Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 23, 2 (1993), 113 - 120

Review of Research
Faculty of Science
Mathematics Series

PERTURBATIONS OF A LINEAR DIFFERENTIAL EQUATION

Mustafa R.S. Kulenović
Faculty of Science, V. Putnika 43
Sarajevo

Mirko Budinčević
Institute of Mathematics, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

The paper deals with an n-th order linear differential equation and gives conditions under which its linear perturbation preserves the same L^p-affiliation.

AMS Mathematics Subject Classification (1991): 34K15.
Key words and phrases: Perturbation, linear differential equation.

1. Introduction

In this paper we will find the conditions implying that all the j-th derivatives $(0 \leq j \leq n-1)$ of the solutions of a perturbed linear differential equation belong to $L^p[0,\infty)$ for some $p \geq 1$, provided that all the j-th derivates of solutions of the unperturbed equation possess the same property.

Such kind of results are originated by Weyl's investigation of singular boundary value problems for a second order equation

$$(1) \qquad y'' + p(t)y = 0, \; t \geq 0,$$

and are given in the well known

Theorem 1. *(Weyl [1],[2]). Consider the equation (1) along with the perturbed linear equation*

$$(2) \qquad y'' + (p(t) + q(t))y = 0, \; t \geq 0,$$

where both p and q are locally integrable on $[0, \infty)$. Then the following Weyl alternative holds: if all the solutions of (1) are (not) in $L^2[0, \infty)$ and $g(t) \in L^\infty[0, \infty)$, then all solutions of (2) are (not) in $L^2[0, \infty)$.

Weyl's alternative result was, in some sense, extended by Patula and Wong [3], to include an arbitrary $L^p[0, \infty)$-perturbation $q(t)$. Wong [4] obtained related results to the theorem for a special case of general self-adjoint equation of the $2n$-th order, while Zettl [5] gave some generalizations for the general n-th order linear equation. All this was improved or generalized in our paper [6].

Recently, Butler and Rao ([7], theorem 4.1) improved the result of Patula and Wong. Using a technique similar to this in paper [6], we are able to generalize and modify this result to a case of the general n-th order linear differential equation

$$(3) \qquad Dy \equiv y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_0(t)y = 0, \; t \geq 0$$

and corresponding linear perturbation

$$(4) \qquad D_L y \equiv Dy + q_{n-1}(t)y^{(n-1)} + \cdots + p_0(t)y = 0, \; t \geq 0.$$

2. Preliminaries

In this section we introduce some notations and list well-known results which will be used in a sequel. Let $\mathcal{D}$ be any of the differential operators D or D_L. Then we define

$$S^k(\mathcal{D}) = \{y^{(k)} : \mathcal{D}y = 0\} \; for \; k = 0, \ldots, n-1.$$

Our first result is a form of the variation of parameters formula.

Lemma 1. *(See Golderg [8] p. 140). Suppose f is a locally L^p, $p \geq 1$, function and y is the solution of $\mathcal{D}y = f(t)$. Then*

$$y^{(j)}(t) = \sum_{i=1}^{n} c_i y_i^{(j)}(t) + \int_0^t \frac{\partial^j}{\partial t^j}(G(s,t))f(s)ds, \; for \; j = 0, 1, \ldots, n-1,$$

where

$$G(s,t) = \frac{\begin{vmatrix} y_1(s) & \dots & y_n(s) \\ \vdots & & \vdots \\ y_1^{(n-2)}(s) & \dots & y_n^{(n-2)}(s) \\ y_1(t) & \dots & y_n(t) \end{vmatrix}}{W_n(s)}$$

and $W_n(s) = \exp(-\int_0^s p_{n-1}(u)du)$ *is a Wronskian of* $y_1, \dots, y_n$.

Our next result is of a technical nature.

Lemma 2. *(See Beckenbach and Bellman [9] p.170). If* $u \in L^p$, $u^{(n)} \in L^r$ *for* $p, r \geq 1$ *and* $n > 1$ *then* $u^{(k)} \in L^m$ *for* $m \geq \max\{p, r\}$ *and* $k = 0, 1, \dots, n-1$.

The next result is a Bihari-type inequality.

Lemma 3. *(See Lakshmikantham, Leela and Martynyuk [10] p.17). Let* $u(t)$, $v(t)$ *be two non-negative functions, locally integrable on* $[0, \infty)$. *Then the following inequalities*

$$u(t) \leq c + \int_0^t v(s)[u(s)]^p ds, \ c \geq 0 \ for \ p \in [0, 1),$$

$$u(t) \leq c + \int_0^t v(s)u(s)ds, \ c \geq 0, \ p = 1,$$

imply that

$$u(t) \leq [C^{1-p} + (1-p)\int_0^t v(s)ds]^{\frac{1}{1-p}}, \ for \ p \in [0; 1)$$

and

$$u(t) \leq c\exp(\int_0^t v(s)ds), \ for \ p = 1,$$

respectively.

An elementary inequality which will be used in the sequal is given in

Lemma 4. *(See Hardy, Littlewood and Polya [11],p.26). Let $a_i \geq 0$ for $i = 1, ..., m$. Then*

$$\sum_{i=1}^{m} a_i^p \leq m^{1-p}(\sum_{i=1}^{m} a_i)^p, \ for\ 0 \leq p \leq 1,$$

and

$$\sum_{i=1}^{m} a_i^p \geq m^{1-p}(\sum_{i=1}^{m} a_i)^p, \ for\ p \geq 1.$$

3. Results

We are now able to give our main result:

Theorem 2. *Let $S(D) \cup S^{n-1}(D) \subseteq L^p \cap L^\infty[0,\infty)$ for $1 \leq p \leq \infty$ and $q_i \in L^k[0,\infty)$ for $k \geq 1$ and $p \leq \frac{2k}{k-1}$. If $p_{n-1}^+(t) \in L[0,\infty)$, then $S(D_L) \cup S^{n-1}(D_L) \subseteq L^p \cap L^\infty[0,\infty)$, where $p_{n-1}^+(t) = \max\{0, p_{n-1}(t)\}$.*

Proof. Starting with the representation given by Lemma 1 and using the fact that $G(t,t) = 0$, we get

$$y^{(k)}(t) = \sum_{i=1}^{n} c_i y_i^{(k)}(t) + \sum_{i=1}^{n} y_i^{(k)}(t) \int_0^t (-1)^{n+1} W_i \cdot \tag{5}$$

$$\cdot \exp(\int_0^s p_{n-1}(u)du) f(s) ds,$$

$$\text{for } k = 0, \ldots, n-1,$$

where $W_i(s)$ is the Wronskian, and

$$f(s) = -\sum_{i=0}^{n-1} q_i(s) y^{(i)}(s).$$

Lemma 2 implies that $\cup_{k=0}^{n-2} S^k(D) \subseteq \cap_{p=2}^{\infty} L^p[0,\infty)$, and we get

$$|W_i(s)| \leq K \sum_{\substack{j=1 \\ j \neq i}}^{n} |y_j(s)| \leq K \sum_{j=1}^{n} |y_j(s)|,$$

for every $i = 1, \ldots, n$ and some $K > 0$. So, by (5) we get

$$(6) \qquad |y^{(k)}(t)| \leq C\Phi_k(t)[1 + \int_0^t \Phi_0(s) \exp(\int_0^s p_{n-1}(u)du)|f(s)|ds],$$

$$t \geq 0, \;\; k = 0, \ldots, n-1 \;,$$

where we denote $\Phi_k(t) = \max\{y_i^{(k)}(t)| : \; i = 1, \ldots, n\}$ which by $p_{n-1}^{+}(t) \in L[0, \infty)$ immediately implies

$$(7) \qquad |y^{(k)}(t) \leq C_1\Phi_k(t)[1 + \int_0^t \Phi_0(s)q(s) \sum_{i=0}^{n-1} |y^{(i)}(s)|ds],$$

$$t \geq 0, \;\; k = 0, \ldots, n-1,$$

where $q(s) = \max\{|q_i(s)| : \; i = 0, \ldots, n-1\}$.

We shall put

$$\frac{1}{p} + \frac{1}{p'} = \frac{1}{k} + \frac{1}{k'} = 1$$

and differ the next cases:

a) $1 \leq k \leq p'$.

Relation (7) and $S(D) \cup S^{n-1}(D) \subseteq L^{\infty}[0, \infty)$ imply

$$\sum_{k=0}^{n-1} |y^{(k)}(t)| \leq C_2[1 + \int_0^t \Phi_0(s)q(s) \sum_{i=0}^{n-1} |y^{(i)}(s)|ds], \; t \geq 0$$

As $\Phi_0(t) \in L^p \cap L^{\infty}[0, \infty)$ implies $\Phi_0(t) \in L^{k'}[0, \infty)$ and because of $q(t) \in L^k[0, \infty)$, $\Phi_0(t)q(t) \in L[0, \infty)$. Now, by Lemma 3, it follows that

$$\sum_{k=0}^{n-1} |y^{(k)}(t)| \in L^{\infty}[0, \infty), \; t \geq 0.$$

Using again relation (7), we obtain

$$|y^{(k)}(t)| \leq C_1\Phi_k(t)[1 + C_3 \int_0^t \Phi_0(s)q(s)ds] \leq C_4\Phi_k(t)$$

$$t \geq 0 \;, \;\; k = 0, \ldots, n-1,$$

which implies that $y^{(k)}(t) \in L^p[0,\infty)$, $k = 0,\dots,n-1$.

b) $1 \le p \le 2 \le p' < k$.
Define the sets A_t and B_t as follow: $B_t = [0,t] \setminus A_t$, $A_t = \{s \in [0,t] : q(s) \le 1\}$. Using Hőlder's inequality, we get

$$\int_0^t \Phi_0(s)q(s)\sum_{i=0}^{n-1}|y^{(i)}(s)|ds = \int_{A_t} + \int_{B_t} \le$$

$$\le \Big(\int_{A_t}\Big(\sum_{i=0}^{n-1}|y^{(i)}(s)|\Big)^p ds\Big)^{\frac{1}{p}}\Big(\int_{A_t}(\Phi_0(s)q(s))^{p'}ds\Big)^{\frac{1}{p'}} + \int_{B_t} \le$$

$$\le C_5\Big(\int_0^t \sum_{i=0}^{n-1}|y^{(i)}(s)|^p ds\Big)^{\frac{1}{p}}\Big(\int_0^t \Phi_0^{p'}(s)ds\Big)^{\frac{1}{p'}} +$$

$$+C_5\Big(\int_0^t \sum_{i=0}^{n-1}|y^{(i)}(s)^p ds\Big)^{\frac{1}{p}}\Big(\int_0^t q^{p'}(s)ds\Big)^{\frac{1}{p'}} \le C_6\Big(\int_0^t \sum_{i=0}^{n-1}|y^{(i)}(s)|^p ds\Big)^{\frac{1}{p}}.$$

Putting the last inequality in relation (7), we obtain

$$|y^{(k)}(t)| \le C_1\Phi_k(t)\Big[1 + C_6\Big(\int_0^t \sum_{i=0}^{n-1}|y^{(i)}(s)|^p ds\Big)^{\frac{1}{p}}\Big], \tag{8}$$

$$t \ge 0\,,\quad k = 0,\dots,n-1,$$

which, according to Lemma 4., implies

$$\sum_{k=0}^{n-1}|y^{(k)}(t)|^p \le C_7\sum_{k=0}^{n-1}\Phi_k^p(t)\Big[1 + C_6\int_0^t \sum_{i=0}^{n-1}|y^{(i)}(s)|^p ds\Big],\ t \ge 0.$$

Using the fact that $\Phi_k^P(t) \in L[0,\infty)$ for $k = 0,\dots,n-1$ and integrating the last inequality from 0 to t we get, applying Lemma 3, that $S(D_L) \cup S^{n-1}(D_L) \subseteq L^p[0,\infty)$ which, according to relation (8), leads to $S(D_L) \cup S^{n-1}(D_L) \subseteq L^\infty[0,\infty)$.

c) $1 < p' < 2 < p$, $p' < k \le \frac{p}{p-2}$.
Now,

$$\Big[\int_{A_t}(\Phi_0(s)q(s))^{p'}ds\Big]^{\frac{1}{p'}} \le \Big(\int_{A_t}\Phi_0^p(s)ds\Big)^{\frac{1}{p}}\Big(\int_{A_t}(q(s))^{\frac{pp'}{p-p'}}ds\Big)^{\frac{p-p'}{pp'}} \le$$

$$\leq \left(\int_{A_t} \Phi_0^p(s)ds\right)^{\frac{1}{p}} \left(\int_{A_t} q^k(s)ds\right)^{\frac{p-p'}{pp'}} \leq C_8,\ t \geq 0,$$

(Because $k \leq \frac{pp'}{p-p'} = \frac{p}{p-2}$) and the proof follows the same line as in case b).

Conditions a),b) and c) cover the whole domain $p, k \geq 1,\ p \leq \frac{2k}{k-1}$ and the proof is complete.

□

In the case u=2 our theorem is an improvement of (theorem 4,[1]). In the same peper was proved that it is the best possible result in the sense that for every couple (p, k), $k > 1$, $p > \frac{2k}{k-1}$ there exists an equation of the form(1) such that all the solutions are in $L^p \cap L^\infty[0, \infty)$ and the function $q(t) \in L^k[0, \infty)$ such that some solution of (2) is not in $L^p[, \infty)$.

References

[1] Hille, E., Lectures on ordinary differential equations, Addison-Wesley Reading, Massachusetts,1969.

[2] Naimark, M.A., Linear differential operators, Nauka, Moscow, 1969.

[3] Patula, W.T., Wong, J.S.W., On L^p-analogue of the Weyl alternative, Math. Ann. 197(1972) 9-28.

[4] Wong, J.S.W., Square integrable solutions of perturbed linear differential equations, Proc. Edinburgh Math. Soc. Sect. A 73 (1974/75), 251-254.

[5] Zettl, A., Square integrable solutions of Ly=f(t,y), Proc. Amer. Math. Soc. 26 (1970) 635-639.

[6] Kulenović, M.R.S., Budinčević, M., L^p-perturbations of linear differential equations, Glasnik mat., 22 (1987) 309-325.

[7] Butler G.J., Rao, V.S.H., Some properties of second order linear differential equations and perturbations that preserve them, SIAM J. Math. Ann.

[8] Goldberg, S., Unbounded linear operators, Mc Graw-Hil, New York, 1986.

[9] Beckenbach, E., Bellman, R., Inequalities, Springer-Verlag, Berlin, 1961.

[10] Lakshmikantham, V., Leela, S., Martynyuk, A.A., Stability analysis of nonlinear systems, Marcel Decker, New York, 1988.

[11] Hardy, G.H., Littlewood, J.E., Polya, G., Inequalites, Cambridge Univ. Press, Cambridge,1952.

REZIME

PERTURBACIJE LINEARNE DIFERENCIJALNE JEDNAČINE

U radu se posmatra linearna diferencijalna jednačina $n-$ tog reda i daju uslovi pod kojima njena linearna perturbacija zadržava istu L^p pripadnost.

Received by the editors June 7, 1991