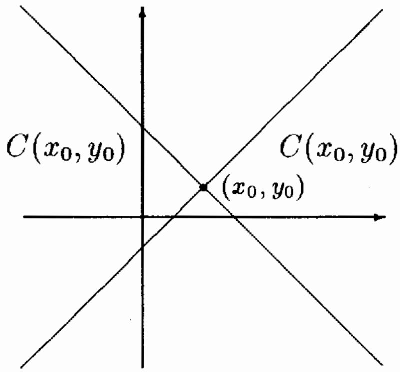
2) $f(x) = -x$ *for all* $x \in \mathbb{R}$,

3) $g(x) = x + a$ *for all* $x \in \mathbb{R}$,

4) $g(x) = -x + a$ *for all* $x \in \mathbb{R}$.

Proof. Step 1. For $(x_0, y_0) \in \mathbb{R}^2$ denote by $C(x_0, y_0)$ the set

$$C(x_0, y_0) := \{(x, y) \in \mathbb{R}^2 : |y - y_0| \le |x - x_0|\}.$$



Putting in (4), (5) x_0 and x in places of x and y, respectively, we obtain the following implications:

(6) if $f(x_0) = y_0$ then $(x, f(x)) \in C(x_0, y_0)$ for $x \in \mathbb{R}$

and

(7) if $g(x_0) = y_0$ then $(x, g(x)) \in C(x_0, y_0)$ for $x \in \mathbb{R}$.

It means that the graphs of f and g are always included in the cone $C(x_0, y_0)$ determined by a given point (x_0, y_0) belonging, respectively, to the graph of f or g. In particular, since $f(0) = 0$ and $g(0) = a$, we have

(8) $\operatorname{graph} f \subset C(0,0)$ and $\operatorname{graph} g \subset C(0, a)$.

Step 2. Fix $x_0 \in \mathbb{R}$. For all x lying between x_0 and 0 we have

(i) if $f(x_0) = x_0$ then $f(x) = x$,

(ii) if $f(x_0) = -x_0$ then $f(x) = -x$,

(iii) if $g(x_0) = x_0 + a$ then $g(x) = x + a$,

(iv) if $g(x_0) = -x_0 + a$ then $g(x) = -x + a$.

Indeed, if, for example, $f(x_0) = x_0$ then, by (6) and (8):

$$(x, f(x)) \in C(0,0) \cap C(x_0, x_0).$$

The intersection of $C(0,0)$ and $C(x_0, x_0)$, over an interval connecting 0 with x_0, coincide with the part of the line $y = x$. Therefore, $f(x) = x$ for x lying between 0 and x_0. The proofs of (ii) – (iv) are similar.

Step 3. There is

$$\text{(i)}\quad \begin{array}{llll} & f(x) = x \text{ for all } x \geq 0, & \text{or} & f(x) = -x \text{ for all } x \geq 0, \\ \text{or} & g(x) = x + a \text{ for all } x \geq 0, & \text{or} & g(x) = -x + a \text{ for all } x \geq 0 \end{array}$$

and

$$\text{(ii)}\quad \begin{array}{llll} & f(x) = x \text{ for all } x < 0, & \text{or} & f(x) = -x \text{ for all } x < 0, \\ \text{or} & g(x) = x + a \text{ for all } x < 0, & \text{or} & g(x) = -x + a \text{ for all } x < 0. \end{array}$$

Let us prove that (i) holds. Putting in (3) 0 in place of y we obtain

$$\text{(9)} \qquad \max\{|f(x)|, |g(x) - a|\} = |x| \quad \text{for } x \in \mathbb{R}.$$

Now, consider two cases:

1° $|f(x)| = x$ for all $x \geq 0$. By continuity of f there is either $f(x) = x$ for $x \geq 0$, or $f(x) = -x$ for $x \geq 0$.

2° There exists an $x_0 > 0$ such that $|f(x_0)| \neq x_0$. Therefore, by (6), $|f(x)| \neq x$ for all $x \geq x_0$ and so, by (9), we have $|g(x) - a| = x$ for $x \geq x_0$. Continuity of g implies that either $g(x) = x + a$ for $x \geq x_0$ or $g(x) = -x + a$ for $x \geq x_0$. This, together with points (iii) and (iv) from Step 2, yields

$$g(x) = x + a \quad \text{for } x \geq 0 \quad \text{or} \quad g(x) = -x + a \quad \text{for } x \geq 0.$$

The proof of (ii) is analogous.

Step 4. For arbitrary $x_0 \in \mathbb{R}$ one of the following equalities is true:

$$\begin{array}{llll} & f(x_0) = -f(-x_0) = x_0 & \text{or} & f(x_0) = -f(-x_0) = -x_0 \\ \text{or} & g(x_0) - a = -g(-x_0) + a = x_0 & \text{or} & g(x_0) - a = -g(-x_0) + a = -x_0. \end{array}$$

To prove this, put, first, in (3) x_0 and $-x_0$ in place of x and y, respectively. Thus we have

$$|f(x_0) - f(-x_0)| = 2|x_0| \quad \text{or} \quad |g(x_0) - g(-x_0)| = 2|x_0|.$$

1° Suppose that $|f(x_0) - f(-x_0)| = 2|x_0|$. In view of (8), it is possible only if $f(x_0) = x_0$ and $f(-x_0) = -x_0$ or if $f(x_0) = -x_0$ and $f(-x_0) = x_0$.

2° Similarly, in the case $|g(x_0) - g(-x_0)| = 2|x_0|$, (8) implies that either $g(x_0) = x_0+a$ and $g(-x_0) = -x_0+a$ or $g(x_0) = -x_0+a$ and $g(-x_0) = x_0+a$.
Step 5. We prove that:

(i) if there exists an $x_0 < 0$ such that $|f(x_0)| < |x_0|$ then either $g(x) = x + a$ for $x \in \mathbb{R}$ or $g(x) = -x + a$ for $x \in \mathbb{R}$;

(ii) if there exists an $x_0 < 0$ such that $|g(x_0) - a| < |x_0|$ then either $f(x) = x$ for $x \in \mathbb{R}$ or $f(x) = -x$ for $x \in \mathbb{R}$.

To prove these statements, suppose that there exists $x_0 < 0$ such that $|f(x_0)| < |x_0|$. By (6) one has $|f(x)| < |x|$ for all $x \leq x_0$ and so, according to Step 4 and the continuity of g, we have either $g(x) - a = -g(-x) + a = x$ for $x \leq x_0$ or $g(x) - a = -g(-x) + a = -x$ for $x \leq x_0$. Using the points (iii),(iv) occuring in Step 2 we can replace "for $x \leq x_0$" by "for $x \leq 0$" and then, by symmetry, we have simply

$$g(x) = x + a \quad \text{for } x \in \mathbb{R} \quad \text{or} \quad g(x) = -x + a \quad \text{for } x \in \mathbb{R}.$$

That is why (i) holds. Similarly, supposing that $|g(x_0) - a| < |x_0|$ for an $x_0 < 0$ we have, by (6), that $|g(x) - a| < |x|$ for $x \leq x_0$ and, according to Step 4 and the continuity of f, we obtain that either $f(x) = -f(-x) = x$ for $x \leq x_0$ or $f(x) = -f(-x) = -x$ for $x \leq x_0$. Using properties (i), (ii) proved in Step 2 we get

$$f(x) = x \quad \text{for } x \in \mathbb{R} \quad \text{or} \quad f(x) = -x \quad \text{for } x \in \mathbb{R}$$

so (ii) is proved as well.
Step 6. Now, we are going to finish the proof of the proposition. According to Step 3, we have four cases to consider.

Case 1°. $f(x) = x$ for $x \geq 0$. Then the following subcases are possible:

- $f(x) = x$ for $x < 0$ so, $f(x) = x$ for all $x \in \mathbb{R}$.

- $f(x) = -x$ for $x < 0$. Then for any $x \in \mathbb{R}\setminus\{0\}$ there is $f(x) \neq -f(-x)$. Therefore, according to Step 4 and the continuity of g, we have either $g(x) = x + a$ for $x \in \mathbb{R}$ or $g(x) = -x + a$ for $x \in \mathbb{R}$.
- There exists $x_0 < 0$ such that $|f(x_0)| < |x_0|$. Then Step 5 provides that either $g(x) = x + a$ for $x \in \mathbb{R}$ or $g(x) = -x + a$ for $x \in \mathbb{R}$.

Case 2°. $f(x) = -x$ for $x \geq 0$. In this case, proceeding similarly as above one obtains that either $f(x) = x$ for $x \in \mathbb{R}$ or $g(x) = x + a$ for $x \in \mathbb{R}$ or $g(x) = -x + a$ for $x \in \mathbb{R}$.

Case 3°. $g(x) = x + a$ for $x \geq 0$. We have to consider three subcases:

- $g(x) = x + a$ for $x < 0$ so, $g(x) = x + a$ for all $x \in \mathbb{R}$.
- $g(x) = -x + a$ for $x < 0$. Therefore, for each $x \in \mathbb{R} \setminus \{0\}$, there is $g(x) - a \neq -g(-x) + a$ and, in view of Step 4 and the continuity of f, we obtain either $f(x) = x$ for $x \in \mathbb{R}$ or $f(x) = -x$ for $x \in \mathbb{R}$.
- There exists an $x_0 < 0$ such that $|g(x_0)| < |x_0|$. According to Step 5 either $f(x) = x$ for $x \in \mathbb{R}$ or $f(x) = -x$ for $x \in \mathbb{R}$.

Case 4°. $g(x) = -x + a$ for $x \geq 0$. In this case, proceeding similarly as in Case 3°, we are able to prove that either $g(x) = -x + a$ for $x \in \mathbb{R}$ or $f(x) = x$ for $x \in \mathbb{R}$ or $f(x) = -x$ for $x \in \mathbb{R}$. This finishes the proof of the proposition. □

Now, we shall admit unbounded intervals as values of a considered set-valued function F. Denote by $ccl(\mathbb{R})$ the family of all nonempty, convex, closed subsets of $\mathbb{R}$ i.e., the family of all closed (but not necessarily bounded) intervals on $\mathbb{R}$. In $ccl(\mathbb{R})$ one can introduce "almost" the same Hausdorff metric, say d'_H, defined at the beginning of this paper. Namely, for $A, B \in ccl(\mathbb{R})$ we put

$$d'_H(A,B) := \inf\{r \geq 0 : \ A \subset B + rI \text{ and } B \subset A + rI\}$$

where $I = [-1,1]$. In the present case infinite distance is possible. The following is a generalization of Lemma 1.

Lemma 2. *For any $A, B \in ccl(\mathbb{R})$, accepting that $(+\infty)-(+\infty) = (-\infty)-(-\infty) = 0$, we have*

$$d'_H(A,B) = \max\{|\inf A - \inf B|, |\sup A - \sup B|\}. \tag{10}$$

Proof. 1. In the case $A, B \in cc(\mathbb{R})$ we can apply Lemma 1, because in this case $d'_H(A,B) = d_H(A,B)$.
2. If exactly one interval, say A, belongs to $cc(\mathbb{R})$ then $d'_H(A,B) = \infty$ and simultanously either $|\inf A - \inf B| = \infty$ or $|\sup A - \sup B| = \infty$.
3. Up to symmetry, the following cases remain:

- $A = (-\infty, a], B = (-\infty, b]$;
- $A = (-\infty, a], B = [b, +\infty)$;
- $A = (-\infty, a], B = (-\infty, +\infty)$;
- $A = B = (-\infty, +\infty)$.

It is simple to check that in each case (10) holds. □
Any set-valued function $F : \mathbb{R} \to ccl(\mathbb{R})$ must have a form:

$$F(x) = \overline{(f(x), g(x))} \quad \text{for } x \in \mathbb{R} \qquad (11)$$
$$\text{where } f : \mathbb{R} \to \mathbb{R} \cup \{-\infty\},\ g : \mathbb{R} \to \mathbb{R} \cup \{+\infty\} \text{ and } f \le g.$$

Set-valued function $F : \mathbb{R} \to ccl(\mathbb{R})$ is called an *isometry* iff

$$d'_H(F(x), F(y)) = |x - y| \quad \text{for } x, y \in \mathbb{R}. \qquad (12)$$

Proposition 2. *Let $F : \mathbb{R} \to ccl(\mathbb{R})$ have form (11) and satisfy (12). Then*

1) *If there exists an $x_0 \in \mathbb{R}$ such that $f(x_0) = -\infty$ then $f(x) = -\infty$ for all $x \in \mathbb{R}$ and (with some $c \in \mathbb{R}$) either $g(x) = x + c$ for $x \in \mathbb{R}$ or $g(x) = -x + c$ for $x \in \mathbb{R}$.*

2) *If there exists an $x_0 \in \mathbb{R}$ such that $g(x_0) = +\infty$ then $g(x) = +\infty$ for all $x \in \mathbb{R}$ and (with some $c \in \mathbb{R}$) either $f(x) = x + c$ for $x \in \mathbb{R}$ or $f(x) = -x + c$ for $x \in \mathbb{R}$.*

Proof. For $x \in \mathbb{R}$ we have $\inf F(x) = f(x)$ and $\sup F(x) = g(x)$. Assume that $f(x_0) = -\infty$ and suppose that there exists an $x_1 \in \mathbb{R}$ such that $f(x_1) > -\infty$. Therefore, by Lemma 2, $d'_H(F(x_0), F(x_1)) = +\infty$, i.e., F does not satisfy (12). Thus we have $f(x) = -\infty$ for all $x \in \mathbb{R}$ and $|\inf F(x) - \inf F(y)| = 0$ for all $x, y \in \mathbb{R}$. This implies $|g(x) - g(y)| = |x - y|$ for $x, y \in \mathbb{R}$, and, finally,

$$g(x) = x + c \quad \text{for } x \in \mathbb{R} \qquad \text{or} \qquad g(x) = -x + c \quad \text{for } x \in \mathbb{R}.$$

The proof of point 2) is similar. □

Corollary 1. *If* $F : \mathbb{R} \to ccl(\mathbb{R})$ *has form (11) and satisfies (12) then either* $F(x) \in cc(\mathbb{R})$ *for all* $x \in \mathbb{R}$ *or* F *has one of the following forms (with an arbitrary* $c \in \mathbb{R}$*):*

1) $F(x) = (-\infty, x + c]$ *for* $x \in \mathbb{R}$,

2) $F(x) = (-\infty, -x + c]$ *for* $x \in \mathbb{R}$,

3) $F(x) = [x + c, +\infty)$ *for* $x \in \mathbb{R}$,

4) $F(x) = [-x + c, +\infty)$ *for* $x \in \mathbb{R}$.

Now, we are going to formulate the main theorem which gives us a complete characterization of the set-valued isometries on the real line.

Theorem 1. *Let a set-valued function* $F : \mathbb{R} \to ccl(\mathbb{R})$ *have form (11). Then* F *is an isometry (i.e. (12) holds) if and only if, for some constant* $c \in \mathbb{R}$*, there is either*

$$1^\circ \quad \left[\begin{array}{c} f(x) = x + c,\ x \in \mathbb{R} \\ or \\ f(x) = -x + c,\ x \in \mathbb{R} \end{array}\right] \quad and \quad \left[\begin{array}{c} |g(x) - g(y)| \le |x - y|,\ x, y \in \mathbb{R} \\ or \\ g(x) = +\infty,\ x \in \mathbb{R} \end{array}\right]$$

or

$$2^\circ \quad \left[\begin{array}{c} g(x) = x + c,\ x \in \mathbb{R} \\ or \\ g(x) = -x + c, x \in \mathbb{R} \end{array}\right] \quad and \quad \left[\begin{array}{c} |f(x) - f(y)| \le |x - y|,\ x, y \in \mathbb{R} \\ or \\ f(x) = -\infty,\ x \in \mathbb{R} \end{array}\right].$$

Proof. To prove the necessity it sufficies to apply Proposition 1 (without the assumption $f(0) = 0$) and Proposition 2 with Corollary 1. The sufficiency is obvious. □

Corollary 2. *If* $F : \mathbb{R} \to ccl(\mathbb{R})$ *has form (11), satisfies (12) and there exists* $x_0 \in \mathbb{R}$ *such that* $F(x_0) = \{y_0\}$ *(one point) then* F *is a single-valued either for all* $x \ge x_0$ *or for all* $x \le x_0$.

Proof. 1. Assume that $f(x) = x + c$ for $x \in \mathbb{R}$. $F(x_0)$ is a single point so, $g(x_0) = f(x_0) = x_0 + c$. By (7) $\operatorname{graph} g \subset C(x_0, x_0 + c)$ which yields, in particular, $g(x) \leq x + c$ for $x \geq x_0$. But as $g(x) \geq f(x) = x + c$ for $x \in \mathbb{R}$, we have $f(x) = g(x)$ for $x \geq x_0$.
2. If $f(x) = -x + c$ for $x \in \mathbb{R}$ then $f(x_0) = g(x_0)$ implies that $\operatorname{graph} g \subset C(x_0, -x_0 + c)$ so, in particular, $g(x) \leq -x + c$ for $x \leq x_0$. As $g(x) \geq f(x) = -x + c$ for $x \in \mathbb{R}$, we have $f(x) = g(x)$ for $x \leq x_0$.
Similarly, in the case 3. where $g(x) = x + c$ for $x \in \mathbb{R}$ and in the case 4. where $g(x) = -x + c$ for $x \in \mathbb{R}$, one obtains that $f(x) = g(x)$, respectively, either for $x \leq x_0$ or for $x \geq x_0$.□

3.

In many papers dealing with the set-valued functions some properties, which generalize corresponding properties of single-valued functions, are considered. In those papers the problem of the existence of suitable selections, appears very often. For example in [2] the existence of additve selections for additive set-valued functions is investigated; in [1] the authors deal with additive selections of subadditive set-valued functions. Additive selections of superadditive set-valued functions are considered in [3] and quadratic selections for subquadratic set-valued functions in [4]. In our considerations concerning set-valued isometries we can ask about the existence of isometric, single-valued, selections. Generally, for mappings $F : X \to cc(Y)$ as at the very beginning of the paper, this question seems to be interesting. However, the answer is unknown to the author. In the case $F : \mathbb{R} \to ccl(\mathbb{R})$ a positive answer is just a simple consequence of Theorem 1. Indeed, we have

Theorem 2. *Let $F : \mathbb{R} \to ccl(\mathbb{R})$ be a set-valued isometry. Then, there exists a selection $i : \mathbb{R} \to \mathbb{R}$ (i.e., $i(x) \in F(x)$ for all $x \in \mathbb{R}$) which is an isometry. Moreover, if there exists an $x_0 \in \mathbb{R}$ such that $F(x_0)$ coincide with a single point then such an isometric selection is unique.*

Proof. Suppose that F has form (11). By Theorem 1 either f or g is an isometry and a selection of F as well. As each isometry $i : \mathbb{R} \to \mathbb{R}$ must have a form $i(x) = x + c$ for $x \in \mathbb{R}$ or $i(x) = -x + c$ for $x \in \mathbb{R}$ in the case $f(x_0) = g(x_0)$, according to Corollary 2, an isometric selection is unique. □

4.

Finally, we give an example showing that a set-valued isometry neither has to be sub-... nor super-additive. Recall that the Mazur-Ulam theorem states that a surjective isometry $I : X \to Y$, with $I(0) = 0$, is additive.

Example 1. Let $F(x) := [f(x), g(x)]$ for $x \in \mathbb{R}$; where

$$f(x) = x \quad \text{for } x \in \mathbb{R}$$

and

$$g(x) = \begin{cases} -x-2 & \text{for} \quad x < -4, \\ 2 & \text{for} \quad -4 \le x < -1, \\ -x+1 & \text{for} \quad -1 \le x < 0, \\ x+1 & \text{for} \quad x \ge 0. \end{cases}$$

We have $|f(x) - f(y)| = |x - y|$ for $x, y \in \mathbb{R}$ and it is easy to check that $|g(x) - g(y)| \le |x - y|$ for $x, y \in \mathbb{R}$. It means that F is a set-valued isometry. However,

$$F(0) + F(-1) = [-1, 3] \supset [-1, 2] = F(-1),$$

$$F(-3) + F(-4) = [-7, 4] \subset [-7, 5] = F(-7).$$

Thus F is neither sub-... nor super-additive.

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REZIME

IZOMETRIJE SKUPOVNIH VREDNOSTI NA REALNOJ LINIJI

Proučavane su funkcije sa vrednostima u skupu zatvorenih intervala definisane na R. Koristeći Hausdorfovu metriku na skupu zatvorenih intervala definisana je izometrija skupovnih vrednosti i data je njena karakterizacija.

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